Polychromatic Colorings of the Integers

We show that if S is any set of 4 integers then there is a 3-coloring of Z such that every additive translate of S gets all 3 colors. This proves a conjecture of Newman in additive number theory that the codensity of any set of 4 integers in Z is at most 1/3 (the codensity of a finite set S of integers is the minimum density of a set T in Z such that S + T = Z).

Loosely speaking, if L is any large structure consisting of some elements, and $\mathscr{F}$ is a family of substructures of L, we say a k-coloring of the elements of L is $\mathscr{F}$-polychromatic if every substructure in $\mathscr{F}$ gets all k colors. The polychromatic number of $\mathscr{F}$ in L is the largest k such that there exists an $\mathscr{F}$-polychromatic k-coloring.

L could be a graph, $\mathscr{F}$ a family of subgraphs of L, and we're coloring the edges of L. Ryan Hansen and I have solved the problem where L is the complete graph K_n, and $\mathscr{F}$ is the set of all subgaphs which are matchings of a specified size, or the set of all cycles of a specified size, or the set of all 2-regular graphs of at least a specified size.

Joe Sampson and I (for his capstone project) solved the problem where L is the set of integers mod n and $\mathscr{F}$ is the set of additive translates of a subset of size 3.

The problem mentioned above is where L is the set of integers, $\mathscr{F}$ is the family of all additive translates of a set S of 4 integers, and we show that the polychromatic number of $\mathscr{F}$ in L is at least 3.