On the netweight of subspaces

by

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Abstract. In this paper we give a (consistent) solution to a problem of A. Hajnal and I. Juhász [3], namely we show a model of set theory with $2^\omega > \omega_1$ in which there exists a regular topological space X with an uncountable netweight and such that every subspace of X of power smaller than that of X has a countable netweight.

Introduction. In [3] A. Hajnal and I. Juhász, in connection with a problem of M. G. Tkačenko, showed that it is consistent with set theory to assume that there exists a Hausdorff space X of power ω_2 with the following properties:

1. $nw(X) = \omega_2$,
2. $nw(Y) = \omega$ for every subspace $Y \subset X$ of power ω_1.

They suggested the natural problem whether an analogous result for regular spaces could be proved. This paper gives a solution to this problem.

We recall that $nw(X)$ is the netweight of X, i.e. the smallest cardinal of a network for X.

Throughout the paper we use the standard set-theoretical notation. We use the forcing technique as described e.g. in [1].

The graph topology. Let $[X]^{\leqslant 2} = \{y \subset X: |y| = 1 \vee |y| = 2\}$.

We say that the function f: $[X]^{\leqslant 2} \to 2$ is a *graph* iff $f(\{x\}) = 0$ for every $x \in X$ (the elements $x, y \in X$ are considered to be connected by an edge in the graph f iff $f(\{x, y\}) = 0$).

For every $x \in X$ and $i<2$ we put

$$U_x^i = \{y \in X: f(\{x, y\}) = i\},$$

in particular $x \in U_x^0$ for every $x \in X$.

We are going to study the topology τ_f on X generated by the subbasis

$$\{U_x^i: x \in X \,\&\, i<2\}.$$

Clearly the space X with the topology τ_f is 0-dimensional.

Let $H(X)$ be the set of all functions from finite subsets of X into 2. For $\varepsilon \in H(X)$ we shall put

$$U_\varepsilon = \bigcap_{x \in \mathrm{dom}(\varepsilon)} U_x^{\varepsilon(x)}.$$

Hence the family $\{U_\varepsilon\colon \varepsilon\in H(X)\}$ is a basis for τ_f.

Let F be a family of sets. We say that the graph $f\colon [X]^{\leq 2}\to 2$ is *ω-full over F* if for every infinite $C\in F\cap P(X)$ and every $\varepsilon\in H(X)$ there exists a $c\in C$ such that $f(\{c,y\})=\varepsilon(y)$ for each $y\in\mathrm{dom}(\varepsilon)$.

Let us note that if $f\colon [X]^{\leq 2}\to 2$ is ω-full over F then for every infinite subspace $Y\subset X$ such that $Y\in F$ we have the equivalence:

$$U_\varepsilon\cap Y\subset U_\delta\cap Y \quad\text{iff}\quad \delta\subset\varepsilon \quad\text{for every } \varepsilon,\delta\in H(X).$$

If Y is a subspace of X then we put

$$Q(Y)=\{\langle A,\varepsilon\rangle\colon A\subset Y\ \&\ A \text{ is finite}\ \&\ \varepsilon\in H(X)\ \&\ A\subset U_\varepsilon\}$$

with the ordering relation

$$\langle A,\varepsilon\rangle\leq\langle B,\delta\rangle \quad\text{iff}\quad A\supset B\ \&\ \varepsilon\supset\delta.$$

We say that a subset Q of a partially ordered set is *compatible* if every two elements of Q are compatible.

Now we can formulate the following

LEMMA 1. *If $f\colon [X]^{\leq 2}\to 2$ is ω-full over F, $Y\in F$ is an infinite subspace of X and $Q(Y)$ is a union of a countable family of compatible sets then* $\mathrm{nw}(Y)\leq\omega$.

The proof of the lemma is contained implicitly in [3].

The idea of the proof. In order to construct a model with a regular topological space having the required properties we first add generically a graph $f\colon [\varkappa]^{\leq 2}\to 2$ for a regular cardinal $\varkappa$ using finite conditions and then add for each $\alpha<\varkappa$ a generic decomposition of $Q(\alpha)$ into a countable family of compatible sets.

The ω-fullness of the graph f over the family $\varkappa=\{\alpha\colon \alpha<\varkappa\}$ follows by the genericity of f. It easily gives the regularity of $\langle\varkappa,\tau_f\rangle$. By Lemma 1 we also get a countable network for each subspace $\alpha\subset\varkappa$, where $\alpha<\varkappa$.

Since our forcing is ccc, it remains to show that $\mathrm{nw}(\varkappa)>\omega$.

It is also very important to mention that in order to define the poset $Q(\alpha)$ for $\alpha<\varkappa$ it suffices to know the values of the graph only for pairs $\{\zeta,\eta\}$ such that $\min\{\zeta,\eta\}<\alpha$.

Now we turn to details.

Construction of the model. Let $\varkappa>\omega_1$ be a regular cardinal. We define several posets:

(i) $$S=\{s\in H([\varkappa]^{\leq 2})\colon \forall\{x\}\in\mathrm{dom}(s)\,[s(\{x\})=0]\}$$

ordered by reverse inclusion,

(ii) $$B_\alpha=\{\langle A,\varepsilon\rangle\colon A\subset\alpha\ \&\ A \text{ is finite}\ \&\ \varepsilon\in H(\varkappa)\}\quad\text{for}\quad \alpha\leq\varkappa$$

with the ordering relation

$$\langle A,\varepsilon\rangle\leq\langle B,\delta\rangle\quad\text{iff}\quad A\supset B\ \&\ \varepsilon\supset\delta,$$

(iii) $$Q=\{q\colon \mathrm{Fnc}(q)\ \&\ \mathrm{dom}(q)=\varkappa\times\omega\ \&\ \forall\alpha<\varkappa\,\forall n<\omega\,[q(\alpha,n)\in B_\alpha]\ \&\ |\mathrm{supp}(q)|<\omega\}$$

where $\mathrm{supp}(q)=\{\langle\alpha,n\rangle\colon q(\alpha,n)\neq\langle 0,0\rangle\}$ with the ordering relation

$$q_1\leq q_2\quad\text{iff}\quad q_1(\alpha,n)\leq q_2(\alpha,n)\quad\text{for any } \alpha<\varkappa \text{ and } n<\omega,$$

(iv) $$P=\{\langle s,q\rangle\in S\times Q\colon \forall\alpha<\varkappa\,\forall n<\omega\,\forall a\,\forall b\,[q(\alpha,n)=\langle A,\varepsilon\rangle\ \&\ a\in A\ \&\ b\in\mathrm{dom}(\varepsilon)\to(\langle\{a,b\},\varepsilon(b)\rangle\in s\vee(a=b\ \&\ \varepsilon(b)=0))]\}$$

with the ordering relation

$$\langle s_1,q_1\rangle\leq\langle s_2,q_2\rangle\quad\text{iff}\quad s_1\leq s_2 \text{ and } q_1\leq q_2.$$

Let us remark that the forcing P can be considered as a product forcing, i.e.

$$P=\{\langle s,q\rangle\in S\times Q\colon \forall\alpha<\varkappa\,\forall n<\omega\,[s\Vdash\text{“}q(\alpha,n)\in Q(\alpha)\text{”}]\}.$$

THEOREM 2. *The forcing P is* ccc.

The proof will be postponed until the last section of this paper.

Let $\alpha<\varkappa$. We fix some notation:

$$D^\alpha=[\varkappa\setminus\alpha]^{\leq 2},\quad D_\alpha=[\varkappa]^{\leq 2}\setminus D^\alpha=\{\{\zeta,\eta\}\in[\varkappa]^{\leq 2}\colon \min\{\zeta,\eta\}<\alpha\},$$
$$S_\alpha=\{s\in S\colon \mathrm{dom}(s)\subset D_\alpha\}\quad\text{and}\quad S^\alpha=\{s\in S\colon \mathrm{dom}(s)\subset D^\alpha\}.$$

The orderings of S_α and S^α are the reverse inclusion.

Next, let

$$Q_\alpha=\{q\restriction\alpha\times\omega\colon q\in Q\}\quad\text{and}\quad Q^\alpha=\{q\restriction(\varkappa\setminus\alpha)\times\omega\colon q\in Q\}$$

both be ordered by

$$q_1\leq q_2\quad\text{iff}\quad q_1(\beta,n)\leq q_2(\beta,n)\quad\text{for every } (\beta,n)\in\mathrm{dom}(q_1).$$

It is clear that $S\simeq S_\alpha\times S^\alpha$ and $Q\simeq Q_\alpha\times Q^\alpha$.

Finally, let

$$\begin{aligned}R_\alpha=\{\langle s_1,q_1,s_2,q_2\rangle\in S_\alpha\times Q_\alpha\times S^\alpha\times Q^\alpha\colon &\forall\beta<\alpha\,\forall n<\omega\,\forall a\,\forall b\,[q_1(\beta,n)\\ &=\langle A,\varepsilon\rangle\ \&\ a\in A\ \&\ b\in\mathrm{dom}(\varepsilon)\to(\langle\{a,b\},\varepsilon(b)\rangle\in s_1\vee\\ &\vee(a=b\ \&\ \varepsilon(b)=0))]\ \&\ \forall\beta\geq\alpha\,\forall n<\omega\,\forall a\,\forall b\,[q_2(\beta,n)=\langle A,\varepsilon\rangle\ \&\\ &\&\ a\in A\ \&\ b\in\mathrm{dom}(\varepsilon)\to(\langle\{a,b\},\varepsilon(b)\rangle\in s_1\cup s_2\vee(a=b\ \&\ \varepsilon(b)=0))]\}\end{aligned}$$

be a poset with the ordering relation

$$\langle s_1,q_1,s_2,q_2\rangle\leq\langle s_1',q_1',s_2',q_2'\rangle\quad\text{iff}\quad s_1\leq s_1'\ \&\ q_1\leq q_1'\ \&\ s_2\leq s_2'\ \&\ q_2\leq q_2'.$$

It is easy to see that a mapping $g_\alpha\colon P\to R_\alpha$ defined by

$$g_\alpha(s,q)=\langle s\restriction D_\alpha,\ q\restriction\alpha\times\omega,\ s\restriction D^\alpha,\ q\restriction(\varkappa\setminus\alpha)\times\omega\rangle\quad\text{for any } \langle s,q\rangle\in P$$

is an order isomorphism of P and R_α.

From now on we shall identify P with R_α.

Let M be a countable transitive model of set theory and let $\varkappa>\omega_1$ be a regular cardinal in M. We consider a forcing P in M defined for $\varkappa$ and let G be an M-generic filter over P.

We define

$$G_\alpha = \{\langle s, q\rangle \in S_\alpha\times Q_\alpha\colon \langle s, q, \langle 0,0\rangle, \mathbf{1}\rangle \in G\}$$

where $\mathbf{1}$ is the maximal element of Q^α,

$$G^\alpha = \{\langle s, q\rangle \in S^\alpha\times Q^\alpha\colon \exists\langle s_1, q_1\rangle[\langle s_1, q_1, s, q\rangle \in G]\}\,.$$

Next,

$$P_\alpha = \{\langle s, q\rangle \in S_\alpha\times Q_\alpha\colon \langle s, q, \langle 0,0\rangle, \mathbf{1}\rangle \in P\}\in M\,,$$

$$P^\alpha = \{\langle s, q\rangle \in S^\alpha\times Q^\alpha\colon \exists\langle s_1, q_1\rangle\in G_\alpha[\langle s_1, q_1, s, q\rangle \in P]\}\in M[G_\alpha]$$

are the posets ordered by

$$\langle s_1, q_1\rangle \leqslant \langle s_2, q_2\rangle \quad\text{iff}\quad s_1\leqslant s_2\ \&\ q_1\leqslant q_2\,.$$

A standard argument shows

PROPOSITION 3. *G_α is M-generic over P_α, G^α is $M[G_\alpha]$-generic over P^α and $M[G] = M[G_\alpha][G^\alpha]$.*

Let $f = \bigcup\{s\colon \langle s, q\rangle\in G\}$ and $X = \langle\varkappa, \tau_f\rangle$.

THEOREM 4. *$M[G]$ is a ccc extension of M such that*

(1) X is regular (even hereditarily normal),

(2) $nw(Y)\leqslant\omega$ for every subspace Y of X of power smaller than $\varkappa$,

(3) $nw(X) = \varkappa$.

Proof. We begin with the following

PROPOSITION 5. *For every $\alpha<\varkappa$ the graph $f\restriction D^\alpha$ is ω-full over $M[G_\alpha]$. In particular f is ω-full over $\varkappa$.*

For the proof it is enough to show that for every infinite $K\in M[G_\alpha]\cap P(\varkappa\setminus\alpha)$ and every $\varepsilon\in H(\varkappa\setminus\alpha)$ the set

$$D = \{\langle s, q\rangle\in P^\alpha\colon \exists\eta\in K\,\forall\zeta\in\operatorname{dom}(\varepsilon)[s(\{\eta,\zeta\}) = \varepsilon(\zeta)]\}\in M[G_\alpha]$$

is dense in P^α.

(1) By Proposition 5 it follows immediately that for every $\zeta<\xi<\varkappa$ there exists $\eta<\varkappa$ such that $\zeta\in U^0_\eta$ and $\xi\in U^1_\eta$, i.e. X is a Hausdorff space. Since X is 0-dimensional, it is also regular.

(2) From an obvious inequality $nw(Y)\leqslant nw(X)$ for a subspace Y of X and from Proposition 5 and Lemma 1 it follows that it is enough to show that for any infinite $\alpha<\varkappa$ the set $Q(\alpha)$ is a union of a countable family of compatible sets.

We take $\alpha<\varkappa$ and let

$$Q^\alpha_n = \{q(\alpha, n)\colon \langle s, q\rangle\in G\}\quad\text{for}\quad n<\omega\,.$$

We show that $Q(\alpha) = \bigcup\{Q^\alpha_n\colon n<\omega\}$. If $\langle A, \varepsilon\rangle\in Q^\alpha_n$ for some $n<\omega$ then there exists $\langle s, q\rangle\in G$ such that $q(\alpha, n) = \langle A, \varepsilon\rangle$. Hence, by the definition of f, $A\subset U_\varepsilon$, i.e. $\langle A, \varepsilon\rangle\in Q(\alpha)$. If $\langle A, \varepsilon\rangle\in Q(\alpha)$ then

$$\forall a\in A\,\forall b\in\operatorname{dom}(\varepsilon)[f(\{a, b\}) = \varepsilon(b)]\,.$$

Hence, by the finiteness of $A\times\operatorname{dom}(\varepsilon)$, there exists $\langle s_0, q_0\rangle\in G$ such that

$$\forall a\in A\,\forall b\in\operatorname{dom}(\varepsilon)[s_0(\{a, b\}) = \varepsilon(b)]\,.$$

It is enough to show that the set

$$\{\langle s, q\rangle\in P\colon \exists n<\omega[q(\alpha, n) = \langle A, \varepsilon\rangle]\}$$

is dense below $\langle s_0, q_0\rangle$.

Let $\langle s, q\rangle\in P$ and $\langle s, q\rangle\leqslant\langle s_0, q_0\rangle$. There exists an $n<\omega$ such that $\langle\alpha, n\rangle\notin\operatorname{supp}(q)$. Let $q'\in Q$ be defined by

$$q'(\beta, m) = \begin{cases} q(\beta, m) & \text{for}\quad \langle\beta, m\rangle\neq\langle\alpha, n\rangle\,,\\ \langle A, \varepsilon\rangle & \text{for}\quad \langle\beta, m\rangle = \langle\alpha, n\rangle\,.\end{cases}$$

It is easy to see that $\langle s, q'\rangle\in P$ and $\langle s, q'\rangle\leqslant\langle s, q\rangle$.

In order to complete the proof of (2) it is enough to verify that each Q^α_n is compatible.

Let $\langle A_0, \varepsilon_0\rangle, \langle A_1, \varepsilon_1\rangle\in Q^\alpha_n$. Then there exist $\langle s_0, q_0\rangle, \langle s_1, q_1\rangle\in G$ such that $q_i(\alpha, n) = \langle A_i, \varepsilon_i\rangle$ for $i<2$. Let $\langle s, q\rangle\in G$ be a common extension of $\langle s_0, q_0\rangle$ and $\langle s_1, q_1\rangle$. Then $q(\alpha, n)$ extends $\langle A_0, \varepsilon_0\rangle$ and $\langle A_1, \varepsilon_1\rangle$, which completes the proof of (2).

Let us note that the space fulfilling condition (2) (where $\varkappa$ is a power of X) is hereditarily Lindelöf. Hence (see [2]) X is hereditarily normal.

(3) To the contrary, let us assume that $\mathrm{nw}(X)<\varkappa$. So there exists a network

$$\{F_\zeta\colon \zeta<\gamma\}\quad\text{where } \gamma<\varkappa\,.$$

By the regularity of X we can assume that all F_ζ are closed for $\zeta<\gamma$. Hence, by hereditary Lindelöfness, we can assume that

$$F_\zeta = \varkappa\setminus\bigcup_{n<\omega} U_{\varepsilon^n_\zeta}\quad\text{for any } \zeta<\gamma\,.$$

Let $E\colon \gamma\times\omega\to H(\varkappa)$ be a mapping defined by

$$E(\zeta, n) = \varepsilon^n_\zeta\quad\text{for any } \zeta<\gamma \text{ and } n<\omega\,.$$

A standard argument shows that there exists an $\alpha<\varkappa$ such that

(i) $$E\in M[G_\alpha]\,.$$

We can also assume that

(ii) $$\bigcup\{\operatorname{dom}(\varepsilon^n_\zeta)\colon \zeta<\gamma\ \&\ n<\omega\}\subset\alpha\,,$$

(iii) $$\bigcup\{F_\zeta\colon \zeta<\gamma\ \&\ |F_\zeta|<\varkappa\}\subset\alpha\,.$$

Since $f \restriction D_\alpha \in M[G_\alpha]$ and the fact that for the definition of U_ε, where $\varepsilon \in H(\alpha)$, the knowledge of $f \restriction D_\alpha$ is sufficient, we have

$$F_\zeta \in M[G_\alpha] \quad \text{for each } \zeta < \gamma .$$

Let $\beta \geqslant \alpha$. We show that

$$\forall \zeta < \gamma [\beta \notin F_\zeta \vee F_\zeta \not\subset U_\beta^0],$$

which contradicts the assumption that $\{F_\zeta \colon \zeta < \gamma\}$ is a network.

Let $\zeta < \gamma$. If $|F_\zeta| < \varkappa$ then, by (iii), $\beta \notin F_\zeta$. If $|F_\zeta| = \varkappa$ then $F_\zeta \setminus \alpha \in M[G_\alpha]$ is an infinite subset of $\varkappa \setminus \alpha$. Hence, by Proposition 5, there exists an $\eta \in F_\zeta$ such that $f(\{\eta, \beta\}) = 1$. So $\eta \notin U_\beta^0$, i.e. $F_\zeta \not\subset U_\beta^0$.

This completes the proof of Theorem 4.

Proof of Theorem 2. Let $y = \{\langle \alpha_0, m_0 \rangle, \ldots, \langle \alpha_{n-1}, m_{n-1} \rangle\}$ be a subset of $\varkappa \times \omega$. We define the posets:

$$Q_y = \{q \colon \operatorname{Fnc}(q) \;\&\; \operatorname{dom}(q) = n \;\&\; \forall i < n [q(i) \in B_{\alpha_i}]\}$$

with ordering relation

$$q_1 \leqslant q_2 \quad \text{iff} \quad q_1(i) \leqslant q_2(i) \quad \text{for every } i < n ,$$

$$P_y = \{\langle s, q \rangle \in S \times Q_y \colon \forall i < n \,\forall a \,\forall b \big[q(i) = \langle A, \varepsilon \rangle \;\&\; a \in A \;\&\; b \in \operatorname{dom}(\varepsilon)$$
$$\to (\langle \{a, b\}, \varepsilon(b) \rangle \in s \vee (a = b \;\&\; \varepsilon(b) = 0))\big]\}$$

with the ordering relation

$$\langle s_1, q_1 \rangle \leqslant \langle s_2, q_2 \rangle \quad \text{iff} \quad s_1 \leqslant s_2 \text{ and } q_1 \leqslant q_2 .$$

We shall repeatedly use the following simple combinatorial

PROPOSITION 6. *If B is finite, C is countable and $h_\zeta \colon B \to C$ for $\zeta < \omega_1$, then there exists an uncountable subset K of ω_1 such that $h_\zeta = h_\xi$ for every $\zeta, \xi \in K$.*

LEMMA 7. *P_y is ccc.*

Proof. Let $\langle \langle s_\zeta, q_\zeta \rangle \colon \zeta < \omega_1 \rangle$ be a sequence of elements of P_y and let

$$q_\zeta(i) = \langle A_\zeta^i, \varepsilon_\zeta^i \rangle \quad \text{for each } i < n \text{ and } \zeta < \omega_1 .$$

We will show that there exist $\xi < \eta < \omega_1$ such that $\langle s_\xi, q_\xi \rangle$ and $\langle s_\eta, q_\eta \rangle$ are compatible.

Without limiting generality we may assume that for every $\zeta < \omega_1$

(1) $$\operatorname{dom}(s_\zeta) = [d_\zeta]^{\leqslant 2} \quad \text{for a certain finite } d_\zeta \subset \varkappa ,$$

(2) $$\bigcup_{i<n} \operatorname{dom}(\varepsilon_\zeta^i) \subset d_\zeta ,$$

(3) $$\bigcup_{i<n} A_\zeta^i \subset d_\zeta .$$

By the Δ-lemma we may assume that

(4) $$d_\zeta = a_\zeta \cup b \quad \text{for any } \zeta < \omega_1 ,$$

where

(5) $$a_\zeta \cap a_\eta = 0 \quad \text{for any } \zeta < \eta < \omega_1 .$$

By applying Proposition 6 to the functions $s_\zeta \restriction [b]^{\leqslant 2}$, we can assume that $s_\zeta \restriction [b]^{\leqslant 2} = s_\eta \restriction [b]^{\leqslant 2}$ for every $\zeta, \eta < \omega_1$. So

(6) $$s_\zeta \cup s_\eta \in S \quad \text{for every } \zeta, \eta < \omega_1 .$$

By applying Proposition 6 to the functions $h_\zeta \colon n \to \omega$ defined by $h_\zeta(i) = |A_\zeta^i|$ for $i < n$, we can assume that for any $i < n$ there exists a t_i such that $|A_\zeta^i| = t_i$ for any $\zeta < \omega_1$. Hence we may assume that

(7) there exist $t_0, \ldots, t_{n-1} \in \omega$ such that
$$A_\zeta^i = \{\alpha_{\zeta,i}^0, \ldots, \alpha_{\zeta,i}^{t_i - 1}\} \quad \text{for any } \zeta < \omega_1 \text{ and } i < n ,$$

and by the same argument

(8) there exist $r_0, \ldots, r_{n-1} \in \omega$ such that
$$|\operatorname{dom}(\varepsilon_\zeta^i)| = r_i \quad \text{and} \quad \operatorname{dom}(\varepsilon_\zeta^i) = \{\delta_{\zeta,i}^0, \ldots, \delta_{\zeta,i}^{r_i - 1}\} \quad \text{for any } \zeta < \omega_1 \text{ and } i < n .$$

By applying the same argument to the functions

$$h'_\zeta \colon \bigcup_{i<n} (\{i\} \times r_i) \to 2$$

defined by

$$h'_\zeta(i, j) = \varepsilon_\zeta^i(\delta_{\zeta,i}^j) \quad \text{for each } i < n \text{ and } j < t_i ,$$

we may assume that

(9) $$\varepsilon_\zeta^i(\delta_{\zeta,i}^j) = \varepsilon_\eta^i(\delta_{\eta,i}^j) \quad \text{for every } \zeta, \eta < \omega_1 .$$

The same argument applied to the functions

$$h''_\zeta \colon b \to P(n \times \bigcup_{i<n} t_i)$$

defined by

$$h''_\zeta(\alpha) = \{\langle i, j \rangle \colon \alpha = \alpha_{\zeta,i}^j\} \quad \text{for any } \alpha \in b$$

allows us to assume that

(10) for every $\alpha \in b$
$$\text{if} \quad \alpha = \alpha_{\zeta,i}^j \quad \text{then} \quad \alpha = \alpha_{\eta,i}^j \quad \text{for every } \eta < \omega_1$$

and similarly

(11) for every $\alpha \in b$
$$\text{if} \quad \alpha = \delta_{\zeta,i}^j \quad \text{then} \quad \alpha = \delta_{\eta,i}^j \quad \text{for every } \eta < \omega_1 .$$

Finally, by applying Proposition 6 to suitable functions we may assume that

(12) $$\text{if} \quad \delta_{\zeta,i}^j = \alpha_{\zeta,k}^l \quad \text{then} \quad \delta_{\eta,i}^j = \alpha_{\eta,k}^l \quad \text{for every } \eta < \omega_1 ,$$

and

(13) $\quad$ if $\alpha^k_{\zeta,i} = \alpha^m_{\zeta,j}$ then $\alpha^k_{\eta,i} = \alpha^m_{\eta,j}$ for every $\eta < \omega_1$.

After having done all these restrictions we show that any $\langle s_\zeta, q_\zeta\rangle$ and $\langle s_\eta, q_\eta\rangle$ are compatible.

Let $\zeta, \eta \in \omega_1$. Since $q_\zeta, q_\eta \in Q_y$, we have $q_\zeta(i), q_\eta(i) \in B_{\alpha_i}$ for $i<n$. Hence $A^i_\zeta \subset \alpha_i$ and $A^i_\eta \subset \alpha_i$, i.e. $A^i_\zeta \cup A^i_\eta \subset \alpha_i$. Moreover, $\operatorname{dom}(\varepsilon^i_\zeta) \cap \operatorname{dom}(\varepsilon^i_\eta) \subset b$. Hence, if $\alpha \in \operatorname{dom}(\varepsilon^i_\zeta) \cap \operatorname{dom}(\varepsilon^i_\eta)$ and $\alpha = \delta^j_{\eta,i}$ then (by (11)) $\alpha = \delta^j_{\zeta,i}$. So, by (9)

$$\varepsilon^i_\zeta(\alpha) = \varepsilon^i_\zeta(\delta^j_{\zeta,i}) = \varepsilon^i_\eta(\delta^j_{\eta,i}) = \varepsilon^i_\eta(\alpha),$$

i.e. $\varepsilon^i_\zeta \cup \varepsilon^i_\eta \in H(\varkappa)$.

Let us put for $i<n$

$$A^i = A^i_\zeta \cup A^i_\eta, \quad \varepsilon^i = \varepsilon^i_\zeta \cup \varepsilon^i_\eta, \quad q(i) = \langle A^i, \varepsilon^i\rangle .$$

We have $q \in Q_y$ and $q \leqslant q_\zeta$, $q \leqslant q_\eta$.

In order to complete the proof it suffices to show that there exists an $s \in S$ such that

(a) $$s_\zeta \cup s_\eta \subset s,$$

(b) $$\langle s, q\rangle \in P_y .$$

In order to show (b) it is enough to show

(c) $$\forall i<n\, \forall a \in A^i\, \forall b \in \operatorname{dom}(\varepsilon^i)\, [s(\{a, b\}) = \varepsilon^i(b)] .$$

If $a \in A^i_\zeta$ and $b \in \operatorname{dom}(\varepsilon^i_\zeta)$ then, by (2) and (3), $\{a, b\} \in \operatorname{dom}(s_\zeta)$. So, by (a)

$$s(\{a, b\}) = s_\zeta(\{a, b\}) = \varepsilon^i_\zeta(b) = \varepsilon^i(b) .$$

Similarly for $a \in A^i_\eta$ and $b \in \operatorname{dom}(\varepsilon^i_\eta)$.

So it is enough to find $s \in S$ such that

(i) $$s_\zeta \cup s_\eta \subset s,$$

(ii) $$\forall i<n\, \forall a \in A^i_\zeta\, \forall b \in \operatorname{dom}(\varepsilon^i_\eta)\, [s(\{a, b\}) = \varepsilon^i_\eta(b)],$$

(iii) $$\forall i<n\, \forall a \in A^i_\eta\, \forall b \in \operatorname{dom}(\varepsilon^i_\zeta)\, [s(\{a, b\}) = \varepsilon^i_\zeta(b)] .$$

Let us define the following functions for $i<n$

$$s'_i(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) = \varepsilon^i_\eta(\delta^k_{\eta,i}) \quad \text{for } j<t_i \text{ and } k<r_i,$$

$$s''_i(\{\alpha^j_{\eta,i}, \delta^k_{\zeta,i}\}) = \varepsilon^i_\zeta(\delta^k_{\zeta,i}) \quad \text{for } j<t_i \text{ and } k<r_i .$$

Let

$$s = s_\zeta \cup s_\eta \cup \bigcup_{i<n} (s'_i \cup s''_i) .$$

Since clearly s satisfies (i), (ii) and (iii), it suffices to show that $s \in S$.

I. $s_\zeta \cup s_\eta \in S$ (by (6)).

II. $s'_i \in S$ for any $i<n$.

Let $\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\} = \{\alpha^l_{\zeta,i}, \delta^m_{\eta,i}\}$. If $\alpha^j_{\zeta,i} = \alpha^l_{\zeta,i}$ and $\delta^k_{\eta,i} = \delta^m_{\eta,i}$ then

$$s'_i(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) = \varepsilon^i_\eta(\delta^k_{\eta,i}) = \varepsilon^i_\eta(\delta^m_{\eta,i}) = s'_i(\{\alpha^l_{\zeta,i}, \delta^m_{\eta,i}\}) .$$

If $\alpha^j_{\zeta,i} = \delta^m_{\eta,i}$ and $\alpha^l_{\zeta,i} = \delta^k_{\eta,i}$ then $\alpha^l_{\zeta,i} \in d_\zeta \cap d_\eta = b$. Hence, by [10], $\alpha^l_{\zeta,i} = \alpha^l_{\eta,i}$. So $\alpha^l_{\eta,i} = \delta^k_{\eta,i}$ and

$$s'_i(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) = \varepsilon^i_\eta(\delta^k_{\eta,i}) = s_\eta(\{\alpha^l_{\eta,i}, \delta^k_{\eta,i}\}) = s_\eta(\{\delta^k_{\eta,i}, \delta^k_{\eta,i}\}) = 0 .$$

Similarly we show that $s'_i(\{\alpha^l_{\zeta,i}, \delta^m_{\eta,i}\}) = 0$, i.e. s'_i is a function.

Moreover if $\alpha^j_{\zeta,i} = \delta^k_{\eta,i}$ then we also have $s'_i(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) = 0$, i.e. $s'_i \in S$.

III. $s''_i \in S$ for any $i<n$.

The proof is similar.

IV. $s'_i \cup s_\zeta \cup s_\eta$ is a function for any $i<n$.

Let $\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\} \in \operatorname{dom}(s_\zeta \cup s_\eta)$. If $\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\} \in \operatorname{dom}(s_\zeta)$ then $\delta^k_{\eta,i} \in b$ and hence, by (11), $\delta^k_{\eta,i} = \delta^k_{\zeta,i}$. So, by (9),

$$s'_i(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) = \varepsilon^i_\eta(\delta^k_{\eta,i}) = \varepsilon^i_\zeta(\delta^k_{\zeta,i}) = s_\zeta(\{\alpha^j_{\zeta,i}, \delta^k_{\zeta,i}\}) = (s_\zeta \cup s_\eta)(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) .$$

If $\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\} \in \operatorname{dom}(s_\eta)$ then $\alpha^j_{\zeta,i} \in b$ and, by (10), $\alpha^j_{\eta,i} = \alpha^j_{\eta,t}$. So

$$s'_i(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) = \varepsilon^i_\eta(\delta^k_{\eta,i}) = s_\eta(\{\alpha^j_{\eta,i}, \delta^k_{\eta,i}\}) = (s_\zeta \cup s_\eta)(\{\alpha^j_{\zeta,i}, \delta^k_{\eta,i}\}) .$$

V. $s''_i \cup s_\zeta \cup s_\eta$ is a function for any $i<n$.

The proof is similar.

VI. $s'_i \cup s'_j$ is a function for any $i, j<n$.

Let $\{\alpha^k_{\zeta,i}, \delta^l_{\eta,i}\} = \{\alpha^m_{\zeta,j}, \delta^s_{\eta,j}\}$. If $\alpha^k_{\zeta,i} = \alpha^m_{\zeta,j}$ and $\delta^l_{\eta,i} = \delta^s_{\eta,j}$ then, by (13), $\alpha^k_{\eta,i} = \alpha^m_{\eta,j}$ and $s'_i(\{\alpha^k_{\zeta,i}, \delta^l_{\eta,i}\}) = \varepsilon^i_\eta(\delta^l_{\eta,i}) = s_\eta(\{\alpha^k_{\eta,i}, \delta^l_{\eta,i}\}) = s_\eta(\{\alpha^m_{\eta,j}, \delta^s_{\eta,j}\}) = \varepsilon^j_\eta(\delta^s_{\eta,j}) = s'_j(\{\alpha^m_{\zeta,j}, \delta^s_{\eta,j}\})$.
If $\alpha^k_{\zeta,i} = \delta^s_{\eta,j}$ and $\alpha^m_{\zeta,j} = \delta^l_{\eta,i}$ then $\alpha^k_{\zeta,i}, \alpha^m_{\zeta,j} \in b$ and by (10)

$$\alpha^k_{\eta,i} = \alpha^k_{\zeta,i} = \delta^s_{\eta,j} \quad \text{and} \quad \delta^l_{\eta,i} = \alpha^m_{\zeta,j} = \alpha^m_{\eta,j} .$$

Hence

$$s'_i(\{\alpha^k_{\zeta,i}, \delta^l_{\eta,i}\}) = \varepsilon^i_\eta(\delta^l_{\eta,i}) = s_\eta(\{\alpha^k_{\eta,i}, \delta^l_{\eta,i}\}) = s_\eta(\{\delta^s_{\eta,j}, \alpha^m_{\eta,j}\}) = \varepsilon^j_\eta(\delta^s_{\eta,j}) = s'_j(\{\alpha^m_{\zeta,j}, \delta^s_{\eta,j}\}) .$$

VII. $s''_i \cup s''_j$ is a function for any $i, j<n$.

The proof is similar.

VIII. $s'_i \cup s''_j$ is a function for any $i, j<n$.

Let $\{\alpha^k_{\zeta,i}, \delta^l_{\eta,i}\} = \{\alpha^m_{\eta,j}, \delta^s_{\zeta,j}\}$. If $\alpha^k_{\zeta,i} = \alpha^m_{\eta,j}$ and $\delta^l_{\eta,i} = \delta^s_{\zeta,j}$ then $\alpha^m_{\eta,j}, \delta^l_{\eta,i} \in b$. So, by (10) and (11)

$$\alpha^k_{\zeta,i} = \alpha^m_{\eta,j} = \alpha^m_{\zeta,j} \quad \text{and} \quad \delta^l_{\zeta,i} = \delta^l_{\eta,i} = \delta^s_{\zeta,j} .$$

Hence, by (9)

$$s'_i(\{\alpha^k_{\zeta,i}, \delta^l_{\eta,i}\}) = \varepsilon^i_\eta(\delta^l_{\eta,i}) = \varepsilon^i_\zeta(\delta^l_{\zeta,i}) = s_\zeta(\{\alpha^k_{\zeta,i}, \delta^l_{\zeta,i}\}) = s_\zeta(\{\alpha^m_{\zeta,j}, \delta^s_{\zeta,j}\})$$
$$= \varepsilon^j_\zeta(\delta^s_{\zeta,j}) = s''_j(\{\alpha^m_{\eta,j}, \delta^s_{\zeta,j}\}) .$$

If $\alpha^k_{\zeta,i} = \delta^s_{\zeta,j}$ and $\alpha^m_{\eta,j} = \delta^l_{\eta,i}$ then, by (12), $\alpha^m_{\zeta,j} = \delta^l_{\zeta,i}$. Hence, by (9) $s'_i(\{\alpha^k_{\zeta,i}, \delta^l_{\eta,i}\}) = \varepsilon^i_\eta(\delta^l_{\eta,i}) = \varepsilon^i_\zeta(\delta^l_{\zeta,i}) = s_\zeta(\{\alpha^k_{\zeta,i}, \delta^l_{\zeta,i}\}) = s_\zeta(\{\delta^s_{\zeta,j}, \alpha^m_{\zeta,j}\}) = \varepsilon^j_\zeta(\delta^s_{\zeta,j}) = s''_j(\{\alpha^m_{\eta,j}, \delta^s_{\zeta,j}\})$.

It is clear that conditions I–VIII give $s \in S$ and our proof of Lemma 7 is complete.

Now we prove Theorem 2. Let $\langle\langle s_\zeta, q_\zeta\rangle: \zeta<\omega_1\rangle$ be a sequence of elements of P. We show that there exist $\zeta<\eta<\omega_1$ such that $\langle s_\zeta, q_\zeta\rangle$ and $\langle s_\eta, q_\eta\rangle$ are compatible. By the Δ-lemma we may assume that

$$\operatorname{supp}(q_\zeta) = y \cup w_\zeta \quad \text{for every } \zeta<\omega_1$$

where

$$w_\zeta \cap w_\eta = 0 \quad \text{for every } \zeta<\eta<\omega_1 .$$

Let $P'_y = \{\langle s, q\restriction y\rangle: \langle s, q\rangle \in P\}$ be a poset with the ordering relation

$$\langle s, q\rangle \leqslant \langle s', q'\rangle \quad \text{iff} \quad s \supset s' \,\&\, \forall\langle\alpha, n\rangle \in y\,[q(\alpha, n) \leqslant q'(\alpha, n)] .$$

Clearly P_y and P'_y are isomorphic.

Let us consider a set $\{\langle s_\zeta, q_\zeta\restriction y\rangle: \zeta<\omega_1\}$ of elements of P'_y. By Lemma 7 there exist $\zeta<\eta<\omega_1$ and $\langle s, q\rangle \in P'_y$ such that $\langle s, q\rangle \leqslant \langle s_\zeta, q_\zeta\restriction y\rangle$ and $\langle s, q\rangle \leqslant s_\eta, q_\eta\restriction y\rangle$.

Let $q' \in Q$ be defined by

$$q'(\alpha, n) = \begin{cases} q(\alpha, n) & \text{for } \langle\alpha, n\rangle \in y , \\ q_\zeta(\alpha, n) & \text{for } \langle\alpha, n\rangle \in w_\zeta , \\ q_\eta(\alpha, n) & \text{for } \langle\alpha, n\rangle \in w_\eta , \\ \langle 0, 0\rangle & \text{otherwise.} \end{cases}$$

It is easy to see that $\langle s, q'\rangle \in P$ and $\langle s, q'\rangle \leqslant \langle s_\zeta, q_\zeta\rangle$ and $\langle s, q'\rangle \leqslant \langle s_\eta, q_\eta\rangle$.

This completes the proof.

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Topological games and products, II

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Abstract. The purpose of this paper is to study the topological games (in the sense of R. Telgársky) of product spaces: Assume that Player *I* has winning strategies in the given topological games of X and Y. Then we consider the conditions of a product space $X\times Y$ under which he has a winning strategy in a certain topological game of $X\times Y$. Moreover, we can apply the results obtained from this kind of argument to the product theorem in dimension theory.

Introduction. R. Telgársky [14] introduced and studied the topological game $G(\boldsymbol{K}, X)$. In our previous paper [19], we have used it to study the covering properties of product spaces. In the present paper, we shall study the topological game on product spaces. If the above $\boldsymbol{K}$ is the class of all one-point spaces, then the game $G(\boldsymbol{K}, X)$ is often abbreviated by $G(X)$, which is called the point-open game. R. Telgársky [15] stated the following: If Player *I* has winning strategies in $G(X)$ and $G(Y)$, then he has a winning strategy in $G(X\times Y)$. This gives the positive answer to [14, Question 14.1]. In this connection, we raise the following natural question: Assume that Player *I* has winning strategies in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, Y)$. What is a topological game of $X\times Y$ which is interesting to investigate? What is a condition on $X\times Y$ under which he has a winning strategy in such a game? In § 2 and § 3, we discuss this question. In § 4, using the result of § 2, we give a product theorem in dimension theory.

Each space considered here is assumed to be a Hausdorff space. N denotes the set of all natural numbers and $\mathfrak{m}$ denotes an infinite cardinal number. For a space or a set X, by $\chi(X)$ we mean the character of X and by $|X|$ the cardinality of X. For a collection $\mathfrak{F}$ of subsets of X, $\bigcup\mathfrak{F}$ denotes $\bigcup\{F: F\in\mathfrak{F}\}$.

§ 1. Topological games. R. Telgársky [15] has introduced an equivalent form of the game $G(\boldsymbol{K}, X)$ defined in [14]. The new form of the game we use below.

Let $\boldsymbol{L}$ be a class of spaces and let X be a space. We define the topological game $G(\boldsymbol{L}, X)$ as follows: There are two players; Player *I* and Player *II*. Player *I* chooses a closed set E_1 of X with $E_1 \in \boldsymbol{L}$, and after that Player *II* chooses an open set U_1 of X with $E_1 \subset U_1$. Again Player *I* chooses a closed set E_2 of X with $E_2 \in \boldsymbol{L}$ and Player *II* chooses an open set U_2 of X with $E_2 \subset U_2$, and so on. Here, the infinite