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# GENERIC FAMILIES AND MODELS OF SET THEORY WITH THE AXIOM OF CHOICE

KRZYSZTOF CIESIELSKI AND WOJCIECH GUZICKI

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ABSTRACT. Let $M$ be a countable transitive model of $ZFC$ and $A$ be a countable $M$-generic family of Cohen reals. We prove that there is no smallest transitive model $N$ of $ZFC$ that either $M \cup A \subseteq N$ or $M \cup \{A\} \subseteq N$. It is also proved that there is no smallest transitive model $N$ of $ZFC^-$ ($ZFC$ theory without the power set axiom) such that $M \cup \{A\} \subseteq N$. It is also proved that certain classes of extensions of $M$ obtained by Cohen generic reals have no minimal model.

## 0. INTRODUCTION

In the paper [1] Blass proved that for a given countable transitive model $M$ of $ZF$ and a countable $M$-generic family $A$ of Cohen reals there exists no smallest transitive model $N$ of $ZF$ such that $M \cup A \subseteq N$. On the other hand Cohen's model $M[A]$ is the smallest transitive model $N$ of $ZF$ such that $M \cup \{A\} \subseteq N$. Similar questions can be asked about models of the theory $ZF^-$ (Zermelo–Fraenkel set theory without the power set axiom and with the collection and comprehension schemes instead of the replacement scheme). Then $M[A]$ is also the smallest transitive model $N$ of $ZF^-$ such that $M \cup \{A\} \subseteq N$. It is an observation of Szczepaniak that the model $M(A) = \bigcup\{M[a] : a \subseteq A \& |a| < \omega\}$ is a model of $ZF^-$ (see [5], for a proof see also [1] or [3]) and it is obvious that $M(A)$ is the smallest model $N$ of $ZF^-$ such that $M \cup A \subseteq N$.

In this paper we ask four analogous questions: If $M$ is a countable transitive model of the theory $ZFC$ (or $ZFC^-$) and $A$ is a countable $M$-generic family of Cohen reals, does there exist the smallest model $N$ of $ZFC$ (or $ZFC^-$, respectively) such that $M \cup \{A\} \subseteq N$ or $M \cup A \subseteq N$? Since $M(A)$ satisfies the axiom of choice it is the smallest transitive model $N$ of $ZFC^-$ such that $M \cup A \subseteq N$. We will show that in the remaining three cases the answers are negative: the respective smallest models do not exist.

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We conclude the paper with investigating extensions of $M$ obtained by adding many Cohen generic reals. We show that among such models there exists no minimal model containing a given generic family as an element.

## 1. Nonexistence of the smallest models

We begin with introducing several notions of forcing. For any sets $A$ and $B$ we define $\mathbb{P}(A, B)$ as the set of functions $p: \mathrm{dom}(p) \to B$, where $\mathrm{dom}(p)$ is a finite subset of $A$, ordered by reverse inclusion. Next

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(A, 2), \\ \mathbb{P} &= \mathbb{P}(\omega), \\ \mathbb{Q}_\alpha &= \mathbb{P}(\alpha \times \omega) \quad \text{for } \alpha \in On, \\ \mathbb{Q} &= \mathbb{Q}_{\omega_1^M} \quad \text{where } M \text{ is a fixed countable transitive model of } ZFC. \end{aligned}$$

For any $M$-generic filter $G \subseteq \mathbb{Q}_\alpha$ we define $M$-generic filters $(G)_\xi \subseteq \mathbb{P}$ for $\xi \in \alpha$ with the formula

$$p \in (G)_\xi \text{ iff } \exists q \in G \forall n \in \mathrm{dom}(p)[\langle \xi, n\rangle \in \mathrm{dom}(q) \& q(\xi, n) = p(n)].$$

Next we define

$$\widetilde{G} = \{(G)_\xi : \xi \in \alpha\}.$$

**Definition 1.1.** A family $\mathcal{G}$ of filters in $\mathbb{P}$ is an $M$-generic family iff for any finite number of filters $G_1, \ldots, G_n \in \mathcal{G}$ the $n$-tuple $\langle G_1, \ldots, G_n\rangle$ is $M$-generic over $\mathbb{P}^n$ and for any forcing condition $p \in \mathbb{P}$ there exists $G \in \mathcal{G}$ such that $p \in G$.

It is well known that for any $M$-generic filter $G \subseteq \mathbb{Q}_\alpha$ the family $\widetilde{G}$ is $M$-generic. For countable generic families the converse is also true (see [2]).

**Lemma 1.2** (Friedman, Cohen). *If $A$ is a countable $M$-generic family then there exists an $M$-generic filter $G \subseteq \mathbb{Q}_\omega$ such that $\widetilde{G} = A$. Moreover $M[A]$ is a model of $ZF + \neg AC$.*

We omit the proof of the lemma. □

A well-known property of the model $M[A]$ is given by the following

**Lemma 1.3.** *If $x \subseteq M$ and $x \in M[A]$ then there exists a finite subset $a \subseteq A$ such that $x \in M[a]$.*

We omit the proof of the lemma. □

In the model $M[A]$ we can define the following notion of forcing $\mathbb{R}_\alpha$ for $\alpha \in On - \omega$:

$$\begin{aligned} \mathbb{R}_\alpha &= \{p \in \mathbb{P}(\alpha, A) : p \text{ is one-to-one}\}, \\ p \leq q &\quad \text{iff } q \subseteq p \quad \text{for} \quad p, q \in \mathbb{R}_\alpha. \end{aligned}$$

Now we can prove the following

**Lemma 1.4.** *If $A$ is a countable $M$-generic family and $\alpha \in (On \cap M) - \omega$ then for any $M[A]$-generic filter $H \subseteq \mathbb{R}_\alpha$ there exists an $M$-generic filter $G \subseteq \mathbb{Q}_\alpha$ such that $\widetilde{G} = A$ and $M[A][H] = M[G]$.*

*Proof.* Let $H \subseteq \mathbb{R}_\alpha$ be an $M[A]$-generic filter. We identify generic filters with generic functions; under this identification $H$ is a one-to-one function from $\alpha$ onto $A$.

We define a filter $G$ in $\mathbb{Q}_\alpha$ by the formula

$$G(\xi, n) = H(\xi)(n).$$

Then obviously $\widetilde{G} = A$ and $M[A][H] = M[G]$. We must only show that the filter $G$ is $M$-generic. Let $D \in M$ be a dense subset of $\mathbb{Q}_\alpha$.

We define a subset $E$ of $\mathbb{R}_\alpha$:

$p \in E$ iff there exists $q \in D$ such that for any $\xi \in \alpha$ and $n \in \omega$ if $\langle \xi, n\rangle \in \mathrm{dom}(q)$ then $\xi \in \mathrm{dom}(p)$ and $p(\xi)(n) = q(\xi, n)$.

Obviously $E \in M[A]$. We will show that the set $E$ is dense in $\mathbb{R}_\alpha$. This will suffice because then $E \cap H \neq 0$ and for $p \in E \cap H$ and $q \in D$ such as in the definition of $E$ we have $q \in G$, thus $D \cap G \neq 0$.

In order to show that $E$ is dense in $\mathbb{R}_\alpha$ we take an arbitrary condition $p_0 \in \mathbb{R}_\alpha$.

Let $a = \mathrm{dom}(p_0) = \{\xi_1, \dots, \xi_n\} \subseteq \alpha$,

$$p_0(\xi_i) = G_i \in A \qquad \text{for} \quad i = 1, \dots, n.$$

We define a set $D_\alpha \subseteq \mathbb{P}(a \times \omega)$ as follows:

$$D_a = \{q \in \mathbb{P}(a \times \omega) \colon \exists p \in D(q = p \restriction a \times \omega)\}.$$

Obviously, $D_a \in M$ and $D_a$ is a dense subset of $\mathbb{P}(a \times \omega)$. Since the tuple $\langle G_1, \dots, G_n\rangle$ is $M$-generic in $\mathbb{P}^n$, there exists $q \in D_a$ such that $\langle G_1, \dots, G_n\rangle$ extends $q$ (under a suitable identification of the notions of forcing $\mathbb{P}^n$ and $\mathbb{P}(a \times \omega)$). Let $p \in D$ be such that $q = p \restriction a \times \omega$.

Suppose that $\mathrm{dom}(p) \subseteq (a \cup b) \times \omega$ and $a \cap b = 0$. Let $b = \{\zeta_1, \dots, \zeta_m\}$.

We find distinct filters $G'_1, \dots, G'_m \in A - \{G_1, \dots, G_n\}$ such that the tuple $\langle G'_1, \dots, G'_m\rangle$ extends $p \restriction b \times \omega$. Then we define a condition $r \in \mathbb{R}_\alpha$:

$$\mathrm{dom}(r) = a \cup b,$$
$$r(\xi_i) = G_i,$$
$$r(\zeta_i) = G'_i.$$

Then obviously $r \leq p_0$ and $r \in E$. □

The following is a special case of a corollary to the product lemma.

**Lemma 1.5.** *If a pair of filters $\langle H_1, H_2\rangle$ is $M[A]$-generic over $\mathbb{R}^2_\alpha$ then $M[A][H_1] \cap M[A][H_2] = M[A]$.*

For a proof see [4]. □

**Corollary 1.6.** *For any $M$-generic family $A$ and any infinite $\alpha \in On \cap M$ there exist $M$-generic filters $G_1, G_2 \subseteq \mathbb{Q}_\alpha$ such that $\widetilde{G}_1 = \widetilde{G}_2 = A$ and $M[G_1] \cap M[G_2] = M[A]$.*

*Proof*. We take an $M[A]$-generic pair of filters $\langle H_1, H_2 \rangle$ over $\mathbb{R}^2_\alpha$ and find $M$-generic filters $G_1, G_2 \subseteq \mathbb{Q}_\alpha$ such that $M[A][H_1] = M[G_1]$ and $M[A][H_2] = M[G_2]$ and $\widetilde{G}_1 = \widetilde{G}_2 = A$.

Then by Lemma 1.5

$$M[G_1] \cap M[G_2] = M[A][H_1] \cap M[A][H_2] = M[A]. \quad \square$$

Now we can prove the following:

**Theorem 1.7.** *For any countable $M$-generic family $A$ there exists no smallest model $N$ of $ZFC$ such that $M \cup A \subseteq N$.*

*Proof*. We take $M$-generic filters $G_1, G_2 \subseteq \mathbb{Q}_\omega$ such that $M[G_1] \cap M[G_2] = M[A]$ and $\widetilde{G}_1 = \widetilde{G}_2 = A$. Now it is enough to show that no submodel $N$ of $M[A]$ such that $M \cup A \subseteq N$ can be model of $ZFC$. Let us assume that $N$ is such a model. Then obviously $P(\omega) \cap N = P(\omega) \cap M[A]$. In $N$ there exists a function $f: \alpha \times \omega \to 2$ such that $\alpha = |P(\omega) \cap N|^N$ and

$$P(\omega) \cap N = \{\{n \in \omega: f(\xi, n) = 0\}: \xi \in \alpha\}.$$

Then there exists a finite set $a \subseteq A$ such that $f \in M[a]$, which gives a contradiction because $A \in M[f]$. $\square$

We also have

**Theorem 1.8.** *For any countable $M$-generic family $A$ there exists no smallest model $N$ of $ZFC$ such that $M \cup \{A\} \subseteq N$.*

*Proof*. We take $M$-generic filters $G_1, G_2 \subseteq \mathbb{Q}_\omega$ such that $M[G_1] \cap M[G_2] = M[A]$ and $\widetilde{G}_1 = \widetilde{G}_2 = A$. If $N$ is a smallest model of $ZFC$ such that $M \cup \{A\} \subseteq N$ then $N \subseteq M[A]$. On the other hand, since $A \in N$, we have $M[A] \subseteq N$. This is a contradiction because $M[A] \models \neg AC$. $\square$

Finally, we have

**Theorem 1.9.** *For any countable $M$-generic family $A$ there exist no smallest model $N$ of $ZFC^-$ such that $M \cup \{A\} \subseteq N$.*

*Proof*. We recall that if $G \subseteq \mathbb{Q}$ is an $M$-generic filter then the model $M^*_G = \bigcup\{M[x, \widetilde{G}]: x \in P(\omega) \cap M[G]\}$ is a model of $ZFC^-$ (for a proof see [3]). Now we take $M$-generic filters $G_1, G_2 \subseteq \mathbb{Q}$ such that $\widetilde{G}_1 = \widetilde{G}_2 = A$ and $M[G_1] \cap M[G_2] = M[A]$. Then $M^*_{G_1} \cap M^*_{G_2} \subseteq M[A]$. On the other hand if $N$ is a model of $ZFC^-$ such that $M \cup \{A\} \subseteq N$ then $M[A] \subseteq N$. Therefore $M^*_{G_1} \cap M^*_{G_2} = M[A]$. This gives a contradiction because the smallest model $N$ of $ZFC^-$ such that $M \cup \{A\} \subseteq N$ would have to be equal to $M[A]$ and $M[A] \models \neg AC$. $\square$

We conclude this section with the observation that a lemma converse to Lemma 1.4 is also true.

**Lemma 1.10.** *If $A$ is a countable $M$-generic family and $\alpha \in On \cap M$ is infinite then for any $M$-generic filter $G \subseteq \mathbb{Q}_\alpha$ such that $\tilde{G} = A$ there exists an $M[A]$-generic filter $H \subseteq \mathbb{R}_\alpha$ such that $M[A][H] = M[G]$.*

*Proof.* The filter $G$ defines the following function $g: \alpha \xrightarrow{1-1} A$

$$g(\xi)(n) = \left(\bigcup G\right)(\xi, n) \qquad \text{for } \xi \in \alpha \text{ and } n \in \omega.$$

The filter $H \subseteq \mathbb{R}_\alpha$ is defined as

$$H = \{g \restriction a : a \subseteq \alpha \,\&\, |a| < \omega\} = \{q \in \mathbb{R}_\alpha : q \subseteq g\}.$$

It is obvious that $M[A][H] = M[G]$. We will show that $H$ is an $M[A]$-generic filter in $\mathbb{R}_\alpha$. The filter $H$ has a canonical name $\dot{H}$ with the following property:

If $\dot{a}_i$ is a canonical name for $g(\xi_i)$, $i = 1, \dots, n$, then

$$1 \Vdash \{\langle \xi_i, \dot{a}_i \rangle : i = 1, \dots, n\} \in \dot{H}.$$

Now let $D \in M[A]$ be a dense subset of $\mathbb{R}_\alpha$. We take a name $\dot{D}$ for $D$ such that there exists a subset $X = \{\xi_1, \dots, \xi_n\} \subseteq \alpha$ such that for any permutation $\pi: \alpha \to \alpha$ such that $\pi \in M$ and $\pi(\xi_i) = \xi_i$ for $i = 1, \dots, n$ we have $\pi(\dot{D}) = \dot{D}$. Now we take a condition $p_0 \in G$ such that $p_0 \Vdash$ "$\dot{D}$ is dense in $\dot{\mathbb{R}}_\alpha$" and define

$$E = \{p \in \mathbb{Q}_\alpha : p \Vdash \dot{D} \cap \dot{H} \neq 0\}.$$

It is enough to show that $E$ is dense in $\mathbb{Q}_\alpha$ below $p_0$. Let $p \in \mathbb{Q}_\alpha$ and $p \le p_0$. Let $\operatorname{dom}(p) \subseteq Z \times \omega$, $Z = \{\zeta_1, \dots, \zeta_m\}$ and $X \subseteq Z$. We take a condition

$$q = \{\langle \zeta_i, g(\zeta_i) \rangle : i = 1, \dots, m\} \in \mathbb{R}_\alpha.$$

Obviously $p \Vdash \dot{q} \in \dot{\mathbb{R}}_\alpha$.

We take $p_1 \le p$ and $q_1 \le q$ such that $p_1 \Vdash \dot{q}_1 \in \dot{D}$. We assume that

$$q_1 = q \cup \{\langle \eta_j, g(\rho_j) \rangle : j = 1, \dots, k\}$$

and $\zeta_i \neq \eta_j$, $\zeta_i \neq \rho_j$ for $i = 1, \dots, m$ and $j = 1, \dots, k$.

We define a permutation $\pi: \alpha \to \alpha$ so that

$$\pi(\rho_j) = \eta_j \quad \text{for} \quad j = 1, \dots, k,$$
$$\pi(\zeta_i) = \zeta_i \quad \text{for} \quad i = 1, \dots, m.$$

Then

$$\pi p_1 \Vdash \pi \dot{q}_1 \in \dot{D}.$$

But $\pi p_1 \le p$ and $\pi \dot{D} = D$. Next

$$\pi q_1 = q \cup \{\langle \eta_i, g(\eta_i) \rangle : j = 1, \dots, k\}$$

and hence

$$\pi p_1 \Vdash \pi \dot{q}_1 \in \dot{H}.$$

Therefore $\pi p_1 \Vdash \pi \dot{q}_1 \in \dot{D} \cap \dot{H}$ and finally $\pi p_1 \in E$. $\square$

## 2. Nonexistence of minimal Cohen extensions

We begin with the following:

**Lemma 2.1.** *For any $M$-generic filter $G \subseteq \mathbb{Q}_\omega$ there exists an $M$-generic filter $H \subseteq \mathbb{Q}_\omega$ such that $\widetilde{G} = \widetilde{H}$ and*

$$P(\omega) \cap M[H] \subsetneq P(\omega) \cap M[G].$$

*Proof.* Let $\mathbb{P}_1$ be the following dense subset of $\mathbb{Q}_\omega$:

$$p \in \mathbb{P}_1 \equiv \exists n, m \in \omega(\mathrm{dom}(p) = n \times m)$$

and let $\mathbb{P}_2$ be the following dense subset of $\mathbb{Q}_\omega \times \mathbb{Q}_\omega \times \mathbb{P}$:

$$\langle q, r, s\rangle \in \mathbb{P}_2 \equiv \exists n_1, n_2, m \in \omega(\mathrm{dom}(q) = n_1 \times m$$
$$\& \ \mathrm{dom}(r) = n_2 \times m \ \& \ \mathrm{dom}(s) = n_1 + n_2 \& |s^{-1}(0)| = n_1 \& |s^{-1}(1)| = n_2).$$

Now we define an isomorphism $h: \mathbb{P}_1 \to \mathbb{P}_2$. For $p \in \mathbb{P}_1$ such that $\mathrm{dom}(p) = n \times m$ we put $h(p) = \langle q, r, s\rangle$, where:

$$\begin{aligned} &\mathrm{dom}(s) = n, \\ &s(i) = p(i, 0) \qquad \text{for} \quad i < n, \\ &n_1 = |s^{-1}(0)|, \\ &n_2 = |s^{-1}(1)| \\ &\mathrm{dom}(q) = n_1 \times (m-1), \\ &\mathrm{dom}(r) = n_2 \times (m-1), \end{aligned}$$

if $s(i) = 0$ and $|\{j < i : s(j) = 0\}| = k$ then

$$q(k, l) = p(i, l+1),$$

if $s(i) = 1$ and $|\{j < i : s(j) = 1\}| = k$ then

$$r(k, l) = p(i, l+1).$$

The definition of $h$ is best illustrated by an example. Let a condition $p$ be represented by the following matrix

$$\begin{array}{|c|c|c|c|c|c|c|c|} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & b & c & d & e & f & g & h \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

Then the following matrices represent conditions $q, r, s$ such that $h(p) = \langle q, r, s\rangle$:

$$q: \begin{array}{|c|c|c|c|c|} \vdots & \vdots & \vdots & \vdots & \vdots \\ a & b & e & g & h \\ \hline \end{array}$$

$$r: \begin{array}{|c|c|c|} \vdots & \vdots & \vdots \\ c & d & f \\ \hline \end{array}$$

$$s: \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

Now let $K = h(G \cap \mathbb{P}_1)$. Then $K$ is an $M$-generic filter in $\mathbb{P}_2$ which generates in a natural way an $M$-generic triple of filters $K_1 \times K_2 \times K_3 \subseteq \mathbb{Q}_\omega \times \mathbb{Q}_\omega \times \mathbb{P}$. Now we split the filter $K_3$ into two $M$-generic filters $H_3$ and $H_4$:

$$p \in H_3 \equiv \exists q \in K_3\ \forall n \in \mathrm{dom}(p)(2n \in \mathrm{dom}(q) \& p(n) = q(2n)),$$
$$p \in H_4 \equiv \exists q \in K_3\ \forall n \in \mathrm{dom}(p)(2n+1 \in \mathrm{dom}(q) \& p(n) = q(2n+1)).$$

We also put $H_1 = K_1$ and $H_2 = K_2$. Finally

$$H = h^{-1}(H_1 \times H_2 \times H_3 \cap \mathbb{P}_2).$$

Then obviously $M[H] = M[H_1, H_2, H_3] \subseteq M[G]$. Since $H_4 \in M[G]$ and $H_4$ is $M[H]$-generic in $\mathbb{P}$ we have

$$P(\omega) \cap M[H] \subsetneq P(\omega) \cap M[G].$$

It remains to show that $\widetilde{G} = \widetilde{H}$. For any $M$-generic filter $F \subseteq \mathbb{P}$ we define a filter $\overline{F}$ with the formula

$$\overline{F}(n) = F(n+1)$$

(we recall that an $M$-generic filter is identified with the corresponding $M$-generic function). Now we observe that

$$\begin{aligned} F \in \widetilde{G} &\equiv (F(0) = 0 \& \overline{F} \in \widetilde{K}_1) \vee (F(0) = 1 \& \overline{F} \in \widetilde{K}_2) \\ &\equiv (F(0) = 0 \& \overline{F} \in \widetilde{H}_1) \vee (F(0) = 1 \& \overline{F} \in \widetilde{H}_2) \\ &\equiv F \in \widetilde{H}. \quad \square \end{aligned}$$

**Corollary 2.2.** *For any $M$-generic family $A$ the class of models*

$$\{M[G] : G \subseteq \mathbb{Q}_\omega \text{ is } M\text{-generic } \& \widetilde{G} = A\}$$

*has no minimal element.* □

Now it is easy to generalize Lemma 2.1 to the case of $M$-generic filters in $\mathbb{Q}$. Namely

$$\mathbb{Q} \cong \mathbb{P}(\omega \times \omega) \times \mathbb{P}((\omega_1^M - \omega) \times \omega).$$

Any $M$-generic filter $G \subseteq \mathbb{Q}$ decomposed as $G_1 \times G_2$ where

$$G_1 = G \cap \mathbb{P}(\omega \times \omega),$$
$$G_2 = G \cap \mathbb{P}((\omega_1^M - \omega) \times \omega)$$

and

$$G_1 \text{ is } M[G_2]\text{-generic}.$$

From Lemma 2.1 it follows that there exists an $M[G_2]$-generic filter $H_1 \subseteq \mathbb{P}(\omega \times \omega)$ such that $\widetilde{H}_1 = \widetilde{G}_1$ and

$$P(\omega) \cap M[G_2][H_1] \subsetneq P(\omega) \cap M[G_2][G_1].$$

We put $H = H_1 \times G_2$ and observe that

$$P(\omega) \cap M[H] \subsetneq P(\omega) \cap M[G]$$

and

$$\widetilde{G} = \widetilde{G}_1 \cup \widetilde{G}_2 = \widetilde{H}_1 \cup \widetilde{G}_2 = \widetilde{H}. \quad \square$$

This obviously gives the following

**Corollary 2.3.** *For any countable $M$-generic family $A$ the class of models*

$$\{M_G^*: G \subseteq \mathbb{Q} \text{ is } M\text{-generic } \& \, \widetilde{G} = A\}$$

*has no minimal element.* □

We conclude the paper with the following

**Problem 2.4.** Does there exist a minimal model $N$ of $ZFC$ (or $ZFC^-$) such that $M \cup \{A\} \subseteq N$? Does there exist a minimal model of $N$ of $ZFC$ such that $M \cup A \subseteq N$?

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*Current address* (*W. Guzicki*): Department of Mathematics, Warsaw University, Warsaw, Poland

*Current address* (*K. Ciesielski*): Department of Mathematics, Methodist College, 5400 Ramsey Street, Fayetteville, North Carolina 28311