BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences mathématiques
Vol. XXVIII, No. 5—6, 1980

Generalized Threshold Logic

by

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Presented by C. RYLL-NARDZEWSKI on March 10, 1979

Summary. A threshold logic formulated in the paper is a generalization of threshold logic with weights and thresholds, which are integers, to the case when weights and thresholds are elements of an abstract group having the relation of linear ordering, which is in agreement with the rules of this grup. A Gentzen-style axiomatization of this logic is presented and it is proved that this axiomatization is a generalization of the axiomatizations presented in [1] and [2]. The condition necessary and sufficient to state the fact that both of these axiomatizations are identical is also given.

1. The first-order formalized language of the threshold logic and its realization. Let G be the set of elements of the nonzero commutative group $(G, +, 0)$ with the relation of linear ordering fulfilling conditions:

(1g) if $a<b$ and $c \leqslant d$, then $a+c<b+d$,

(2g) if $a<b$, then $-a>-b$,
for every $a, b, c, d \in G$.

Since G is non-zero and by (2g) we can distinguish an element $g_0 \in G$ such that $g_0>0$. Let

(3g) $$1=\begin{cases}\min\{g \in G\colon g>0\} & \text{if this minimum exists}\\ g_0 & \text{otherwise.}\end{cases}$$

By definition we have:

(4g) $1>0$.

The denotation of the element defined in (3g) by the symbol 1 has a deeper sense. If we assume that G is the additive group of the ordered field, we can assume that g_0 is the unitary element of this field. In this case 1 defined in (3g) is also a unitary element of this field.

By the first-order formalized language $\mathscr{L}_G$ of the threshold logic we will understand the triple $\langle A, T, F\rangle$. An alphabet

$$A=V \cup \bigcup_{m=0}^{\infty} \Phi_m \cup \bigcup_{m=1}^{\infty} P_m \cup G \cup \{\forall, \exists\} \cup \{(,),'\},$$

where

$V=\{x, x_1, x_2, \dots\}$ is a set of the individual variables,

Φ_m—a set of m-ary functors, $m=0, 1, 2, \dots$,

$P_m=\{\rho, \rho_1, \rho_2, \dots\}$—a set of m-ary predicates, $m=1, 2, 3, \dots$,

G—the set of elements of the above-defined group,

$\{\forall, \exists\}$ the set of quantifiers,

$\{(,),'\}$—the set of auxiliary signs.

The set T of terms is defined in the usual way. We will assume that T is enumerable.

The set F of formulas is the least set satisfying the following conditions:

if $\rho \in P_m$ and $\tau_1, \dots, \tau_m \in T$, then $\rho(\tau_1, \dots, \tau_m) \in F$, for $m=1, 2, \dots$,

if $a_1, \dots, a_m \in F$ and $n_1, \dots, n_m \in G \setminus \{0\}$ and $t \in G$, then $(n_1, \dots, n_m, t)(a_1, \dots, a_m) \in F$, for $m=1, 2, \dots$,

if $a(x) \in F$, then $\forall x a(x) \in F$ and $\exists x a(x) \in F$, where x is a free individual variable appearing in a.

The elements $n_1, \dots, n_m$ are called the weights of the formulas $a_1, \dots, a_m$, respectively, in the formula $(n_1, \dots, n_m, t)(a_1, \dots, a_m)$. The element t is called its threshold.

Let R be a realization of the language $\mathscr{L}_G$ in a non-empty set J and a generalized complete Boolean algebra $\mathscr{B}=\{\{\mathbf{0}, \mathbf{1}\}, \bigcup, \bigcap\}$, where $\bigcup$ and $\bigcap$ are the symbols of infinite meet and join, respectively.

For every term τ the mapping $\tau_R: J^V \dashrightarrow J$ is defined in the usual way.

If a is a formula, then the mapping $a_R: J^V \dashrightarrow \{\mathbf{0}, \mathbf{1}\}$ is defined as follows:

(1R) $$(\rho(\tau_1, \dots, \tau_m))_R(v)=\rho_R(\tau_{1R}(v), \dots, \tau_{mR}(v)),$$

(2R) $$((n_1, \dots, n_m, t)(a_1, \dots, a_m))_R(v)=\begin{cases} \mathbf{1} & \text{if } \sum_{i=1}^{m} f_{R,v}(n_i, a_i)<t \\ \mathbf{0} & \text{if } \sum_{i=1}^{m} f_{R,v}(n_i, a_i)\geqslant t, \end{cases}$$

where $f_{R,v}: G \times F \dashrightarrow G$ is the following function:

$$f_{R,v}(n, a)=\begin{cases} n & \text{if } a_R(v)=\mathbf{1} \\ 0 & \text{if } a_R(v)=\mathbf{0}, \end{cases}$$

(3R) $$(\exists x a(x))_R(v)=\bigcup_{j\in J} a_R(\omega_j),$$

(4R) $$(\forall x a(x))_R(v)=\bigcap_{j\in J} a_R(\omega_j),$$

where $\omega_j: V \dashrightarrow J$ is the valuation of individual variables in the set J defined as follows:

$$\omega_j(z)=\begin{cases} v(z) & \text{if } z\neq x \\ j & \text{if } z=x, \end{cases}$$

for every valuation $v: V \dashrightarrow J$.

If G is the field described previously, then the elements $\mathbf{0}, \mathbf{1}$ of the algebra $\mathscr{B}$ can be treated as $0, 1 \in G$, respectively. Then the function $f_{R,v}$ has the following interpretation:

$$f_{R,v}(n, a) = n \cdot a_R(v).$$

A formula a is said to be a tautology of the language $\mathscr{L}_G$ if $a_R(v) = \mathbf{1}$ for every realization R and every valuation v.

2. The schemas of rules. A formula a is said to be indecomposable if it is of the form $\rho(\tau_1, \ldots, \tau_m)$ or $(1,1)(\rho(\tau_1, \ldots, \tau_m))$.

Let the letter Γ (with indices, if necessary) denote a finite sequence

$$a_1, a_2, \ldots, a_m$$

of formulas in $\mathscr{L}_G$. The empty sequence is also being examined.

A sequence Γ composed only of indecomposable formulas or the empty sequence is said to be indecomposable. A sequence Γ is said to be fundamental if it contains simultaneously a formula a and $(1,1)(a)$.

Let us define the extension of realization R onto the set of all finite sequences of formulas in the following way:

(5R) if $\Gamma = a_1, \ldots, a_m$, then

$$\Gamma_R(v) = \begin{cases} \mathbf{0} & \text{if } a_{iR}(v) = \mathbf{0} \text{ for every } i = 1, 2, \ldots, m \\ \mathbf{1} & \text{otherwise} \end{cases}$$

(6R) if Γ is an empty sequence, then $\Gamma_R(v) = \mathbf{0}$ for every valuation $v: V \dashrightarrow J$.

By a schema of rule we mean the expression of the form:

$$\frac{\Gamma}{\Gamma_1; \ldots; \Gamma_r} \quad \text{where } r = 1, 2, 3, \ldots$$

satisfying the following condition

(1s) $\Gamma_R(v) = \mathbf{1}$ iff $\Gamma_{iR}(v) = \mathbf{1}$ for every $i = 1, 2, \ldots, r$ for every realization R and every valuation v.

Γ is called the conclusion of the schema of rule, $\Gamma_1, \ldots, \Gamma_r$ are called its premises.

We will use the following schemas of rules, where Γ' always denotes an indecomposable sequence:

(S1) $$\frac{\Gamma', (n_1, \ldots, n_m, t)(a_1, \ldots, a_m), \Gamma''}{\Gamma', \Gamma_1, \Gamma''; \Gamma', \Gamma_2, \Gamma''} \quad \text{for } m > 1 \text{ and } n_m < 0,$$

where

$$\Gamma_1 = a_m, (n_1, \ldots, n_{m-1}, t)(a_1, \ldots, a_{m-1}),$$
$$\Gamma_2 = (n_1, \ldots, n_{m-1}, t - n_m)(a_1, \ldots, a_{m-1}), (n_1, \ldots, n_{m-1}, t)(a_1, \ldots, a_{m-1}),$$

(S2) $$\frac{\Gamma', (n_1, \ldots, n_m, t)(a_1, \ldots, a_m), \Gamma''}{\Gamma', \Gamma_1, \Gamma''; \Gamma', \Gamma_2, \Gamma''} \quad \text{for } m > 1 \text{ and } n_m > 0,$$

where

$$\Gamma_1 = (n_1, \ldots, n_{m-1}, t - n_m)(a_1, \ldots, a_{m-1}), (1, 1)(a_m),$$

$$\Gamma_2 = (n_1, \ldots, n_{m-1}, t - n_m)(a_1, \ldots, a_{m-1}), (n_1, \ldots, n_{m-1}, t)(a_1, \ldots, a_{m-1}),$$

(S3) $$\frac{\Gamma', (n, t)(a), \Gamma''}{\Gamma', a_0, (1, 1)(a_0), \Gamma''} \quad \text{for } n < t \text{ and } t > 0,$$

where a_0 is an arbitrary fixed elementary formula in $\mathscr{L}_G$,

(S4) $$\frac{\Gamma', (n, t)(a), \Gamma''}{\Gamma', a, \Gamma''} \quad \text{for } n < t \text{ and } t \leqslant 0,$$

(S5) $$\frac{\Gamma', (n, t)(a), \Gamma''}{\Gamma', \Gamma''} \quad \text{for } n \geqslant t \text{ and } t \leqslant 0,$$

(S6) $$\frac{\Gamma', (n, t)(a), \Gamma''}{\Gamma', (1, 1)(a), \Gamma''} \quad \text{for } n \geqslant t \text{ and } t > 0 \text{ and } \neg(n = t = 1),$$

(S7) $$\frac{\Gamma', (1, 1)\big((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)\big), \Gamma''}{\Gamma', (-n_1, \ldots, -n_m, \varepsilon(n_1, \ldots, n_m, t) - t)(a_1, \ldots, a_m), \Gamma''},$$

where $\varepsilon(n_1, \ldots, n_m, t) = \min\big(\{1\} \cup S(n_1, \ldots, n_m, t)\big)$ and

$$S(n_1, \ldots, n_m, t) = \Big\{g \in G : g > 0 \text{ and } g = t - \sum_{i=1}^{m} \chi_Q(i) \text{ for } Q \subset \{1, \ldots, m\}\Big\},$$

where

$$\chi_Q(i) = \begin{cases} n_i & \text{for } i \in Q \\ 0 & \text{for } i \notin Q \end{cases} \quad \text{for every } Q \subset \{1, \ldots, m\},$$

(S8) $$\frac{\Gamma', (1, 1)\big(\forall x a(x)\big), \Gamma''}{\Gamma', \exists x (1, 1)\big(a(x)\big), \Gamma''},$$

(S9) $$\frac{\Gamma', (1, 1)\big(\exists x a(x)\big), \Gamma''}{\Gamma', \forall x (1, 1)\big(a(x)\big), \Gamma''},$$

(S10) $$\frac{\Gamma', \forall x a(x), \Gamma''}{\Gamma', a(y), \Gamma''},$$

where y is a free individual variable which does not appear in any formula in the conclusion.

(S11) $$\frac{\Gamma', \exists x a(x), \Gamma''}{\Gamma', a(\tau), \Gamma'', \exists x a(x)},$$

where τ is a term.

LEMMA 1. *The following conditions are equivalent for every* $r=1, 2, 3, \ldots$

(1) $$\frac{\Gamma', \Gamma, \Gamma''}{\Gamma', \Gamma_1, \Gamma''; \ldots; \Gamma', \Gamma_r, \Gamma''}$$ *is a schema of rule (i.e* (1s) *holds)*

for every sequence Γ' *and* Γ'',

(2) $$\frac{\Gamma}{\Gamma_1; \ldots; \Gamma_r}$$ *is a schema of rule.*

Proof. If (1) holds and Γ', Γ'' are the empty sequences, then we obtain condition (2).

Let us assume that (2) holds and let $(\Gamma', \Gamma, \Gamma'')_R(v)=\mathbf{1}$. Then, according to (5R), at least one of the conditions: (a) $\Gamma'_R(v)=\mathbf{1}$, (b) $\Gamma_R(v)=\mathbf{1}$, (c) $\Gamma''_R(v)=\mathbf{1}$ is satisfied. If cases (a) or (c) occur, then $(\Gamma', \Gamma_i, \Gamma'')_R(v)=\mathbf{1}$ for $i=1, \ldots, r$. If (b) holds, then, according to (2) and (1s), $\Gamma_{iR}(v)=\mathbf{1}$ and hence $(\Gamma', \Gamma_i, \Gamma'')_R(v)=\mathbf{1}$ for $i=1, \ldots, r$.

Otherwise, if $(\Gamma', \Gamma, \Gamma'')_R(v)=\mathbf{0}$, then $\Gamma'_R(v)=\Gamma_R(v)=\Gamma''_R(v)=\mathbf{0}$. From (2) and from the fact that $\Gamma_R(v)=\mathbf{0}$ it follows that $\Gamma_{iR}(v)=\mathbf{1}$ is not satisfied for every $i=1, \ldots, r$, i.e. there is an index i from the set $\{1, \ldots, r\}$ such that $\Gamma_{iR}(v)=\mathbf{0}$. Hence $(\Gamma', \Gamma_i, \Gamma'')_R(v)=\mathbf{0}$.

LEMMA 2. $((1, 1)(a))_R(v)=\mathbf{1}$ iff $a_R(v)=\mathbf{0}$
for every: formula a, *realization* R *and valuation* v.

Proof. From (2R) it follows

$$((1, 1)(a))_R(v)=\mathbf{1} \text{ iff } f_{R,v}(1, a)<1 \text{ iff } a_R(v)=\mathbf{0}.$$

THEOREM 1. *Expressions* (S1)–(S11) *are the schemas of rules, i.e. they satisfy condition* (1s).

Proof. From Lemma 1 it follows that it suffices to prove this theorem in the case, when the sequences Γ' and Γ'' are empty.

(S1). Let $((n_1, \ldots, n_m, t)(a_1, \ldots, a_m))_R(v)=\mathbf{1}$. It follows from (2R) that $\sum_{i=1}^{m} f_{R,v}(n_i, a_i)<t$. If $a_{mR}(v)=\mathbf{0}$, then $f_{R,v}(n_m, a_m)=0$ and $\sum_{i=1}^{m-1} f_{R,v}(n_i, a_i)= \sum_{i=1}^{m} f_{R,v}(n_i, a_i)<t$, i.e. $((n_1, \ldots, n_{m-1}, t)(a_1, \ldots, a_{m-1}))_R(v)=\mathbf{1}$. Hence $\Gamma_{1R}(v)= \Gamma_{2R}(v)=\mathbf{1}$.

If $a_{mR}(v)=\mathbf{1}$, then $\Gamma_{1R}(v)=\mathbf{1}$ and $f_{R,v}(n_m, a_m)=n_m$. Hence by (1g) we obtain $\sum_{i=1}^{m-1} f_{R,v}(n_i, a_i)<t-n_m$ and by (2R) we have $((n_1, \ldots, n_{m-1}, t-n_m)(a_1, \ldots, a_{m-1}))_R(v)= =\mathbf{1}$. Hence $\Gamma_{1R}(v)=\Gamma_{2R}(v)=\mathbf{1}$.

Let us now suppose that $((n_1, \ldots, n_m, t)(a_1, \ldots, a_m))_R(v)=\mathbf{0}$. It means that $\sum_{i=1}^{m} f_{R,v}(n_i, a_i)\geqslant t$. If $a_{mR}(v)=\mathbf{0}$, then $f_{R,v}(n_m, a_m)=0$ and $\sum_{i=1}^{m-1} f_{R,v}(n_i, a_i)= \sum_{i=1}^{m} f_{R,v}(n_i, a_i)\geqslant t$. Hence $((n_1, \ldots, n_{m-1}, t)(a_1, \ldots, a_{m-1}))_R(v)=\mathbf{0}=a_{mR}(v)$, i.e. $\Gamma_{1R}(v)=\mathbf{0}$.

If $a_{mR}(v)=\mathbf{1}$, then $f_{R,v}(n_m, a_m)=n_m<0$ and from (1g) and (2g) we have $\sum_{i=1}^{m-1} f_{R,v}(n_i, a_i)\geqslant t-n_m>t$. Hence $((n_1, \ldots, n_{m-1}, t-n_m)\ (a_1, \ldots, a_{m-1}))_R(v)=\mathbf{0}=((n_1, \ldots, n_{m-1}, t)\ (a_1, \ldots, a_{m-1}))_R(v)$ and $\Gamma_{2R}(v)=\mathbf{0}$.

(S2). The proof is analogous to the proof for the schema of rules (S1).

(S3). To prove condition (1s) it suffices to show that $((n, t)\ (a))_R(v)=(a_0, (1, 1)\ (a_0))_R(v)$ for $n<t$, $t>0$ and for every formula a_0, realization R and valuation v of the language $\mathscr{L}_G$.

It follows from Lemma 2 and from (5R) that for every a_0, R and v we have $(a_0, (1, 1)\ (a_0))_R(v)=\mathbf{1}$.

On the other hand, if $a_R(v)=\mathbf{0}$, then $f_{R,v}(n, a)=0<t$, i.e. $((n, t)\ (a))_R(v)=\mathbf{1}$. If $a_R(v)=\mathbf{1}$, then $f_{R,v}(n, a)=n<t$, i.e. also $((n, t)\ (a))_R(v)=\mathbf{1}$. Hence $((n, t)\ (a))_R(v)=\mathbf{1}=(a_0, (1, 1)\ (a_0))_R(v)$ for every a_0, R and v.

(S4)–(S6). The proofs are analogous to the proof for (S3).

(S7). For simplicity in the following proof we will write ε instead of $\varepsilon(n_1, \ldots, n_m, t)$ and S instead $S(n_1, \ldots, n_m, t)$.

Let us observe that the following facts are true:

(1) ε is well defined, because S is finite and nonempty,

(2) $\varepsilon>0$.

It suffices to show that for every realization R and every valuation v

$$\big((1, 1)\ ((n_1, \ldots, n_m, t)\ (a_1, \ldots, a_m))\big)_R(v)=\mathbf{1}$$
$$\text{iff } ((-n_1, \ldots, -n_m, \varepsilon-t)\ (a_1, \ldots, a_m))_R(v)=\mathbf{1}.$$

Let us observe that the following conditions are equivalent:

(i) $\big((1, 1)\ ((n_1, \ldots, n_m, t)\ (a_1, \ldots, a_m))\big)_R(v)=\mathbf{1}$,

(ii) $((n_1, \ldots, n_m, t)\ (a_1, \ldots, a_m))_R(v)=\mathbf{0}$,

(iii) $\sum_{i=1}^{m} f_{R,v}(n_i, a_i)\geqslant t$,

(iv) $t-\sum_{i=1}^{m} f_{R,v}(n_i, a_i)\leqslant 0$.

The equivalences follow from Lemma 2, (2R) and (1g), respectively.

Likewise the following conditions are equivalent:

(v) $((-n_1, \ldots, -n_m, \varepsilon-t)\ (a_1, \ldots, a_m))_R(v)=\mathbf{1}$,

(vi) $\sum_{i=1}^{m} f_{R,v}(-n_i, a_i)<\varepsilon-t$,

(vii) $t+\sum_{i=1}^{m} f_{R,v}(-n_i, a_i)<\varepsilon$,

(viii) $t-\sum_{i=1}^{m} f_{R,v}(n_i, a_i)<\varepsilon$.

The equivalences follow from (2R), (1g) and from the fact that $f_{R,v}(-n, a) = -f_{R,v}(n, a)$, respectively. To finish the proof it suffices to prove the equivalence of conditions (iv) and (viii).

If (iv) holds, then from (2) and (1g) we have (viii).

Let us assume that (iv) does not hold, i.e. that

$t - \sum_{i=1}^{m} f_{R,v}(n_i, a_i) > 0$. Hence, from the definition of S we have

$t - \sum_{i=1}^{m} f_{R,v}(n_i, a_i) \in S \subset \{1\} \cup S$, i.e., from the definition of ε,

$t - \sum_{i=1}^{m} f_{R,v}(n_i, a_i) \geqslant \varepsilon$, which contradicts (viii).

(S8)–(S11). These schemas of rules are exactly of the form of the respective schemas of rules for the classical logic, because it follows from Lemma 2 that the connective (1, 1) plays the same role as the classical connective of negation.

3. Completeness theorem. Let D be the set of all finite sequences of integers 0 and 1. Its elements are denoted by i, j, k. The empty sequence is admitted and will be denoted by $\mathbf{o}$.

We shall write $i \leqslant j$ if **i** is an initial (proper or improper) segment of j. By definition $\mathbf{o} \leqslant j$ for every j.

Let

$$\tau_1, \tau_2, \tau_3, \ldots \tag{1t}$$

be a fixed infinite sequence containing every term in $\mathscr{L}_G$ exactly once.

Let us fix an elementary formula appearing in (S3). By the diagram of a formula a_0 in $\mathscr{L}_G$ we shall mean any mapping which associates certain finite sequences Γ_i of formulas with some finite sequences **i**, and which is defined by induction as follows:

1) Γ_0 is the sequence formed only of the formula a_0.

2) If Γ_i is defined but is either fundamental or indecomposable, then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

3) If Γ_i is defined and is neither indecomposable nor fundamental, then Γ_i is the conclusion of exactly one of the schemas of rules (S1)–(S11). If Γ_i is the conclusion of schema of rule (S1) or of (S2), then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are premises $\Gamma', \Gamma_1, \Gamma''$ and $\Gamma', \Gamma_2, \Gamma''$ of this schema of rule, respectively. If Γ_i is the conclusion of one of schemas of rules (S3)–(S11), then $\Gamma_{i,0}$ is the only premise of this schema of rule and $\Gamma_{i,1}$ is not defined. Moreover, if this schema of rule is (S10), we assume additionally that the variable y mentioned in (S10) is the first individual variable in sequence (1t) such that y does not appear in any formula in Γ_i. If this schema of rule is (S11), we assume additionally that the term τ mentioned in (S11) is the first term in the sequence (1t) such that $a(\tau)$ does not appear in any sequence Γ_j with $\mathbf{j} \leqslant \mathbf{i}$.

4) If Γ_i is not defined, then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

The diagram $\{\Gamma_i\}$ of a_0 is uniquely determined by a_0. The diagram $\{\Gamma_i\}$ is said to be finite (infinite) if the set of all sequences i for which Γ_i is defined is finite (infinite).

Γ_i is said to be an end sequence of the diagram of a_0 if Γ_i is either fundamental or indecomposable.

THEOREM 2. *A formula a_0 is a tautology of the language $\mathscr{L}_G$ if and only if the diagram of a_0 is finite and all the end sequences are fundamental.*

The proof is the same as in [3], pp. 303, 304.

4. **Discussion of schema of rule** (S7). Schema of rule (S7) is very inconvenient to use because of the numerical way of defining the magnitude of $\varepsilon(n_1, \ldots, n_m, t)$. Of course, a schema of rule in which all the group elements occurring in the premises would be obtained by simple rules of arithmetic from $n_1, \ldots, n_m, t$ would be much more helpful. But, generally, such a result cannot be obtained. To prove this, we must introduce some notions.

Let B be the class of sets $\{(g_1, g_2)\}$ where $g_1, g_2 \in G$, $g_1 < g_2$ and by definition $(g_1, g_2) = \{g \in G\colon g_1 < g < g_2\}$ and let $\mathfrak{O}$ be a topology conformable with the ordering relation in the set G given by the basis B.

An element $g \in G$ is said to be an accumulation point if

(1a) $(g_1, g_2) \setminus \{g\} \neq \emptyset$ for every (g_1, g_2) such that $g \in (g_1, g_2)$.

From this definition we have

(2a) g is an accumulation point iff $\{g\} \notin \mathfrak{O}$.

THEOREM 3. *If the space $(G, \mathfrak{O})$ contains an accumulation point g_0, then a schema as follows is not a schema of rule*:

(1w)
$$\frac{\Gamma', (1,1)\big((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)\big), \Gamma''}{\Gamma', \Gamma_1, \Gamma''; \ldots; \Gamma', \Gamma_r, \Gamma''},$$

$$\text{where } \Gamma_i = b^{i,1}, \ldots, b^{i,s_i} \quad \text{for } i = 1, \ldots, r,$$

$$b^{i,j} = \big(f_1^{i,j}(p), \ldots, f_{m+1}^{i,j}(p)\big)(a_1, \ldots, a_m) \quad \text{for } j = 1, \ldots, s_i,$$

$$p = (n_1, \ldots, n_m, t) \in G^{m+1},$$

and all the mappings $f_k^{i,j}$ are continuous in the space $(G, \mathfrak{O})$.

Proof. Let us assume that schema (1w) satisfies condition (1s) and consider a formula a, a realization R and a valuation v such that

(1) $a_R(v) = 1$.

If

(2) $U = \{g \in G\colon \big((1,1)\big((g, g_0)(a)\big)\big)_R(v) = 1\}$,

then from (1w), (1s) and (5R) we obtain that

$$U = \{g \in G\colon \big[\big((f_1^{1,1}(p), f_2^{1,1}(p))(a)\big)_R(v) = 1 \vee \ldots \vee \big((f_1^{1,s_1}(p), f_2^{1,s_1}(p))(a)\big)_R(v) = 1\big] \vee \ldots \vee \big[\big((f_1^{r,1}(p), f_2^{r,1}(p))(a)\big)_R(v) = 1 \vee \ldots \vee \big((f_1^{r,s_r}(p), f_2^{r,s_r}(p))(a)\big)_R(v) = 1\big]\},$$

where $p = (g, g_0)$.

Hence we have that

$$U=\bigcap_{i=1}^{r}\left(\bigcup_{j=1}^{s_i} U_{i,j}\right), \tag{3}$$

where

$$U_{i,j}=\{g\in G: ((f_1^{i,j}(p), f_2^{i,j}(p))(a))_R(v)=\mathbf{1}\}. \tag{4}$$

From (4) by (2R) and (1) we have that

$$U_{i,j}=\{g\in G: f_1^{i,j}(p)<f_2^{i,j}(p)\}$$

or

$$U_{i,j}=\{g\in G: f_1^{i,j}(p)-f_2^{i,j}(p)<0\}. \tag{5}$$

Let $h_{i,j}\colon G\to G$ be a continuous function defined as follows

$$h_{i,j}(g)=f_1^{i,j}(g, g_0)-f_2^{i,j}(g, g_0). \tag{6}$$

Since from (5) and (6) $U_{i,j}=h_{i,j}^{-1}(\{g\in G: g<0\})$, then from the fact that the set $\{g\in G: g<0\}$ is open we have that $U_{i,j}\in\mathfrak{O}$, and by (3) $U\in\mathfrak{O}$.

Similarly, we prove that

$$W=\{g\in G: ((1, 1)((g_0, g)(a)))_R(v)=\mathbf{1}\}\in\mathfrak{O}.$$

Hence

$$U\cap W\in\mathfrak{O}. \tag{7}$$

On the other hand, by Lemma 2 we obtain that

$$U\cap W=\{g\in G: ((g, g_0)(a))_R(v)=\mathbf{0}\wedge((g_0, g)(a))_R(v)=\mathbf{0}\}.$$

Hence from (2R) and (1) we have that

$$U\cap W=\{g\in G: g\geqslant g_0\wedge g_0\geqslant g\}$$

i.e., by (7), $\{g_0\}\in\mathfrak{O}$.

On the other hand, by (2a) we have that $\{g_0\}\notin\mathfrak{O}$. Thus the hypothesis that (1w) satisfies (1s) leads to a contradiction.

THEOREM 4. *If the space* $(G, \mathfrak{O})$ *does not contain any accumulation point, then in the schema of rule* (S7) $\varepsilon(n_1, \ldots, n_m, t)=1$ *for every* $n_1, \ldots, n_m, t$, *i.e.* (S7) *is of the form*

$$\frac{\Gamma', (1, 1)((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)), \Gamma''}{\Gamma', (-n_1, \ldots, -n_m, 1-t)(a_1, \ldots, a_m), \Gamma''}. \tag{2w}$$

Proof. By the assumption $(G, \mathfrak{O})$ does not contain any accumulation point, hence in particular 0 is not an accumulation point.

Thus by (3g) we have $1=\min\{g\in G: g>0\}$ and since $S(n_1, \ldots, n_m, t)\subset\{g\in G: g>0\}$ for every $n_1, \ldots, n_m, t$ we have $\varepsilon(n_1, \ldots, n_m, t)=1$ for every $n_1, \ldots, n_m, t$.

By Theorems 3 and 4, either (S7) is of the form (2w), i.e. it is very simple to use, or any schema in which the conclusion is the same as in (S7) and the weights and thresholds of the premises are obtained from threshold t and weights $n_1, \ldots, n_m$ of the conclusion only by arithmetical operation in G is not a schema of rule.

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К. Чешельски, Обобщенная порогажовая логика

Содержание. В работе сформулирована пороговая логика. Она является обобщением пороговой логики с весами и порогами будущими целыми числами на случай, когда веса и пороги являются элементами некоторой абстрактной группы с линейным порядком, согласованным с алгебраической структурой. Также представлена генценовская аксиоматика этой логики и доказано, что она является обобщением аксиоматики, сформулированной в [1] и [2.. Кроме того найдено необходимое и достаточное условие для того, чтобы обе аксиоматики совпадали друг с другом.