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THRESHOLD LOGIC

The results presented in this paper constitute generalization of the results published in [1]. In the logic considered in [1] the weights and the thresholds were the elements of the group of integers. What is taken into consideration in this paper, is the logic in which the weights and the thresholds are the elements of an ordered group such that every subgroup of the additive group of the real numbers' ring is its particular case.

§1. The language of the threshold logic

Let G be the set of elements of non-zero commutative group $(G, +, 0)$ with the relation of linear ordering $\leqslant$ fulfilling the following conditions:

(1g) if $a < b$ and $c \leqslant d$, then $a + c < b + d$
(2g) if $a < b$, then $-a > -b$

for every $a, b, c, d \in G$, where by definition

$$a < b \text{ iff } \neg(a \geqslant b)$$

Since G is non-zero and by (2g) we have that there is an element $g \in G$ such that $g > 0$. Let

(3g) $1 = \begin{cases} \text{minimal element from the set } \{a \in G : a > 0\} \text{ if it exists} \\ g \quad \text{otherwise.} \end{cases}$

By definition we have

(4g) $1 > 0$.

By a formalized language L of the threshold logic we will understand the pair $\langle A, F\rangle$, where the set A of the symbols is defined as follows: $A =$

$V \cup G \cup \{(\ ,\ ),\ '\ \}$ where $V = \{x, x_1, x_2, x_3, \ldots\}$ is the set of propositional variables, $\{(\ ,\ ),\ '\ \}$ is the set of auxiliary signs, G – is the set of the above defined group.

The set F of formulas of the language L is the least set satisfying the following conditions:

$(1F)$ if $x \in V$, then $x \in F$
$(2F)$ if $a_1, \ldots, a_m \in F$ and $n_1, \ldots, n_m \in G \backslash \{0\}$ and $t \in G$, then $(n_1, \ldots, n_m, t)(a_1, \ldots, a_m) \in F$, where $m = 1, 2, 3, \ldots$.

The elements $n_1, \ldots, n_m$ are called the weights of the formulas $a_1, \ldots, a_m$ respectively in the formula $(n_1, \ldots, n_m, t)$ $(a_1, \ldots, a_m)$. An element t is called its threshold.

Let $v : V \to \{\underline{0}, \underline{1}\}$ be a valuation of propositional variables in a two-element algebra. The extention of a valuation v onto the whole set F of formulas is defined as follows:

$$(1v) \quad v((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)) = \begin{cases} \underline{0} & \text{if} \quad \sum_{i=1}^{m} f_v(n_i, a_i) \geqslant t \\ \underline{1} & \text{if} \quad \sum_{i=1}^{m} f_v(n_i, a_i) < t \end{cases}$$

where $f_v : (G \backslash \{0\}) \times F \to G$ is the following function:

$$(2v) \quad f_v(n, a) = \begin{cases} n & \text{if} \quad v(a) = \underline{1} \\ 0 & \text{if} \quad v(a) = \underline{0} \end{cases}$$

Note that if we treat the group G as an additive group of the ring $(R, +, \cdot)$ with a standard ordering relation in R, and if the elements of algebra $\underline{0}, \underline{1}$ are treated as $0, 1 \in R$ respectively, then a mapping f_v can be interpreted as follows:

$$f_v(n, a) = n \cdot v(a)$$

A formula $a \in F$ is said to be a tautology of the language L if $v(a) = \underline{1}$ for every valuation v.

§2. The schemas of threshold logic

A formula $a \in F$ is said to be indecomposable if it is of the form x or $(1, 1)(x)$.

Let the letter Γ (with indices, if necessary) denote a finite sequence

$$a_1, \ldots, a_m$$

of formulas in L. The empty sequence is also being examined.

A sequence Γ composed only of indecomposable formulas is said to be indecomposable. A sequence Γ is said to be fundamental if it contains simultaneously a formula a and $(1,1)(a)$.

Let us define the extension of the valuation v onto the set of all finite sequences of formulas in the following way:

$(3v)$ if $\Gamma = a_1, \ldots, a_m$, then $v(\Gamma) = \begin{cases} \underline{0} & \text{if } v(a_1) = \ldots = v(a_m) = \underline{0} \\ \underline{1} & \text{otherwise} \end{cases}$

$(4v)$ if is empty, then $v(\Gamma) = \underline{0}$.

By a schema of a rule we mean the following expression:

$$\frac{\Gamma}{\Gamma_1; \ldots; \Gamma_r} \qquad \text{where} \qquad r = 1, 2, 3, \ldots$$

with the following condition

$(1r)$ for every valuation v

$$v(\Gamma) = \underline{1} \text{ iff } v(\Gamma_1) = \ldots = v(\Gamma_r) = \underline{1}$$

Γ is called the conclusion of the schema, $\Gamma_1, \ldots, \Gamma_r$ are called its premises.

We will use the following schemas:

$(R1)$ $\dfrac{\Gamma', (n_1, \ldots, n_m, t)(a_1, \ldots, a_m), \Gamma''}{\Gamma', \Gamma_1, \Gamma''; \Gamma', \Gamma_2, \Gamma''}$ for $m > 1$ and $n_m < 0$ where

$\Gamma_1 = a_m, (n_1, \ldots, n_{m-1}, t)(a_1, \ldots, a_{m-1})$

$\Gamma_2 = (n_1, \ldots, n_{m-1}, t - n_m)(a_1, \ldots, a_{m-1}), (n_1, \ldots, n_{m-1}, t)$
$(a_1, \ldots, a_{m-1})$

$(R2)$ $\dfrac{\Gamma', (n_1, \ldots, n_m, t)(a_1, \ldots, a_m), \Gamma''}{\Gamma', \Gamma_1, \Gamma''; \Gamma', \Gamma_2, \Gamma''}$ for $m > 1$ and $n_m > 0$ where

$\Gamma_1 = (n_1, \ldots, n_{m-1}, t - n_m)(a_1, \ldots, a_{m-1}), (1,1)(a_m)$

$\Gamma_2 = (n_1, \ldots, n_{m-1}, t - n_m)(a_1, \ldots, a_{m-1}), (n_1, \ldots, n_{m-1}, t)$
$(a_1, \ldots, a_{m-1})$

$(R3)$ $\dfrac{\Gamma', (n, t)(a), \Gamma''}{\Gamma', x, (1,1)(x), \Gamma''}$ for $n < t$ and $t > 0$ where $x \in V$

$(R4)$ $\dfrac{\Gamma', (n,t)(a), \Gamma''}{\Gamma', a, \Gamma''}$ for $n < t$ and $t \leqslant 0$

$(R5)$ $\dfrac{\Gamma', (n,t)(a), \Gamma''}{\Gamma', \Gamma''}$ for $n \geqslant t$ and $t \leqslant 0$

$(R6)$ $\dfrac{\Gamma', (n,t)(a), \Gamma''}{\Gamma', (1,1)(a), \Gamma''}$ for $n \geqslant t$ and $t > 0$ and $\neg(n = t = 1)$

$(R7)$ $\dfrac{\Gamma', (1,1)((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)), \Gamma''}{\Gamma', (-n_1, \ldots, -n_m, \varepsilon - t)(a_1, \ldots, a_m), \Gamma''}$ where $\varepsilon = min\{1\} \cup S$

$S = \{g \in G : g > 0$ and $(\exists v\text{-valuation})$ $(g = t - \sum_{i=1}^{m} f_v(n_i, x_i))$ where $x_1, \ldots, x_m \in V$ and $x_i \neq x_j$ for $i \neq j\}$

Theorem 1. *The schemas* $(R1) - (R7)$ *satisfy the condition* $(1r)$.

§3. Completeness theorem

The notion of a diagram of a formula and an end sequence of the diagram are defined in the usual way. The reader can find their definitions in [2], pp. 264–266.

Theorem 2. *A formula* a_0 *is a tautology if and only if all end sequences in the diagram of* a_0 *are fundamental.*

The proof is taken from [1]. This proving method is also in [2], pp. 264–269.

§4. Discussion of a schema $(R7)$

The schema $(R7)$ is very inconvenient because of the numerical way of defining the magnitude of ε. Of course, another schema would be much more helpful, in which all the group elements mentioned in the premises would be obtained by simple rules of arithmetic. But generally, such a result cannot be obtained. To prove this, we must introduce some notions. Let B be the class of sets $\{(g_1, g_2)\}$ where $g_1, g_2 \in G$, $g_1 < g_2$ and by definition $(g_1, g_2) = \{g \in G : g_1 < g < g_2\}$.

Let τ be a topology conformable with an ordering relation in the set G given by the basis B.

An element $g \in G$ is said to be an accumulation point, if

(1s) $(g_1, g_2)\backslash\{g\} \neq \emptyset$ for every $(g_1, g_2) \ni g$.

From this definition we have

(2s) $g \in G$ is an accumulation point iff $\{g\} \notin \tau$.

Note also that

(1f) $U = \{g \in G : g < g_0\} \in \tau$ for every $g_0 \in G$.

THEOREM 3. *If the space (G, τ) contains an accumulation point g_0, then a schema as follows does not exist:*

(1w) $\dfrac{\Gamma', (1,1)((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)), \Gamma''}{\Gamma', \Gamma_1, \Gamma'', \ldots, \Gamma', \Gamma_r, \Gamma''}$ *where*

$\Gamma_i = b^{i,j}, \ldots, b^{i,s_i}$ *for* $i = 1, \ldots, r$

$b^{i,j} = (f_1^{i,j}(p), \ldots, f_{m+1}^{i,j}(p))(a_1, \ldots, a_m)$ *for* $j = 1, \ldots, s_i$,

$p = (n_1, \ldots, n_m, t) \in G^{m+1}$

and all the mappings $f_k^{i,j}$ are continuous in the space (G, τ).

SKETCH OF PROOF. It suffices to examine the set $U = \{g \in G : v((1,1)((g, g_0(x))) = \underline{1}$ and $v((1,1)((g_0, g)(x))) = \underline{1}\}$ where $x \in V$ and v is a valuation such that $v(x) = \underline{1}$.

On the one hand $U = \{g_0\}$. But then, if we assume that a schema (1w) exists, then U will be an open set, i.e. we will obtain that $\{g_0\} \in \tau$, which is contradictory to (2s).

THEOREM 4. *If G has an accumulation point, then there is no such a schema in which the weights and the thresholds of the premises are obtained from the threshold and the weights of the conclusion by the arithmetical operations of G.*

THEOREM 5. *For the group G there exists a schema* (1w) *iff*

(2w) $\dfrac{\Gamma', (1,1)((n_1, \ldots, n_m, t)(a_1, \ldots, a_m)), \Gamma''}{\Gamma', (-n_1, \ldots, -n_m, 1-t)(a_1, \ldots, a_m), \Gamma''}$

is a schema.

To prove this it suffices to note that $1 = min\{a \in G : a > 0\}$. Hence $\varepsilon = 1$ and $(2w)$ coincides with the schema $(R7)$.

From Theorems 4 and 5 it follows that if we can talk about schema described in Theorem 4, then $(R7)$ takes the form of $(2w)$, i.e. it is very simple to use.

References

[1] E. Orłowska, *Threshold logic*, **Studia Logica**, Vol. XXXIII, No. 1 (1974).

[2] H. Rasiowa and R. Sikorski, **The mathematics of metamathematics**, PWN, Warszawa 1970.