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FUNCTIONAL ANALYSIS

A Weakly Lindelöf Function Space $C(K)$ without Any Continuous Injection into $c_0(\Gamma)$

by

Krzysztof CIESIELSKI and Roman POL

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Summary. The main result of this note is the following example: Assuming GCH, there exists a function space $C(K)$, K being a compact scattered space, such that $C(K)$ is Lindelöf under the weak topology, but there is no continuous injection of the space $C(K)$ endowed with the weak, or the pointwise topology, into $c_0(\Gamma)$ endowed with the same topology, where Γ is an arbitrary set.

1. Terminology and notation. Our terminology related to topology, set theory and function spaces follows [7], [14] and [18], respectively.

1.1 We let $E(\omega) = \{\alpha : \alpha$ is an ordinal of countable cofinality$\}$. We denote the generalized continuum hypothesis by GCH and MA stand for Martin's Axiom [14, Ch. II, § 2].

1.2. Given a compact space K, we denote by $C(K)$ the space of the real-valued continuous functions on K with the sup-norm; this function space endowed with the pointwise topology is denoted by $C_p(K)$ and $C(K, D)$ is the subspace of $C_p(K)$ consisting of the functions whose range is contained in $D = \{0, 1\}$. Recall, that if K is scattered, then the weak and the pointwise topology coincide on each norm-bounded subet of $C(K)$; in particular, the two topologies in $C(K, D)$ are the same [18; 19.7.7].

1.3. Given a set Γ we denote by $c_0(\Gamma)$ the space of all real-valued functions on Γ vanishing at infinity, i.e. such functions f that for each $\varepsilon > 0$ the set $\{\gamma : |f(\gamma)| > \varepsilon\}$ is finite. We shall consider $c_0(\Gamma)$ with either norm, or pointwise topology, and it will be clear from the context which one is taken into account.

2. The example. The main result of this note is the following

2.1. Example. *Assuming GCH, there exists a function space $C(K)$, K being a compact scattered space, such that $C(K)$ is Lindelöf under the weak topology (and hence, so is under the pointwise topology), but there is no continuous injection of the space $C(K)$ endowed with the weak, or the pointwise topology, into $c_0(\Gamma)$ endowed with the same topology, where Γ is an arbitrary set.*

This example is related to a general problem raised by Gulko [9; Problem 1 and 2, p. 37] for what function spaces $C_p(K)$ there exists a continuous injection into $c_0(\Gamma)$; Gulko [9], [10; p. 41] proved that for a wide class of compact spaces—including compact subspaces of Σ-products of the real intervals (i.e. Corson's compacta, see [2; sec. 4.1]), continuous images of Cantor cubes (i.e. dyadic compacta [7]), and the ordinal intervals $[0, \alpha]$ – this is the case. An example answering Gulko's question [9; Problem 1] without any additional axioms for set theory, but with properties much weaker than provided by Example 2.1, is given in sec. 7.1.

Example 2.1 shows also that the assertion of the well-known Amir–Lindenstrauss theorem that a WCG function space $C(K)$ has a linear bounded injection into $c_0(\Gamma)$ [6; Theorem 1, p. 144] fails, if WCG is replaced by a weaker assumption that $C(K)$ is Lindelöf under the weak topology (while the assumption that the weak topology is K-analytic is sufficient, see Gulko [9], Vašák [19]).

The space $C(K)$ in Example 2.1 is a function space associated in a standard way with a "ladder system" on ω_2 (see sec. 3). The function spaces associated with ladder systems on ordinals are always Lindelöf under the weak topology (sec. 4), and for some ladder systems (closely related to Ostaszewski's principle club †) which can be obtained from GCH following Gregory [8], they fail to have any continuous injection into $c_0(\Gamma)$ (sec. 5); but, on the other hand, if we assume MA, then the function spaces associated with ladder systems on $\lambda < 2^{\omega}$ have always a continuous, even linear, injection into $c_0(\Gamma)$ (sec. 6).

3. Ladder systems of ordinals and associated function spaces. Given an ordinal α of countable cofinality, a ladder on α is a set $S_\alpha = \{\alpha(1), \alpha(2), \ldots\}$ of ordinals from $\alpha \setminus E(\omega)$ such that $\alpha(1) < \alpha(2) < \ldots$ and $\alpha = \sup_i \alpha(i)$; a ladder system on an ordinal λ is a collection $\mathcal{S} = \langle S_\alpha : \alpha \in E(\omega) \cap \lambda \rangle$, where S_α is a ladder on α, cf. [5, sec. 3], [4].

We associate with a ladder system $\mathcal{S} = \langle S_\alpha : \alpha \in E(\omega) \cap \lambda \rangle$ a compact space $\lambda_{\mathcal{S}}$ in the following standard way (cf. [18; 8.5.10 (G)], [5: sec. 3]): we give the set λ a locally compact topology by making the points in $\lambda \setminus E(\omega)$ isolated and taking as a base of neighbourhoods of a point $\alpha \in E(\omega) \cap \lambda$ the sets $\{\alpha\} \cup (S_\alpha \setminus F)$, where F is a finite set, and let $\lambda_{\mathcal{S}}$ be the one-point compactification of this space, λ being the point at infinity.

We shall call $C(\lambda_{\mathscr{S}})$ the function space associated with the ladder system $\mathscr{S}$; a more direct description of $C(\lambda_{\mathscr{S}})$ is that this is the algebra generated in $l_\infty(\lambda \backslash E(\omega))$ by the characteristic functions of the finite subsets of $\lambda \backslash E(\omega)$, the characteristic functions of the ladders S_α and the unit element of $l_\infty(\lambda \backslash E(\omega))$.

4. The Lindelöf property of function spaces associated with ladder systems. Let $\mathscr{S} = \langle S_\alpha : \alpha \in E(\omega) \cap \lambda \rangle$ be a ladder system on λ and let $\lambda_{\mathscr{S}}$ be the compact space associated with $\mathscr{S}$. We shall show that $C_p(\lambda_{\mathscr{S}})$ is a Lindelöf space, and since $\lambda_{\mathscr{S}}$ is scattered, this will prove also that the function space $C(\lambda_{\mathscr{S}})$ endowed with the weak topology is Lindelöf, see sec. 1.2 (for $\lambda = \omega_1$ this was verified in [16]).

Let E_0 be the set of the characteristic functions of finite subsets of $\lambda \backslash E(\omega)$, let f_α, for each $\alpha \in E(\omega) \cap \lambda$, be the characteristic function of the set $\{\alpha\} \cup S_\alpha$, and let $E = E_0 \cup \{f_\alpha : \alpha \in E(\omega) \cap \lambda\}$.

Similarly, as in [16; sec. 3, cf. (3), (4)] one reduces our task to the verification that the countable product E^ω is Lindelöf; cf. also [2; 4.1.17]. We shall prove this fact following a reasoning of Alster [1; proof of Lemma 2].

Let us consider in E the topology generated by G_δ-subsets of E and let us denote by E^* the resulting space. Since by a result of Noble [15] the countable product of Lindelöf spaces in which G_δ-sets are open is a Lindelöf space, it is enough to check that E^* is Lindelöf. Let us put $A_\xi = \{f \in E^* : f^{-1}(1) \subset \xi\}$, for $\xi \leqslant \lambda$. We shall verify by transfinite induction that A_ξ is Lindelöf. Assume that A_ξ is Lindelöf for $\xi < \alpha \leqslant \lambda$ and consider A_α. If $\alpha \in E(\omega)$, then A_α is a countable union of Lindelöf subspaces A_{α_n}, where $\alpha_n \to \alpha$, and if $\alpha = \beta + 1$, then A_α is a union of A_β and the subspace $\{f \in A_\alpha : f(\beta) = 1\}$, which is homeomorphic either to A_β or to $A_\beta \cup \{f_\beta\}$, depending on cofinality of β, and so in both cases A_α is Lindelöf. The remaining case, $\mathrm{cf}(\alpha) > \omega$, requires a little bit more effort. Recall, that basic neighbourhoods of a point f in E^* are of the form $V(f, C) = \{g \in E^* : f \mid C = g \mid C\}$, $C \subset \lambda$ being a countable set. Let $\mathscr{U}$ be an open cover of A_α in E^*. By induction, choose countable collections $\mathscr{V}_1, \mathscr{V}_2, \ldots$ of neighbourhoods $V(f, C)$, with $f \in A_\alpha$ and countable $C \subset \lambda$, such that $\mathscr{V}_i$ refines $\mathscr{U}$ and $A_{\gamma_i} \subset \bigcup \mathscr{V}_{i+1}$, where $\gamma_0 = 0$ and $\gamma_{i+1} = \bigcup \{C \cap \alpha : V(f, C) \in \mathscr{V}_i\}$; this can be done, as at each stage i, $\gamma_i < \alpha$ since $\mathrm{cf}(\alpha) > \omega$, and therefore A_{γ_i} is Lindelöf by inductive assumption. Let us put $\gamma = \sup_i \gamma_i$ and $\mathscr{V} = \bigcup_i \mathscr{V}_i$.

We claim that $\mathscr{V}$ covers $A_\alpha \backslash \{f_\gamma\}$. Let $u \in A_\alpha \backslash \{f_\gamma\}$. If $u = f_\beta$ for $\beta \in E(\omega) \cap \gamma$, then $u \in A_{\gamma_i}$ for some i, and hence $u \in \bigcup \mathscr{V}_{i+1}$. If $u \in E_0$ or $u = f_\beta$ for $\beta \in E(\omega) \cap (\alpha \backslash \gamma)$, then the set $u^{-1}(1) \cap \gamma = F$ is finite, and hence the characteristic function g of the det F belongs to some A_{γ_i}, so $g \in V(f, C) \in \mathscr{V}_{i+1}$, and since $C \cap \alpha \subset \gamma$ and $f \in A_\alpha$, the functions f and u coincide on C, i.e. $u \in V(f, C)$.

5. The function spaces associated with ladder systems under GCH. In this section we shall use the following fact, being a certain substitute for Ostaszewski's principle † [4; Theorem 4.1], which can be easily derived from a particular case of a result of Gregory [8; Lemma 2.1]; see sec. 7.3 for more details.

5.1. LEMMA. *If $\varkappa$ is a cardinal such that $\varkappa^{\omega} = \varkappa$ and $\lambda = \varkappa^{+} = 2^{\varkappa}$, then there exists a ladder system $\mathcal{S} = \langle S_{\alpha}: \alpha \in E(\omega) \cap \lambda \rangle$ such that each subset Z of $\lambda \backslash E(\omega)$ of cardinality λ contains some S_{α}.*

5.2. LEMMA. *Let $\mathcal{S}$ be a ladder system such as in Lemma 5.1. Then the function space $C(\lambda_{\mathcal{S}}, D)$ does not admit any continuous injection into $c_0(\Gamma)$ endowed with the pointwise topology, $\lambda_{\mathcal{S}}$ being the compact space associated with the ladder system $\mathcal{S}$, and Γ being an arbitrary set.*

Proof. Assume on the centrary that there exists a continuous injection $T: C(\lambda_{\mathcal{S}}, D) \to c_0(\Gamma)$. For each $\alpha \in \lambda \backslash E(\omega)$, let f_{α} be the characteristic function of the singleton $\{\alpha\}$ and let $f_0 \equiv 0$. Then f_0 is the one-point compactification of the discrete subspace $A = \{f_{\alpha}: \alpha \in \lambda \backslash E(\omega)\}$ of $C(\lambda_{\mathcal{S}}, D)$, and hence $\{T(f_{\alpha}): :\alpha \in \lambda \backslash E(\omega)\}$ is a discrete subspace of $c_0(\Gamma)$, the restriction $T \mid A \cup \{f_0\}$ being an embedding. We claim that, on one hand,

(I) there exists a set $X \subset \lambda \backslash E(\omega)$ of cardinality λ and, for each $\alpha \in X$, a neighbourhood V_{α} of $T(f_{\alpha})$ in $c_0(\Gamma)$ such that $\bigcap \{V_{\alpha}: \alpha \in I\} = \emptyset$ for every infinite set $I \subset X$,

but, on the other hand,

(II) if $X \subset \lambda \backslash E(\omega)$ is of cardinality λ and W_{α}, for $\alpha \in X$, is a neighbourhood of f_{α} in $C(\lambda_{\mathcal{S}}, D)$, then there exists an infinite set $I \subset X$ such that $\bigcap \{W_{\alpha}: \alpha \in I\} \neq \emptyset$.

These two claims provide a contradiction, since if V_{α}, $\alpha \in X$, are such as in claim (I), then $W_{\alpha} = T^{-1}(V_{\alpha})$, $\alpha \in X$, contradict the assertion of claim (II).

Proof of (I). Put $u_{\alpha} = T(f_{\alpha})$ and let $\Gamma_{\alpha n} = \{\gamma \in \Gamma: |u_{\alpha}(\gamma)| > 1/n\}$. The set $\{u_{\alpha}: \alpha \in \lambda \backslash E(\omega)\}$ being a discrete subspace of $c_0(\Gamma)$, there exists an n with $\left|\bigcup_{\alpha} \Gamma_{\alpha n}\right| = \lambda$, and one can choose inductively a set $X \subset \lambda \backslash E(\omega)$ of cardinality λ such that $\Gamma_{\alpha n} \backslash \bigcup \{\Gamma_{\beta n}: \beta \in X \cap \alpha\}$ contains some γ_{α}, for each $\alpha \in X$. Then the neighbourhoods $V_{\alpha} = \{u \in c_0(\Gamma): |u(\gamma_{\alpha})| > 1/n\}$ have the desired property.

Proof of (II). Let $X \subset \lambda \backslash E(\omega)$ be of cardinality λ and let, for $\alpha \in X$, W_{α} be a neighbourhood of f_{α}. One can assume that $W_{\alpha} = \{f \in C(\lambda_{\mathcal{S}}, D): :f(\alpha) = 1$ and $f \mid F_{\alpha} \equiv 0\}$, F_{α} being a finite subset of $\lambda + 1 \backslash \{\alpha\}$. Put $L_{\alpha} = F_{\alpha} \backslash (E(\omega) \cup \{\lambda\})$ and $M_{\alpha} = F_{\alpha} \cap (E(\omega) \cup \{\lambda\})$.

At first, use Lázár's Theorem [13; A.3.4] to choose a set $Y \subset X$ of cardinality λ such that $Y \cup \bigcup \{L_{\alpha}: \alpha \in Y\} = \emptyset$, and next, λ being regular, apply the Δ-system lemma [14; Theorem 1.6, Ch. II] to choose a set $Z \subset Y$ of cardinality λ and a set M such that for any pair of distinct $\alpha, \beta \in Z$, $M_\alpha \cap M_\beta = M$. Now, since the ladder system $\mathcal{S}$ satisfies the assertion of Lemma 5.1, there exists an $\alpha \in E(\omega) \cap \lambda \setminus M$ such that $S_\alpha \subset Z$. Let f be the characteristic function of the closed-and-open set $S_\alpha \cup \{\alpha\}$ and let $I = S_\alpha \setminus \{\beta \in Z : \alpha \in M_\beta\}$. Then, the set I is infinite (since we removed from S_α at most one point) and $f \in \bigcap \{W_\alpha : \alpha \in I\}$.

5.3. Corollary. *For each cardinal $\lambda = \varkappa^+$, where $\varkappa^+ = 2^\varkappa$ and $\varkappa^\omega = \varkappa$, there exists a ladder system $\mathcal{S}$ on λ such that the function space $C(\lambda_{\mathcal{S}})$ equipped with the weak or pointwise topology does not admit any continuous injection into $c_0(\Gamma)$ considered with the same topology.*

In particular we obtain

5.4. Corollary. *Assuming GCH, for each cardinal $\lambda = \varkappa^+$, where $\mathrm{cf}(\varkappa) > \omega$ (in particular, for $\lambda = \omega_2$) there exists a ladder system $\mathcal{S}$ on λ such that, for the associated compact space $K = \lambda_{\mathcal{S}}$, the function space $C(K)$ satisfies the assertion of Example 2.1.*

5.5. Remark. Our reasonig in this section was based on the combinatorial fact in 5.1, which is a version of Ostaszewski's (†). The principle (†) itself can be used in the same way to demonstrate that, in $V = L$, the assertion of Corollary 5.4 still holds true for $\lambda = \omega_1$.

6. The function spaces associated with ladder systems under MA.

We shall show that, under MA, the image is quite different from that we have obtained at the end of the preceding section assuming GCH, or (†); cf. [3].

6.1. Proposition. *Under MA, if $\mathcal{S} = \langle S_\alpha : \alpha \in E(\alpha) \cap \lambda \rangle$ is an arbitrary ladder system on $\lambda < 2^\omega$, then there exists a bounded linear injection $T: C(\lambda_{\mathcal{S}}) \to c_0(\Gamma)$. Moreover, the operator T embeds the function space $C_p(\lambda_{\mathcal{S}})$ with the pointwise topology into $c_0(\Gamma)$ with the pointwise topology.*

Proof. We shall use the following fact closely related to a result of Hajnal and Matè [11; Theorem 8.2] (for completeness sake, we indicate briefly a proof in sec. 7.4):

(*) Under MA, there exists a decomposition of $\lambda \setminus E(\omega)$ into disjoint sets $\Gamma_1, \Gamma_2, \ldots$ such that, for each i and each ladder S_α from the system $\mathcal{S}$, the intersection $\Gamma_i \cap S_\alpha$ is finite.

Let us put $\Gamma_0 = \{\lambda\} \cup (\lambda \cap E(\omega))$, and let us define an operator $T: C(\lambda_{\mathcal{S}}) \to$ $\to c_0(\lambda + 1)$ by the formula

$$Tf(\lambda) = f(\lambda), \quad \text{and} \quad Tf(\gamma) = (1/i+1)\,(f(\gamma) - f(\lambda)) \quad \text{for} \quad \gamma \in \Gamma_i.$$

To check that, indeed, $Tf \in c_0(\Gamma)$ for $f \in C(\lambda_{\mathcal{S}})$, assume on the contrary, that for some $\varepsilon > 0$ the set $A = \{\gamma \in \lambda : |Tf(\gamma)| > \varepsilon\}$ is infinite. Since f

is bounded, there exists an i such that the set $A \cap \Gamma_i$ is infinite, and thus there exists an accumulation point α of this set in $\lambda_{\mathscr{S}}$. Now, $\alpha \neq \lambda$, hence $\alpha \in E(\omega) \cap \lambda$, $i > 0$, and the intersection $S_\alpha \cap (A \cap \Gamma_i)$ is infinite, contrary to (*).

Evidently, T is an injective, linear, bounded operator which embeds $C(\lambda_{\mathscr{S}})$ into $c_0(\Gamma)$, when both spaces carry the pointwise topology.

7. Remarks.

7.1. Example. There exists a compact scattered space S such that there is no continuous injection of $C_p(S)$ into $c_0(\Gamma)$ endowed with the pointwise topology, Γ being an arbitrary set.

To define S, let us consider Bernstein's set B in the real line R (i.e. both B and $R \setminus B$ intersect each perfect set in R), and let $\langle A_\alpha : \alpha < 2^\omega \rangle$ be an enumeration of all countable subsets of $R \setminus B$ with uncountable closure in R. Choose by transfinite induction distinct points $a_\alpha \in \bar{A}_\alpha \cap B$ and, for each $\alpha < 2^\omega$, choose a sequence $C_\alpha = \{a_\alpha(i) : i \in \omega\} \subset A_\alpha$ converging to a_α. Give the set $R \cup \{\infty\}$ a compact topology in the same standard way as in Sec. 3, where the sets C_α play now the role of the ladders and ∞ is the point at infinity. The resulting space is our S.

One easily checks that each uncountable subset $Z \subset R \setminus B$ contains some C_α (cf. the assertion of Lemma 5.1). This property allows one to repeat the reasoning of the proof of Lemma 5.2 to make sure that $C_p(S)$ satisfies the assertion of Example 7.1.

However, a construction of this sort never yields a function space $C_p(S)$ which is Lindelöf, cf. [17].

7.2. The compact spaces $\lambda_{\mathscr{S}}$ associated with ladder systems $\mathscr{S}$ are, among compact spaces K we were able to detect in the literature, the only ones whose function space $C(K)$ might satisfy the assertion of Example 2.1. The results of this note left open the question whether or not, one can construct in ZFC a ladder system $\mathscr{S}$ on an ordinal λ providing the space $\lambda_{\mathscr{S}}$ with the desired properties. Let us mention, however, that the reasoning in Sec. 5 shows, that the negative answer yields an existence of $0^\#$, which is a strong assumption about large cardinals [12; Corollary, p. 357].

7.3. Let us derive the combinatorial fact 5.1 we used in Sec. 5 from the following version of diamond: "there exists a collection $\langle T_\alpha : \alpha \in E(\omega) \cap \lambda \rangle$ such that $T_\alpha \subset \alpha$ and for all subsets X of λ, $\{\alpha \in E(\omega) \cap \lambda : X \cap \alpha = T_\alpha\}$ in stationary in λ", proved by Gregory [8; Lemma 2.1] for $\lambda = \varkappa^+ = 2^\varkappa$, with $\varkappa^\omega = \varkappa$; cf. [4; Theorem 4.1].

Put $A = \{\alpha \in E(\omega) \cap \lambda : \bigcup (T_\alpha \setminus E(\omega)) = \alpha\}$ and choose for each $\alpha \in A$ a ladder $S_\alpha \subset T_\alpha$ on α, and for $\alpha \in E(\omega) \cap \lambda \setminus A$, let S_α be an arbitrary ladder on α. We claim that $\langle S_\alpha : \alpha \in E(\omega) \cap \lambda \rangle$ satisfies the assertion of Lemma 5.1. Let Z be a set in $\lambda \setminus E(\omega)$ of cardinality λ and let $Z' = \{\alpha \in \lambda : \bigcup (Z \cap \alpha) = \alpha\}$.

Then, Z' being a closed, unbounded set in λ, there exists an $\alpha \in Z' \cap E(\omega)$ with $Z \cap \alpha = T_\alpha$. Since $\alpha \in Z'$, $\bigcup (T_\alpha \backslash E(\omega)) = \alpha$ and hence $\alpha \in A$, so the ladder S_α is contained in $T_\alpha \subset Z$.

7.4. Let us indicate briefly how one can derive from MA the combinatorial fact (*) in the proof of Proposition 6.1, cf. [11; Theorem 8.2]. Let $\mathcal{S} = \langle S_\alpha : \alpha \in E(\omega) \cap \lambda \rangle$ be a ladder system on $\lambda < 2^\omega$. What follows, is a standard application of MA to the countable product $\mathbf{P} = (\mathbf{P}_{\mathcal{S}})^\omega$ of the almost disjoint sets partial order $\mathbf{P}_{\mathcal{S}}$, see [14; Def. 2.7 in Ch. II, Def. 1.1 in Ch. VIII], [12; p. 189].

Put $\Lambda = (\lambda_{\mathcal{S}} \backslash \{\lambda\}) \times \omega$, $\lambda_{\mathcal{S}}$ being as in Sec. **3**, and let $\mathbf{P} = \{\langle F, V \rangle : F$ is a finite set of isolated points in Λ and V is a compact-open set in $\Lambda\}$, where $\langle F', V' \rangle \leqslant \langle F, V \rangle$ iff $F \subset F'$, $V \subset V'$ and $V \cap F' \subset F$.

At first, let us make sure that $\mathbf{P}$ is c.c.c. One can apply here the Δ-system lemma, but we rather use a fact we have already stated in Sec. 4. Let E^* and E_0 be the spaces defined in Sec. 4; identifying a set with its characteristic function, one can consider $\mathbf{P}$ as a subset of the Lindelöf space $L = \{(f_1, g_1, f_2, g_2, \ldots) \in E^* \times E^* \times \ldots : f_i \in E_0$ and all but finitely many f_i and g_i vanish$\}$. Now, given an uncountable set $A = \{\langle F_\alpha, V_\alpha \rangle : \alpha < \omega_1\}$ in L, choose an accumulation point $\langle F, V \rangle$ of A in L and then (recall, what are the basic neighbourhoods in E^*, Sec. 4), choose for $C_1 = F \cup V$ a pair $\langle F_\alpha, V_\alpha \rangle$ such that $F_\alpha \cap C_1 = F$ and $V_\alpha \cap C_1 = V$, and next, choose for $C_2 = C_1 \cup (F_\alpha \cup V_\alpha)$ a pair $\langle F_\beta, V_\beta \rangle$ with $\beta \neq \alpha$, such that $F_\beta \cap C_2 = F$ and $V_\beta \cap C_2 = V$. The pairs $\langle F_\alpha, V_\alpha \rangle$ and $\langle F_\beta, V_\beta \rangle$ are compatible in $\mathbf{P}$, as $\langle F_\alpha \cup F_\beta, V_\alpha \cup V_\beta \rangle$ is their common extension; cf. [14; Ch. II, 2.8]. Define in $\mathbf{P}$ dense sets of two sorts, cf. [14; proof of 2.15 in Ch. II]:

$$D_\alpha^n = \{\langle F, V \rangle \in \mathbf{P} : S_\alpha \times \{n\} \subset V\}, \alpha \in E(\omega) \cap \lambda, n \in \omega,$$

$$E_\alpha = \{\langle F, V \rangle \in \mathbf{P} : \exists_n (\alpha, n) \in F\}, \alpha \in \lambda \backslash E(\omega),$$

and let G be a filter in $\mathbf{P}$ intersecting all these dense sets. Put, for $n \in \omega$, $\Gamma_n = \bigcup \{A \subset \lambda : A \times \{n\} \subset F$ for some $\langle F, V \rangle \in G\}$. Then the condition $G \cap D_\alpha^n \neq \emptyset$ guarantees that $\Gamma_n \cap S_\alpha$ is finite, cf. [14; 2.10 in Ch. II], while the condition $G \cap E_\alpha \neq \emptyset$, for $\alpha \in \lambda \backslash E(\omega)$, implies that $\bigcup_n \Gamma_n = \lambda \backslash E(\omega)$.

INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, PKiN, 00-901 WARSAW
(INSTYTUT MATEMATYKI, UNIWERSYTET WARSZAWSKI)

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К. Циесельски, Р. Поль, **Слабое линделефо пространство** $C(K)$ **без непрерывных вложений в** $c_0(\Gamma)$

В предположении общей гипотезы континуума строится пример $C(K)$ — пространства (K — рассеянный бикомпакт), которое линделефо в слабой топологии. Нет однако непрерывного вложения этого пространства в $c_0(\Gamma)$-пространство, также обладающее слабой топологией (Γ — произвольное множество).