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Topologies generated by graphs

by

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Introduction

Every topological space can be identified in a natural way with the family of characteristic functions of sets belonging to a certain subbasis of this space. Hence in order to construct a space $\langle X, \tau \rangle$ with some desired topological properties it suffices to define a function $f : \varkappa \times X \longrightarrow 2$ satisfying certain combinatorial requirements. Using the above method A.Hajnal and I.Juhász produced several interesting examples of spaces during the last decade. In this paper we show applications of this technique in the case when the coding function is a generic function or its simple modification. Apart from one example /see [Ha, Ju 3]/ the paper contains a complete review of results obtained in this way. Two of them, i.e. the solution of one of the problems of R.E. Hodel /Thm. 6.1/ and the result on spectra of cardinalities of closed sets in L-spaces /Thm.4.2./ are published for the first time. All others are known but their proofs in the formalism presented in this paper seem more clear than the original ones.

1. Preliminaries

Throughout the paper we use standard set-theoretical notation.

Ordinals are identified with sets of their predecessors. The class of all ordinals is denoted by Ord and Lim denotes the class of all limit ordinals. The successor of an ordinal α is denoted by $\alpha + 1$. Cardinals are initial ordinals. The class of all cardinals is denoted by Card. The cardinality od a set x is denoted by $|x|$. The successor cardinal of $\varkappa$ is denoted by $\varkappa^{+}$. If $\varkappa$ is a cardinal then by $[x]^{\varkappa}$ we denote the set of all subsets of x of cardinality $\varkappa$, $[x]^{\varkappa}=\{y\subset x: |y| = \varkappa\}$. Analogously we define $[x]^{\leq\varkappa}$ and $[x]^{<\varkappa}$. The powerset of a set x is denoted by $\mathcal{P}(x)$. Functions are always sets of ordered pairs i.e. they are identified with their graphs. The class of all functions is denoted by Fnc. The domain of a function f is denoted by $D(f)$ and its range by $R(f)$. By $f\restriction x$ we denote the restriction of f to the set x. We shall often use the notation $H(x,y)=\{f\in \mathrm{Fnc}: D(f)\subset x \ \&\ |D(f)|<\omega \ \&\ R(f)\subset y\}$. For $y = 2 = \{0,1\}$ we simply write $H(x)$ instead of $H(x,2)$.

A generic extension of a countable transitive model M of ZFC will be denoted by $M[G]$. A notion of forcing will be a partially ordered set $\langle P,\leq\rangle$ and we say that a condition q is stronger then p iff $q\leqslant p$. A set $Q\subset P$ is compatible in P iff for every $p,q\in Q$ there exists $r\in P$ stronger then p and q. The conditions p and q are compatible iff the set $\{p,q\}$ is compatible. A notion of forcing P satisfies $\varkappa$-chain condition /$\varkappa$ - cc/ iff for any sequence $\langle p_\zeta : \zeta<\varkappa\rangle$ of elements of P there exist $\zeta<\eta<\varkappa$ s.t. p_ζ and p_η are compatible. P satisfies ccc if it satisfies ω_1 - cc.

The Cohen forcing for a set x /denoted by $P(x)$/ there is a set $H(x)$ ordered by reverse inclusion. Thus if G is M-generic over $P(x)$ then $f=\bigcup G$ is a function and $f : x\to 2$. In this case we shall write $M[f]$ instead $M[G]$.

The topological notation used in the paper is that of [Ju 1] or [En]. In what fallows X is an arbitrary topological space with a topology τ.

The family $\mathcal{B} \subset \tau$ is a basis of X if every $U \in \tau$ is a union of some members of $\mathcal{B}$. The cardinal number $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a basis of $X\} + \omega$ is called the weight of X.

The family $\mathcal{N} \subset \mathcal{P}(X)$ is a network of X if every $U \in \tau$ is a union of some members of $\mathcal{N}$. Then $nw(X) = \min\{|\mathcal{N}| : \mathcal{N}$ is a network of $X\} + \omega$ is called the netweight of X.

Let $x \in X$. The family $\mathcal{U} \subset \tau$ is a neighbourhood basis of x in X if for every open set U s.t. $x \in U$ there is $W \in \mathcal{U}$ with $x \in W \subset U$.

$\chi(x,X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a neighbourhood basis of x in $X\} + \omega$

$\chi(X) = \sup\{\chi(x,X) : x \in X\}$ is called the character of X.

Let $x \in X$ and $X \in T_1$. The family $\mathcal{V} \subset \tau$ is a local ψ-basis of x in X iff $\{x\} = \bigcap \mathcal{V}$.

$\psi(x,X) = \min\{|\mathcal{V}| : \mathcal{V}$ is a local ψ-basis of x in $X\} + \omega$

$\psi(X) = \sup\{\psi(x,X) : x \in X\}$ is called the pseudo character of X.

A set $Y \subset X$ is dense in X if Y meets every nonempty open set in X. Then $d(X) = \min\{|Y| : Y$ is dense in $X\} + \omega$ is called the density of X.

Let $\varkappa \in$ Card, $\varkappa \geqslant \omega$. X is said to be $\varkappa$-Lindelöf /resp. $\varkappa$-compact/ if every open cover of X has a subcover of cardinality at most $\varkappa$ /resp. less then $\varkappa$/. X is hereditarily $\varkappa$-Lindelöf if every subspace of X is $\varkappa$-Lindelöf.

A subspace Y of X is weekly separated if with every point $x \in Y$ we can associate an open neighbourhood U_x of the point x in such a way that for every distinct points $x,y \in Y$ either $x \notin U_y$ or $y \notin U_x$.

A set $Y \subset X$ is right /left/ separated in the space X if there exist a well-ordering $\prec$ of the set Y and an open neighbourhood U_x of x for every $x \in Y$ such that for all $x,y \in Y$ if $x < y$ then $y \notin U_x$ /resp. $x \notin U_y$/.

We denote

$R(X) = \sup \{|Y| : Y$ is weekly separated in $X\} + \omega$

$h(X) = \sup \{|Y| : Y$ is right separated in $X\} + \omega$

$z(X) = \sup \{|Y| : Y$ is left separated in $X\} + \omega$

A regular space X is an L-space /resp. S-space/ iff $h(X) = \omega$ and $z(X) > \omega$ /resp. $z(X) = \omega$ and $h(X) > \omega$ /.

The spectrum of X will be the set $Sp(X) = \{|F| : X \setminus F \in \tau\}$

The space X omits a cardinal $\varkappa$ iff $\varkappa < |X|$ and $\varkappa \notin Sp(X)$.

2. The graph topology.

The following definitions are generalizations of definitions from [Ha, Ju 2]. A graph will be a function $f : A \times K \longrightarrow \alpha$, where $\alpha \in \mathrm{Ord}$, $\alpha \geqslant 2$.

Let $f : A \times K \longrightarrow \alpha$ be a graph and let $P \subset \mathcal{P}(\alpha)$ be a field of sets s.t. $P \neq \{0, \alpha\}$. For $a \in A$ and $p \in P$ we denote

$U_a^p = \{k \in K : f(a,k) \in p\}$

In the case when $p = \{\eta\}$ is a singleton we shall write U_a^η instead $U_a^{\{\eta\}}$.

Given a graph $f : A \times K \longrightarrow \alpha$ and a field P we define a graph topology $\tau_{f,p}$ on K by subbasis $\{U_a^p : a \in A \ \& \ p \in P\}$.

For $\varepsilon \in H(A,P)$ we denote

$U_\varepsilon = \bigcap \{U_a^{\varepsilon(a)} : a \in D(\varepsilon)\}$

$F_\varepsilon = \bigcup \{U_a^{\varepsilon(a)} : a \in D(\varepsilon)\}$

Then the family

$$\mathcal{B}_{f,P} = \{U_\varepsilon : \varepsilon \in H(A, P \setminus \{0,\alpha\})\}$$

is a basis for $\langle K, \tau_{f,P} \rangle$.

For $\alpha = 2$ we have $P = \mathcal{P}(2)$ and then we omit the index P. Moreover in this case $\mathcal{B}_f = \{U_\varepsilon : \varepsilon \in H(A)\}$. Sometimes we shall write U^f_ε instead U_ε and F^f_ε instead F_ε in order to stress that definitions of these sets depend on the graph f.

By closure of P under complements follows that every graph space is zerodimensional. The reverse implication is also true. Thus our definition has only technical meaning.

Let us also note that P is a basis of some topological space $Y(P)$ on α. So, if $\langle K, \tau_{f,P} \rangle \in T_1$ and $g : K \to \alpha^A$ is given by formula $g(k,a) = f(a,k)$ for $a \in A$ and $k \in K$ then g is an embedding of $\langle K, \tau_{f,P} \rangle$ into $Y(P)^A$. In particular, if $\alpha = 2$ then $\langle K, \tau_f \rangle \in T_1$ can be treated as a subspace of 2^A /see also [Ju 2]/.

3. The basis space in the Cohen model.

In this section we construct in the Cohen model the basic space of this paper. It will be obtained from a generic function as a graph topology spaces. The properties presented here are proved by A.Hajnal and I.Juhàsz /see [Ha, Ju 1]/ and they play a fundamental role in the next sections.

Let $\varkappa$ and λ be infinite cardinals in M and let $M[f]$ be a Cohen extension of M obtained by forcing with a p.o. set $P(\lambda \times \varkappa)$. So $f : \lambda \times \varkappa \to 2$.

Let $X = \langle \varkappa, \tau_f \rangle$.

Before the formulation of the theorem we prove some technical lemmas. The first two of them concerne the structure of basic sets of X and they are absolute for transitive models of ZFC.

3.1. **Lemma.** For any infinite set $A \subset \varkappa$ from M and for any $\varepsilon \in H(\lambda)$

$|U_\varepsilon \cap A| = |A|.$

So, in particular, $|U_\varepsilon| = \varkappa$ for any $\varepsilon \in H(\lambda)$.

Proof. In M we can decompose any infinite set A into $|A|$ - many infinite subsets. So, it is enough to prove that $U_\varepsilon \cap A \neq 0$ for any $\varepsilon \in H(\lambda)$ and any such A. However the required property follows immediately from the density of the set $\{p \in P(\lambda \times \varkappa) : p \Vdash \text{"} U_\varepsilon \cap A \neq 0 \text{"}\}$ ∎

Analogously we prove

3.2. **Lemma.** For any infinite set $A \subset \lambda$ from M and any two distinct elements $\zeta, \eta \in \varkappa$ there exists $a \in A$ s.t. $\zeta \in U_a^0$ and $\eta \in U_a^1$. ∎

The following lemma explicitely describes some closed subsets of X. Next we show that any closed subset of X satisfies the hypothesis of the lemma.

3.3. **Lemma.** Let $F = \bigcap_{n<\omega} F_{\varepsilon_n}$ where $\varepsilon_n \in H(\lambda)$ for $n < \omega$ and let subset $K \subset \varkappa$ from M be s.t.

$(*)$ $\langle \varepsilon_n : n < \omega \rangle \in M[f \restriction \lambda \times K]$

Then either $F \subset K$ or there exists $\varepsilon \in H(\bigcup_{n<\omega} D(\varepsilon_n))$ s.t. $U_\varepsilon \subset F$.

Proof. We work in the model $M[f \restriction \lambda \times K]$.

Let $F \not\subset K$ and let $\eta \in F \setminus K$. So, by product lemma, there exists $s \in P(\lambda \times (\varkappa \setminus K))$ s.t. $s \Vdash \text{"} \eta \in F \text{"}$. Hence, by $(*)$,

$s \Vdash$ " $\eta \in F_{\varepsilon_n}$ " for any $n < \omega$, i.e.
$\forall n < \omega\ \exists a \in D(\varepsilon_n)\,[\, < a, \eta, \varepsilon_n(a) > \in s\,]$

Let

$$\varepsilon = \{ < a, s(a,\eta) > \quad < a, \eta > \in D(s)\ \&\ a \in \bigcup_{n<\omega} D(\varepsilon_n) \}$$

Then $\quad \varepsilon \in H(\bigcup_{n<\omega} D(\varepsilon_n)) \quad$ and $U_\varepsilon \subset F$ ∎

The following lemma has only technical meaning.

3.4. Lemma. Let the sequences $< b_\zeta : \zeta < \omega_1 >$, $< \varepsilon_\zeta : \zeta < \omega_1 > \in M[f]$ satisfy the following properties:

(1) $b_\zeta \in \varkappa$, $\varepsilon_\zeta \in H(\lambda)$;

(2) $\zeta \neq \eta \longrightarrow b_\zeta \neq b_\eta$;

(3) $\zeta \neq \eta \longrightarrow D(\varepsilon_\zeta) \cap D(\varepsilon_\eta) = 0$.

Then there exist $\zeta < \eta < \omega_1$ s.t. $b_\zeta \in U_{\varepsilon_\eta}$ and $b_\eta \in U_{\varepsilon_\zeta}$.

Proof. Let $A \subset \lambda$ and $B \subset \varkappa$ be countable sets from M s.t.

(∗) $< b_n : n < \omega >$, $< \varepsilon_n : n < \omega > \in M[f \upharpoonright A \times B]$

Then there exist ν, $\omega \leq \nu < \varkappa$ s.t. $b_\nu \notin B$ and $D(\varepsilon_\nu) \cap A = 0$. We show that there exists $n < \omega$ s.t. $b_n \in U_{\varepsilon_\nu}$ and $b_\nu \in U_{\varepsilon_n}$.

Let

$D = \{ s \in P(\lambda \times \varkappa \setminus A \times B) : (\exists n < \omega)(s \Vdash$ " $b_n \in U_{\varepsilon_\nu}\ \&\ b_\nu \in U_{\varepsilon_n}$ " $)\}$

Hence, by (∗), $D \in M[f \upharpoonright A \times B]$ i.e. it is enough to show that D is dense in $P(\lambda \times \varkappa \setminus A \times B)$.

Let $s \in P(\lambda \times \varkappa \setminus A \times B)$. Then there exists $n < \omega$ s.t. $(\lambda \times \{b_n\}) \cap D(s) = 0$ and $(D(\varepsilon_n) \times \varkappa) \cap D(s) = 0$.

Let

$q_1 = \{ < a, b_n, \varepsilon_\nu(a) > : a \in D(\varepsilon_\nu) \}$

$q_2 = \{ < a, b_\nu, \varepsilon_n(a) > : a \in D(\varepsilon_n) \}$

Hence, if $t = a \cup q_1 \cup q_2$ then $t \in P(\lambda \times \varkappa \setminus A \times B)$, $t \leqslant a$ and $t \Vdash$ " $b_n \in U_{\varepsilon_\nu}$ & $b_\nu \in U_{\varepsilon_n}$ " ∎

3.5. Theorem. The space X satisfies the following properties:

(1) X is a zerodimensional, Hausdorff space;

(2) $R(X) = h(X) = z(X) = \omega$;

(3) $\chi(X) = w(X) = \lambda$;

(4) $\pi w(X) = \min\{\varkappa, \lambda\}$.

Proof. (1) follows immediately from Lemma 3.2.

(2) By the inequality $\omega \leqslant h(X) . z(X) \leqslant R(X)$ it is enough to show that $R(X) < \omega_1$.

Let $\langle b_\zeta : \zeta < \omega_1 \rangle$ and $\langle \delta_\zeta : \zeta < \omega_1 \rangle$ be one - to - one sequences of elements from $\varkappa$ and $H(\lambda)$ respectively s.t. $b_\zeta \in U_{\delta_\zeta}$ for any $\zeta < \omega_1$. We find $\zeta < \eta < \omega_1$ s.t. $b_\zeta \in U_{\delta_\eta}$ and $b_\eta \in U_{\delta_\zeta}$.

By Δ-lemma we can assume that there exists a finite set $a \subset \lambda$ s.t. $D(\delta_\zeta) \cap D(\delta_\eta) = a$ and $\delta_\zeta \restriction a = \delta_\eta \restriction a$ for any $\zeta < \eta < \omega_1$. So, if $\varepsilon_\zeta = \delta_\zeta \restriction D(\delta_\zeta) \setminus a$ then it is enough to find $\zeta < \eta < \omega_1$ s.t. $b_\zeta \in U_{\varepsilon_\eta}$ and $b_\eta \in U_{\varepsilon_\zeta}$. But this follows from Lemma 3.4.

(3) By the inequality $\chi(X) \leqslant w(X) \leqslant |\mathcal{B}_1| \leqslant \lambda$ it is enough to show that $\chi(X) \geqslant \lambda$.

Let us assume that $\chi(X) = \mu < \lambda$. So there exists a family $\mathcal{U} = \{U_{\varepsilon_\zeta} : \zeta < \mu\}$ which is a neighbourhood basis of a point 0 in X. However if $a \in \lambda \setminus \bigcup_{\nu < \mu} D(\varepsilon_\nu)$ and $i < 2$ then, by Lemma 3.1., $U_{\varepsilon_\nu} \cap U_a^{1-i} \neq 0$ for $\nu < \mu$. Hence, if $0 \in U_a^i$ then $U_{\varepsilon_\nu} \not\subset U_a^i$ for any $\nu < \mu$, i.e. $\mathcal{U}$ is not a neighborhood basis of 0 in X.

(4). By inequality $nw(X) \leq \min\{|X|, \pi(X)\} = \min\{\varkappa, \lambda\}$ it is enough to show that $nw(X) \geqslant \min\{\varkappa, \lambda\}$.

Let us assume that $nw(X) = \mu < \min\{\varkappa, \lambda\}$ and let $\mathcal{N} = \{F_\zeta : \zeta < \mu\}$ be a network of X. By the regularity of X we can assume that all F_ζ's are closed for $\zeta < \mu$. So, by hereditary Lindelöfness of X /i.e. $h(X) = \omega$ /, for any $\zeta < \mu$ there exists a sequence $\langle \varepsilon^\zeta_n : n < \omega \rangle$ of elements of $H(\lambda)$ s.t.

$$F_\zeta = \bigcap_{n<\omega} F_{\varepsilon^\zeta_n}$$

Let a subset $K \subset \varkappa$ from M be s.t. $|K| \leq \mu$ and $\langle \varepsilon^\zeta_n : \zeta < \mu \ \&\ n < \omega \rangle \in M[f \restriction \lambda \times K]$.

Let $a \in \lambda \setminus \bigcup \{D(\varepsilon^\zeta_n) : \zeta < \mu \ \&\ n < \omega\}$, $\eta \in \varkappa \setminus K$

and $i < 2$ be s.t. $\eta \in U^i_a$. We show that for any $\zeta < \mu$ either $\eta \notin F_\zeta$ or $F_\zeta \not\subset U^i_a$.

Let $\zeta < \mu$ and let $\eta \in F_\zeta$. Hence $F_\zeta \not\subset K$. So, by Lemma 3.3., there exists $\varepsilon \in H(\bigcup_{n<\omega} D(\varepsilon^\zeta_n))$ s.t. $U_\varepsilon \subset F_\zeta$. However, by Lemma 3.1., $U_\varepsilon \cap U^{1-i}_a \neq 0$. Hence $F_\zeta \not\subset U^i_a$ ∎

Let us also note that for any infinite set $K \subset \varkappa$ from M with a subspace topology $nw(K) = \min\{\lambda, |K|\}$. The proof is the same as for the space X. Therefore the construction which we present in the section 5 is essential.

4. L - spaces ans S - spaces in the Cohen model.

In this section we construct L - spaces and S - spaces in the Cohen model /see also [Ru],[Ju 2],[Sz]/. The method of construction is due to A. Hajnal and I. Juhász i.e. the required spaces will be obtained as graph spaces for suitable modifications of a Cohen generic graph. We shall also study the spectra of those L - spaces.

Let $\varkappa$ be uncountable regular cardinal in M and let $M[f]$ be a Cohen extension of M obtained by forcing with $P(\varkappa \times \varkappa)$. So $f : \varkappa \times \varkappa \longrightarrow 2$.

We start with the construction of the S - space. For this purpose we define a function $g : \varkappa \times (\varkappa + 1) \longrightarrow 2$ by

$$g(\zeta, \eta) = \begin{cases} 0 & \text{for} \quad \zeta \leqslant \eta \\ f(\zeta, \eta) & \text{for} \quad \zeta > \eta \end{cases}$$

Let us also note that for $\zeta < \varkappa$

$$U^{g}_{\{<\zeta, i>\}} = \begin{cases} U^{f}_{\{<\zeta, 0>\}} \cup \{\xi : \zeta \leq \xi \leq \varkappa\} & \text{for} \quad i = 0 \\ U^{f}_{\{<\zeta, 1>\}} \setminus \{\xi : \zeta \leq \xi \leq \varkappa\} & \text{for} \quad i = 1 \end{cases}$$

4.1. Theorem. Let $Y = \langle \varkappa + 1, \tau_g \rangle$. Then

(1) Y is a zerodimensional, Hausdorff space;

(2) $z(Y) = \omega$;

(3) $\psi(Y) = h(Y) = \varkappa$.

In particular Y is an S -space. Moreover Y is $\varkappa$ -compact.

Proof. (1) Zerodimensionality of Y is obvious.

Let $\zeta < \eta \leq \varkappa$. Then, by Lemma 3.2. / with $A = \{\xi : \zeta < \xi < \varkappa\}$ / there exists $\xi > \zeta$ s.t.

$$\zeta \in U^{f}_{\{<\xi, 1>\}} \quad \& \quad [\ \eta = \varkappa \quad \vee \quad \eta \in U^{f}_{\{<\xi, 0>\}}]$$

Then $\zeta \in U^{g}_{\{<\xi, 1>\}}$ and $\eta \in U^{g}_{\{<\xi, 0>\}}$, i.e. Y is a Hausdorff space.

(2) We show that $z(Y) < \omega_1$.

Let $\langle b_\zeta : \zeta < \omega_1 \rangle$ and $\langle \delta_\zeta : \zeta < \omega_1 \rangle$ be one - to - one sequences of elements from $\varkappa$ and $H(\varkappa)$ respectively s.t. $b_\zeta \in U^g_{\delta_\zeta}$ for any $\zeta < \omega_1$. We find $\zeta < \eta < \omega_1$ s.t. $b_\zeta \in U^g_{\delta_\eta}$.

Wlog we can assume that the sequence $\langle b_\zeta : \zeta < \omega_1 \rangle$ is increasing. By Δ-lemma we can also assume that there exists a finite set $a \subset \varkappa$ s.t. $D(\delta_\zeta) \cap D(\delta_\eta) = a$ and $\delta_\zeta \restriction a = \delta_\eta \restriction a$ for any $\zeta < \eta < \omega_1$. So, if $\varepsilon_\zeta = \delta_\zeta \restriction D(\delta_\zeta) \setminus a$ then it is enough to find $\zeta < \eta < \omega_1$ s.t. $b_\zeta \in U^g_{\varepsilon_\eta}$.
However, by Lemma 3.4., there exist $\zeta < \eta < \omega_1$ s.t. $b_\zeta \in U^f_{\varepsilon_\eta}$.

Let $a \in D(\varepsilon_\eta)$.

If $a > b_\zeta$ then $b_\zeta \in U^f_{\{\langle a, \varepsilon_\eta(a)\rangle\}}$ implies $b_\zeta \in U^g_{\{\langle a, \varepsilon_\eta(a)\rangle\}}$.

If $a \leq b_\zeta$ then $a < b_\eta$. So $b_\eta \in U^g_{\{\langle a, \varepsilon_\eta(a)\rangle\}}$ implies $\varepsilon_\eta(a) = 0$. Hence $b_\zeta \in U^g_{\{\langle a, \varepsilon_\eta(a)\rangle\}}$.

So $b_\zeta \in U^g_{\varepsilon_\eta}$.

(3) For a Hausdorff space Y it holds that $h(Y) \geq \psi(Y)$ /see [Ju 1]/. So, it is enough to prove that $\psi(Y) \geq \varkappa$. For this purpose we show that $\psi(\varkappa, Y) \geq \varkappa$.

By regularity of $\varkappa$ it is enough to prove

(*) if $\varkappa \in U^g_\varepsilon$ then there exists $\xi < \varkappa$ s.t. $\{\zeta : \xi \leq \zeta \leq \varkappa\} \subset U^g_\varepsilon$

Let $\varkappa \in U^g_\varepsilon$ and $a \in D(\varepsilon)$. Then $\varepsilon(a) = 0$ and $\{\zeta : a \leq \zeta \leq \varkappa\} \subset U^g_{\{\langle a, \varepsilon(a)\rangle\}}$. Hence, for $\xi = \max D(\varepsilon)$ we have $\{\zeta : \xi \leq \zeta \leq \varkappa\} \subset U^g_\varepsilon$

By (*) we also have that $\varkappa \in U \in \tau_g$ implies $|Y \setminus U| < \varkappa$

So Y is $\varkappa$-compact.

In the case, when $\varkappa = \omega_1$ we obtaine a Lindelöf S-space. So, in particular, it is a normal space. A.Hajnal and I.Juhász proved /see [Ha, Ju 1]/ that in this case the space is also hereditarily normal. A natural problem arises whether for

$\varkappa > \omega_1$ the space defined as above is normal. This problem is connected with the following problem of A.Hajnal and I.Juhász:

" If X is compact T_2, is $h(X) \leq z(X)^+$? "

Now we construct the L - spaces.

For $T \subset \varkappa \cap \mathrm{Lim}$ we define $\omega(T) = \{\alpha + n : \alpha \in T \cup \{0\}\ \&\ n < \omega\}$ and let $f_T : \omega(T) \times \varkappa \longrightarrow 2$ be s.t. for $\alpha \in T \cup \{0\}, n < \omega, \zeta < \varkappa$

$$f_T(\alpha + n, \zeta) = \begin{cases} f(\alpha + n, \zeta) & \text{for } \alpha \leq \zeta \\ 0 & \text{for } \alpha > \zeta \end{cases}$$

In order to stress that definition of a set U_ε depend on the graph f_T we shall write U^T_ε. Let us also note that for $n < \omega$ and $\alpha \in T \cup \{0\}$

$$U^T_{\{<\alpha + n, i>\}} = \begin{cases} U^f_{\{<\alpha + n, 0>\}} \cup \alpha & \text{for } i = 0 \\ U^f_{\{<\alpha + n, 1>\}} \setminus \alpha & \text{for } i = 1 \end{cases}$$

In particular, for $i < 2$ and $n < \omega$

$U^T_{\{<n, i>\}} = U^f_{\{<n, i>\}}$

4.2. Theorem. Let $T \subset \varkappa \cap \mathrm{Lim}$ and let $X_T = < \varkappa, \tau_{f_T} >$. Then

(1) X_T is a zerodimensional , Hausdorff space;

(2) $h(X_T) = \omega$;

(3) $z(X_T) = |T| + \omega$;

(4) if T is closed under countable unions and $\{|\beta \setminus \alpha| : \alpha, \beta \in T\} \cup \{\omega\} \subset T$ then $Sp(X_T) = \{|\alpha| : \alpha \in T\} \cup \omega \cup \{\varkappa\}$.

Proof. (1) is obvious by Lemma 3.2. and the fact that $U^T_{\{<n, i>\}} = U^f_{\{<n, i>\}}$.

(2) The proof is analogous to the proof of (2) of Theorem 4.1.

(3) By the inequality $\omega \leq z(X_T) \leq w(X_T) \leq |T| + \omega$ it is enough to show that $z(X_T) \geq |T|$.

Let $T = \{t_\zeta : \zeta < \gamma\}$ be an increasing enumeration of T. So $\gamma \geq |T|$. By Lemma 3.2. for any $\zeta < |T|$ there exists $n_\zeta < \omega$ s.t. $t_\zeta \in U^{f}_{\{<t_\zeta + n_\zeta, 1>\}}$. So $t_\zeta \in U^{T}_{\{<t_\zeta + n_\zeta, 1>\}}$ and $t_\eta \notin U^{T}_{\{<t_\zeta + n_\zeta, 1>\}}$ for any $\zeta < \eta < |T|$. Therefore $z(X_T) \geq |T|$.

(4) Obviously $\{\varkappa\} \cup \omega \subset Sp(X_T)$. Moreover, for $\alpha \in T$ we have by Lemma 3.2. that $\bigcap_{n<\omega} U^{T}_{\{<\alpha+n, 0>\}} =$ $= \alpha \cup \bigcap_{n<\omega} U^{f}_{\{<\alpha+n, 0>\}} = \alpha \cup 0 = \alpha$, so α is a closed set. Hence $\{|\alpha| : \alpha \in T\} \cup \omega \cup \{\varkappa\} \subset Sp(X_T)$.

In order to prove the reverse inclusion we show that for any closed set F in X_T we have $|F| \in T \cup \omega \cup \{\varkappa\}$

Let F be a closed subset of X_T.

By hereditary Lindelöfness of X_T there exists a sequence $<\varepsilon_n : n < \omega>$ of elements of $H(\omega(T))$ s.t. $F = \bigcup\{F_{\varepsilon_n} : n < \omega\}$.

Let $S' = \{\alpha < \varkappa : (\exists n < \omega)(\alpha + n \in \bigcup_{m<\omega} D(\varepsilon_m))\}$ and let S be a closure of $S' \cup \{0\} \cup \{\varkappa\}$ under countable unions. So $S \subset T \cup \{0\} \cup \{\varkappa\}$ and S is countable. Let $<a_\zeta : \zeta \leq \xi>$ be an increasing enumeration of S. Since $\bigcup_{\zeta<\xi}(a_{\zeta+1} \setminus a_\zeta) = \varkappa$ we get

$F = \bigcup_{\zeta<\xi} [F \cap (a_{\zeta+1} \setminus a_\zeta)]$.

Let $\zeta < \xi$. It is enough to show that

$|F \cap (a_{\zeta+1} \setminus a_\zeta)| \in T \cup \omega \cup \{\varkappa\}$

In order to prove this show that for some closed subset F' of space $<\varkappa, \tau_f>$ we have

(*) $F \cap (a_{\zeta+1} \setminus a_\zeta) = F' \cap (a_{\zeta+1} \setminus a_\zeta)$

This is enough because, by Lemma 3.3. and 3.1, we have that

$|F' \cap (a_{\xi+1} \setminus a_\xi)| \in \{|a_{\xi+1} \setminus a_\xi|\} \cup \omega \cup \{\omega\} \subset T \cup \omega \cup \{\varkappa\}$

In order to prove $(*)$ it enough to show that for any $n < \omega$ and any $a \in D(\mathcal{E}_n)$ there exists a closed subset H of the space $\langle \varkappa, \tau_f \rangle$ s.t.

$$U^T_{\{\langle a, \mathcal{E}_n(a)\rangle\}} \cap (a_{\xi+1} \setminus a_\xi) = H \cap (a_{\xi+1} \setminus a_\xi)$$

If $a \le a_\xi$ then

$$U^T_{\{\langle a, \mathcal{E}_n(a)\rangle\}} \cap (a_{\xi+1} \setminus a_\xi) = U^f_{\{\langle a, \mathcal{E}_n(a)\rangle\}} \cap (a_{\xi+1} \setminus a_\xi)$$

If $a > a_\xi$ then, by definition of S, $a \geqslant a_{\xi+1}$.

Hence

$$U^T_{\{\langle a, 0\rangle\}} \cap (a_{\xi+1} \setminus a_\xi) = (a_{\xi+1} \setminus a_\xi)$$

$$U^T_{\{\langle a, 1\rangle\}} \cap (a_{\xi+1} \setminus a_\xi) = 0 \quad \blacksquare$$

From this theorem we can easily obtain the following

4.3. Corollary. For any set of cardinals $S \subseteq \varkappa$ which is closed under countable unions and s.t. $\omega \in S$, and for any infinite cardinal $\mu \leqslant \sup S$ s.t. $\mu \geqslant |S|$ there exists a heraditarily Lindelöf, zerodimensional space Y s.t. $Sp(Y) = S \cup \{\varkappa\}$ and $z(Y) = \mu$. In particular, if $\mu > \omega$ then Y is an L - space. $\blacksquare$

All examples presented here have the spectrum closed under countable unions. In [Ju 2] I.Juhàsz showed that if $\varkappa = \mathfrak{c}$ then a slight modification of the basic space gives an S-space with a spectrum $\omega \cup \{\varkappa\}$.

5. The basic space in the extensions of the Cohen model.

There is a well-known theorem /see [Ju 1]/ that for any cardinal $\varkappa$ and any topological space X the following implication holds

$$(\forall Y \in [X]^{\leq \varkappa})(w(Y) < \varkappa) \longrightarrow w(X) < \varkappa .$$

The analogous implication for a netweight is not a theorem of ZFC. In [Ha, Ju 2] A.Hajnal and I.Juhàsz constructed a consistant counterexample in the class of Hausdorff space. In this section we construct such a counterexample in the class of regular, hereditarily Lindelöf space. The proof presented here is a slight simiplification of that contained in [Ci].

Let $\varkappa > \omega_1$ be a regular cardinal in M and let $M[f]$ be a Cohen extension of M obtained by forcing $P(\varkappa \times \varkappa)$. So $f : \varkappa \times \varkappa \longrightarrow 2$. Our counterexample will be the basic space in some generic extension of the model $M[f]$.

In the proof we use the forcing form of the property of having a countable network.

For any family of a sets $\mathcal{B}$ we define a partialy ordered set

$Q(\mathcal{B}) = \{\langle A,U\rangle : A \subset U \ \& \ |A| < \omega \ \& \ U \in \mathcal{B}\}$

with the ordering relation

$\langle A,U\rangle \leq \langle B,W\rangle$ iff $A \supset B \ \& \ U \subset W$

The paper [Ha,Ju 2] implicitly containes the following

5.1. Lemma. Let $\mathcal{B}$ be a basis of a topological space X. If $Q(\mathcal{B}) = \bigcup_{n<\omega} Q^n$ where the sets Q^n are compatible in $Q(\mathcal{B})$ then $nw(X) = \omega$. Moreover, if $\mathcal{B}$ is closed under finite intersections then the reverse implication is also true. ∎

Let us note now that for any subspace Y of $\langle \varkappa, \tau_f \rangle$ the family $\mathcal{B}_{f \restriction \varkappa \times Y}$ is a standard basis of Y. In the case when $Y = \alpha$ is an infinite ordinal the set $Q(\mathcal{B}_{f\restriction \varkappa \times \alpha})$ has more useful representation. Namely let the set

$Q_\alpha = \{\langle A, \varepsilon\rangle : \varepsilon \in H(\varkappa) \ \& \ A \subset U_\varepsilon \cap \alpha \ \& \ |A| < \omega\}$

be partialy ordered with the relation

$<A, \mathcal{E}> \leq <B, \delta>$ iff $A \supset B$ & $\mathcal{E} \supset \delta$

By Lemma 3.1. we immediately obtaine the following

5.2. Proposition. Q_α and $Q(\mathcal{B}_{f \restriction \varkappa \times \alpha})$ are order isomorfic for $\omega \leq \alpha < \varkappa$ ∎

Now we can define a generic extension of $M[f]$.

Let for $\alpha \leq \varkappa$

$$P_\alpha = \{p \in Fnc : D(p) \subseteq \alpha \times \omega \ \& \ |D(p)| < \omega \ \& \ (\forall <\beta, n> \in D(p))(p(\beta, n) \in Q_\beta)\}$$

be a partialy ordered set with the ordering relation

$$p \leq q \quad iff \quad D(p) \supset D(q) \ \& \ (\forall <\beta, n> \in D(q))(p(\beta, n) \leq q(\beta, n))$$

Let us notice that for the definition of Q_β we used only the restriction of f to $\varkappa \times \beta$. Hence

5.3. Proposition. $P_\alpha \in M[f \restriction \varkappa \times \alpha]$ for $\alpha \leq \varkappa$ ∎

Let $M[f][G]$ be a generic extension of $M[f]$ obtained by forcing with $P_\varkappa$. We first show

5.4. Lemma. $M[f] \models$ " $P_\varkappa$ satisfies ccc".

Proof. Let $<p_\xi : \xi < \omega_1>$ be a sequence of elements of $P_\varkappa$. We show that there exist $\xi < \eta < \omega_1$ s.t. s.t. p_ξ and p_η are compatible.

By Δ-lemma we can assume that for some finite set $S \subseteq \varkappa \times \omega$ and for any $\xi < \eta < \omega_1$ we have $D(p_\xi) \cap D(p_\eta) = S$. However if $p_\xi \restriction S$ and $p_\eta \restriction S$ are compatible then p_ξ and p_η are compatible as well. Hence wlog we can assume that $D(p_\xi) = S$ for any $\xi < \omega_1$. For $s \in S$ and $\xi < \omega_1$ we denote $p_\xi(s) = <B^s_\xi, \delta^s_\xi>$

By Δ-lemma we can assume that for any $s \in S$ there exist sets B_s and D_s s.t.

$$B^s_\xi \cap B^s_\eta = B_s \ \& \ D(\delta^s_\xi) \cap D(\delta^s_\eta) = D_s \ \& \ \delta^s_\xi \restriction D_s = \delta^s_\eta \restriction D_s$$

for any $\xi < \eta < \omega_1$.

Let for any $s \in S$ and $\zeta < \omega_1$

$K^s_\zeta = B^s_\zeta \setminus B_s$ & $\varepsilon^s_\zeta = \delta^s_\zeta \upharpoonright D(\delta^s_\zeta) \setminus D_s$

In order to complete the proof it is enough to show that there exist $\zeta < \eta < \omega_1$ s.t.

(*) $K^s_\zeta \subset U_{\varepsilon^s_\eta}$ & $K^s_\eta \subset U_{\varepsilon^s_\zeta}$ for any $s \in S$, because then $p = \{<s, <B^s_\zeta \cup B^s_\eta, \delta^s_\zeta \cup \delta^s_\eta>>: s \in S\}$ is easily seen to be an element of $P_{\varkappa}$ and obviously extend p_ζ and p_η.

By Δ-lemma we can also assume that for some finite set a and any $\zeta < \eta < \omega_1$ $\left(\bigcup_{s \in S} K^s_\zeta\right) \cap \left(\bigcup_{s \in S} K^s_\eta\right) = a$.

Moreover we can also assume that all functions $g_\zeta : S \longrightarrow \mathcal{P}(a)$ given by formula $g_\zeta(s) = a \cap K^s_\zeta$ are equal $/\zeta < \omega_1/$. Hence $g_\zeta(s) = g_\zeta(s) \cap g_\eta(s) \subset K^s_\zeta \cap K^s_\eta = 0$ for any $\zeta < \eta < \omega_1$ and $s \in S$, i.e. $a = 0$. So, we can assume that

(1) $\left(\bigcup_{s \in S} K^s_\zeta\right) \cap \left(\bigcup_{s \in S} K^s_\eta\right) = 0$ for any $\zeta < \eta < \omega_1$

Analogously we can assume that

(2) $K^s_\zeta \cap K^{s'}_\zeta \neq 0$ iff $K^s_\eta \cap K^{s'}_\eta \neq 0$ for any $\zeta < \eta < \omega_1$ and $s, s' \in S$

Let us also note that

(3) $K^s_\zeta \cap K^{s'}_\zeta \neq 0 \longrightarrow \varepsilon^s_\zeta \cup \varepsilon^{s'}_\zeta \in Fnc$ for any $\zeta < \omega_1$ and $s, s' \in S$

This follows from the fact that for any $k \in K^s_\zeta \cap K^{s'}_\zeta$ and for any $a \in D(\varepsilon^s_\zeta) \cap D(\varepsilon^{s'}_\zeta)$ we have $\varepsilon^s_\zeta(a) = f(a,k) = \varepsilon^{s'}_\zeta(a)$.

Now we can find $\zeta < \eta < \omega_1$ satisfying (*).

Let $A \subset \varkappa$ be a countable set from M s.t.

(4) $<K^s_n : n < \omega \;\&\; s \in S>, <\varepsilon^s_n : n < \omega \;\&\; s \in S> \in M[f \upharpoonright A^2]$

So there exists η s.t. $\omega \leq \eta < \varkappa$ and

(5) $K^s_\eta \cap A = 0$ & $D(\varepsilon^s_\eta) \cap A = 0$ for any $s \in S$

We show that there exists $n < \omega$ s.t. n and η satisfy (*). Let us define the set

$D = \{r \in P(\varkappa^2 \setminus A^2): \exists n < \omega \ \forall s \in S \ \ r \Vdash \text{"} K^s_n \subset U_{\varepsilon^s_\eta} \ \& \ K^s_\eta \subset U_{\varepsilon^s_n} \text{"}\}$

By (4) it belongs to $M[f \restriction A^2]$, so it is enough to show that it is dense in $P(\varkappa^2 \setminus A^2)$.

Let $t \in P(\varkappa^2 \setminus A^2)$ and B be a finite set s.t. $D(t) \subset B^2$. Then there exists $n < \omega$ s.t.

(6) $K^s_n \cap B = 0$ & $D(\varepsilon^s_n) \cap B = 0$ for any $s \in S$

Let

$t_1 = \{\langle a, k, \varepsilon^s_\eta(a)\rangle: \ s \in S \ \& \ a \in D(\varepsilon^s_\eta) \ \& \ k \in K^s_n\}$

$t_2 = \{\langle a, k, \varepsilon^s_n(a)\rangle: \ s \in S \ \& \ a \in D(\varepsilon^s_n) \ \& \ k \in K^s_\eta\}$

We prove that $t \cup t_1 \cup t_2 \in P(\varkappa^2 \setminus A^2)$.

If $\langle a, k\rangle \in (D(\varepsilon^s_\eta) \times K^s_n) \cap (D(\varepsilon^{s'}_\eta) \times K^{s'}_n)$ for some $s, s' \in S$ then, by (2) and (3), $\varepsilon^s_\eta(a) = \varepsilon^{s'}_\eta(a)$.
So $t_1 \in$ Fnc. Analogously we prove that $t_2 \in$ Fnc. Moreover, from (6) it follows that $D(t) \cap D(t_1) = D(t) \cap D(t_2) = 0$ and from (1) it follows that $D(t_1) \cap D(t_2) = 0$. So $t \cup t_1 \cup t_2 \in$ Fnc. Hence, by (5), we conclude that $t \cup t_1 \cup t_2 \in P(\varkappa^2 \setminus A^2)$.

Now it is enough to note that for any $s \in S$

$t_1 \Vdash \text{"} K^s_n \subset U_{\varepsilon^s_\eta} \text{"}$

$t_2 \Vdash \text{"} K^s_\eta \subset U_{\varepsilon^s_n} \text{"}$ ∎

Now we can prove the main theorem of this section.

5.5. Theorem. In the model $M[f][G]$ the space $X = \langle \varkappa, \tau_f \rangle$ has the following properties:

(1) X is a zerodimensional, Hausdorff space; in particular X is regular;

(2) $nw(Y) = \omega$ for every $Y \in [\varkappa]^{<\varkappa}$;

(3) $h(X) = \omega$;

(4) $nw(X) = \varkappa$.

Proof.(1) follows immediately from Lemma 3.2. and the absoluteness of the definition of $\mathcal{P}_f$.

(2) Let $Y \in [\varkappa]^{<\varkappa}$. By regularity of $\varkappa$ there exists α, $\omega \leq \alpha < \varkappa$ s.t. $Y \subset \alpha$. Hence it is enough to prove that $nw(\alpha) = \omega$. However, by Lemma 5.1. and Proposition 5.2. we must only to show that the set Q_α is a union of countable many compatible subsets.

We put

$Q^n = \{p(\alpha, n) : p \in G \ \& \ \langle \alpha, n \rangle \in D(p)\}$

and leave to the reader to verify that $Q_\alpha = \bigcup_{n<\omega} Q^n$ and that the Q^n's are compatible for $n < \omega$.

(3) By (2) and inequalities $\varkappa > \omega_1$ /see Lemma 5.4./ and $h(Y) \leq nw(Y)$ we conclude that $h(Y) = \omega$ for any $Y \in [\varkappa]^{\omega_1}$ Hence $h(X) = \omega$.

(4) It is enough to prove that $nw(X) \geq \varkappa$. Let us assume to the contrary that $nw(X) = \mu < \varkappa$ and let $\{F_\zeta : \zeta < \mu\}$ be a network of X. By the regularity of X we can assume that the sets F_ζ for $\zeta < \mu$ are closed. Hence, by (3), for any $\zeta < \mu$ there exists a sequence $\langle \varepsilon^\zeta_n \ n < \omega \rangle$ of elements of $H(\varkappa)$ s.t. $F_\zeta = \bigcap_{n<\omega} F_{\varepsilon^\zeta_n}$

Let us observe that $P_\varkappa = \bigcup_{\alpha<\varkappa} P_\alpha$. Hence $G_\alpha = G \cap P_\alpha$ is $M[f]$-generic over P_α for $\alpha < \varkappa$. So, by the regulariry of $\varkappa$, there exists $\beta < \varkappa$ s.t.

$\langle \varepsilon^\zeta_n \ \zeta < \mu \ \& \ n < \omega \rangle \in M[f][G_\beta]$

However, by product lemma and Proposition 5.3.

$M[f][G_\beta] = M[f \upharpoonright \varkappa \times \beta][f \upharpoonright \varkappa \times (\varkappa \setminus \beta)] \ [G_\beta]$

$= M[f \upharpoonright \varkappa \times \beta][G_\beta] \ [f \upharpoonright \varkappa \times (\varkappa \setminus \beta)]$

So there exists α s.t. $\beta \le \alpha < \varkappa$ and

$$\langle \varepsilon_n^\zeta : \zeta < \mu \ \&\ n < \omega \rangle \in M[f\restriction \varkappa\times\beta][G_\beta][f\restriction\varkappa\times(\alpha\setminus\beta)]$$
$$= M[f\restriction\varkappa\times\alpha][G_\beta] \subset M[f\restriction\varkappa\times\alpha][G_\alpha]$$

Now let $\alpha < \eta < \varkappa$ and let $a \in \varkappa \setminus \bigcup\{D(\varepsilon_n^\zeta) : \zeta < \mu \ \&\ n < \omega\}$. Let us assume that $\eta \in U_a^0$ /for $\eta \in U_a^1$ the proof is similar/. We prove that for any $\zeta < \mu$

(*) $M[f\restriction\varkappa\times\alpha][G_\alpha][f\restriction\varkappa\times(\varkappa\setminus\alpha)] \models$ " $\eta \notin F_\zeta \ \vee\ F_\zeta \not\subset U_a^0$ ".

This will be enought becausethen, by absoluteness of the above formula,

$M[f][G] \models$ " $(\forall \zeta < \mu)(\eta \notin F_\zeta \ \vee\ F_\zeta \not\subset U_a^0)$ "

which contradicts the assumption that the family $\{F_\zeta : \zeta < \mu\}$ is a network of X.

Let $\eta \in F_\zeta$. Hence /compare Lemma 3.3. / there exists $\xi \in E(\bigcup_{n<\omega} D(\varepsilon_n^\zeta))$ s.t. $U_\xi \subset F_\zeta$.

However $a \notin D(\xi)$. So, by Lemma 2.1., $U_\xi \cap U_a^1 \neq 0$.
Hence $F_\zeta \not\subset U_a^0$. ■

6. A regular space omitting $\mathfrak{c}$ with countable pseudo character.

It is easy to see that no Hausdorff space omits $\mathfrak{c}$. A natural question posed by R.E.Hodel /see [Ju 2]/ is whether the same is true for pseudo character instead of the character. In this section we construct in some model of ZFC + $\neg$CH a zerodimensional Hausdorff space with countable pseudo character which omits $\mathfrak{c}$.

Let M satisfy CH and let $\varkappa$ and λ be regular cardinals in M s.t. $\varkappa > \lambda > \omega_1$.

For $Z \subseteq M$ let

$H_\omega(Z) = \{\delta \in M : \delta \in Fnc \;\&\; D(\delta) \subset Z \;\&\; |D(\delta)| \le \omega \;\&\; R(\delta) \subset \omega\}$

and let $R = H(\lambda)$ and $Q = H_\omega(\varkappa \times \varkappa)$ be sets partialy ordered by reverse inclusion. Let G be M-generic over R and let f be an $M[G]$-generic function over Q. So $f : \varkappa \times \varkappa \longrightarrow \omega$.

Since $R \times Q$ is an Easton-type forcing then

6.1. Proposition

(1) $M \models$ " $R \times Q$ satisfies ω_2 - cc"

(2) $M \models$ " $R \times Q$ preserves the cardinals"

(3) $M[G][f] \models$ " $\omega_1 < \lambda = \mathfrak{c} < \varkappa$ " ■

Similarly to the Lemma 3.1. we prove

6.2. Lemma. In the model $M[G][f]$ the following hold

(1) $(\forall\, \xi < \eta < \omega_1)(\forall A \in M)\big[(A \subset \varkappa \;\&\; |A| \ge \omega_1) \longrightarrow$

$\longrightarrow (\exists\, a \in A)(\, f(a, \xi) = 0 \;\&\; f(a, \eta) = 1)\big]$

(2) $(\forall\, \delta \in H_\omega(\varkappa))(\forall A \in M)\big[(A \subset \varkappa \;\&\; |A| \ge \omega_1) \longrightarrow$

$\longrightarrow (\exists\, \xi \in A)(\forall d \in D(\delta))\,(\, f(d, \xi) = \delta(d))\big]$ ■

Let us define a function $g : \varkappa \times \varkappa \longrightarrow \omega + 1$ by a formula

$$g(\xi, \eta) = \begin{cases} f(\xi, \eta) & \text{for } \xi \ne \eta \\ \omega & \text{for } \xi = \eta \end{cases}$$

and let

$P = \bigcup \{\{a, \omega + 1 \setminus a\} : a \subset \omega \;\&\; |a| < \omega\}$

Let $X = \langle \varkappa, \tau_{g,P} \rangle$.

For $\delta \in H_\omega(\varkappa)$ we define

$$U_\delta = \bigcap_{d \in D(\delta)} U_d^{\delta(d)}$$

Hence, by Lemma 6.2., we obtain

6.3. Corollary. In the model $M[G][f]$ we have

$(\forall\, \delta \in H_\omega(\varkappa))(\forall A \in M)\big[(A \subset \varkappa \;\&\; |A| \ge \omega_1) \longrightarrow U_\delta \cap A \ne 0\big]$

Hence, in particular, for any $\delta \in H_\omega(\varkappa)$ we have $|U_\delta| = \varkappa$ ∎

6.4. Theorem. In the model $M[G][f]$ the space $X = \langle \varkappa, \tau_{g,P} \rangle$ is a zerodimensional, Hausdorff space with countable pseudo character which omits all cardinals bigger then ω_1 and less then $\varkappa$. So, in particular, X omits $\mathfrak{c}$.

Proof. The zerodimensionality of X is obvious. The fact that X is a Hausdorff space follows immediately from Lemma 6.2.

Moreover X has a countable pseudo character because for any $\zeta < \varkappa$

$$\{\zeta\} = \bigcap_{n<\omega} U_\zeta^{(\omega+1\setminus n)}$$

Now it is enough to show that for any closed subset F of X either $|F| = \varkappa$ or $|F| \le \omega_1$.

Let F be a closed subset of X s.t. $|F| \ne \varkappa$. We show that $|F| \le \omega_1$

By inequality $h(X) \le w(X) \le \varkappa$ we conclude that there exists $\delta : \varkappa \longrightarrow H(\varkappa, P \setminus \{0, \omega+1\})$ s.t. $F = \bigcap_{\zeta<\varkappa} F_{\delta(\zeta)}$

Hence, by Corollary 6.3.,

$(\ast)$ $\forall \delta \in H_\omega(\varkappa)\ \exists \eta < \varkappa\ \ U_\delta \not\subset F_{\delta(\eta)}$

Let us define by induction two increasing sequences $\langle A_\zeta : \zeta < \omega_1 \rangle$ and $\langle Z_\zeta : \zeta < \omega_1 \rangle$ of subsets of $\varkappa$ s.t. for every $\zeta < \omega_1$

(i) $|Z_\zeta| \le \omega_1$, $|A_\zeta| \le \omega_1$;

(ii) $Z_\zeta \in M$;

(iii) $\bigcup\{D(\delta(\eta)) : \eta \in A_\zeta\} \subset Z_{\zeta+1}$.

(iv) $\forall \delta \in H_\omega(Z_\zeta)\ \exists a \in A_\zeta\ \ U_\delta \not\subset F_{\delta(a)}$.

Let $Z_0 = 0$

If for some $\xi < \omega_1$ we have already defined $\langle A_\zeta : \zeta < \xi \rangle$ and $\langle Z_\zeta : \zeta \le \xi \rangle$ then we define A_ξ as follows: by $(\ast)$ we can define for every $\delta \in H_\omega(Z_\xi)$ an ordinal

$\eta_\delta = \min\{\eta : U_\delta \not\subset F_{\delta(\eta)}\}$ and let $B_\xi = \{\eta_\delta : \delta \in H_\omega(Z_\xi)\}$ hence, by $|Z_\xi| \leq \omega_1$ and $M \models$ "CH", we have that $|B_\xi| \leq \omega_1$, so the set $A_\xi = B_\xi \cup \bigcup_{\zeta<\xi} A_\zeta$ satisfies (i) - (iv).

If $\langle A_\zeta : \zeta < \xi\rangle$ and $\langle Z_\zeta : \zeta < \xi\rangle$ are defined for some $\xi < \omega_1$ then we define

$$Y_\xi = \begin{cases} Z_\alpha \cup \bigcup\{D(\delta(\eta)) : \eta \in A_\alpha\} & \text{for } \xi = \alpha+1 \\ \bigcup_{\zeta<\xi} Z_\zeta & \text{for } \xi \in \mathrm{Lim} \end{cases}$$

Hence, because $R \times Q$ satisfies ω_2- cc, then we can choose $Z_\xi \in M$ s.t. $Y_\xi \subset Z_\xi \subset \kappa$ and $|Z_\xi| \leq \omega_1$. By construction, Z_ξ satisfies (i) -(iv).

Let $A = \bigcup_{\zeta<\omega_1} A_\zeta$ and $Z = \bigcup_{\zeta<\omega_1} Z_\zeta$

By (iii) we obtain immediately

(v) $\bigcup\{D(\delta(a)) : a \in A\} \subset Z$

Let

$$F' = \bigcap_{a\in A} F_{\delta(a)}$$

So $F' \supset F$. Hence it is enough to prove that $|F'| \leq \omega_1$

The forcing $R \times Q$ satisfies ω_2- cc and $|A| \leq \omega_1$. So there exists $X \in M$ s.t. $X \subset \kappa$, $|X| \leq \omega_1$ and

(1) $\bigcup\{D(\delta(a)) : a \in A\} \subset X$

(2) $\delta \restriction A \in M[G][f \restriction \kappa \times X]$

We show that $F' \subset X$.

Let us assume that there exists $\eta \in F' \setminus X$.

Then, by product Lemma, there exists $s \in H_\omega(\kappa \times (\kappa \setminus X))$ s.t. $s \Vdash$ "$\eta \in F'$". Hence, by (2), for any $a \in A$ we have $s \Vdash$ "$\eta \in F_{\delta(a)}$". However $\delta(\zeta)(\xi) \neq \omega+1$ for any $\zeta < \kappa$ and $\xi \in D(\delta(\zeta))$ and, by (1), $\eta \notin \bigcup\{D(\delta(a)): a \in$

So, for every $a \in A$ there exists $\xi_a \in D(\delta(a)) \subset Z$ s.t.

$\langle \xi_a, \eta \rangle \in D(s)$ & $s(\xi_a, \eta) \in \delta(a)(\xi_a)$

Hence, if

$\delta' = \{ \langle \xi, s(\xi, \eta) \rangle : \langle \xi, \eta \rangle \in D(s) \}$

and $\delta = \delta' \restriction Z$ then

(o) $U_\delta \subset F_{\delta(a)}$ for every $a \in A$.

On the other hand $|D(\delta)| \leq |D(s)| \leq \omega$. So there exists

$\xi < \omega_1$ s. t. $D(\delta) \subset Z_\xi$. Hence

$\delta = \delta' \restriction Z = \delta' \restriction Z_\xi \in M$

However then $\delta \in H_\omega(Z_\xi)$. So, by (iv), $U_\delta \not\subset F_{\delta(a)}$

for some $a \in A_\xi \subset A$ which contradicts (o) ∎

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