Paradoxes in Mathematics and the Meaning of Truth

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First of all, in what sense the word *paradox* in the title is used? The Webster's Dictionary [4] gives the following relevant interpretation for it:

(1) a statement that seems contradictory, unbelievable, or absurd but that may be true in fact;

(2) a statement that is self-contradictory and, hence, false.

Most of the paradoxes mentioned in this talk will be of the first kind, though some of the second kind will also be mentioned.

One of the first paradoxes recorded in a written history is a Zeno's of Elea (495–435 B.C.) paradox known as *Achilles* or *Achilles and the Turtle* Paradox:

> Achilles running to overtake a crawling turtle can never overtake it because he must first reach the place from which the turtle started; when Achilles reaches that place, the turtle has already departed, and so it is still ahead. Repeating the argument we easily see that the turtle will be always ahead.

Although the argument sounds convincing, we know from experience that Achilles will easily overtake a turtle. So, what is wrong?

The problem with Zeno's argument is that he uses the word *always* in a non-standard way. He describes the "race" as follows. If Achilles starts at a point P_1 and turtle at a point P_2, by the time Achilles reaches point P_2 turtle will be already at some point P_3; by the time Achilles reaches point P_3, turtle will move to a point P_4, and so we can repeat this process to construct consecutive points $P_1, P_2, \ldots, P_n, \ldots$ for all natural numbers n. Also, it takes Achilles a positive time length t_1 to get from point P_1 to P_2, a positive time length t_2 to get from point P_2 to P_3, and, in general, a positive time length t_n to get from point P_n to P_{n+1}. So, *always* in the Paradox refers to *all* infinitely many time periods of the process, each of a positive length t_n. Thus, Paradox is true when *always* in its statements refers to the entire time period described which has length $t_1 + t_2 + t_3 + \cdots + t_n + \cdots$. It is very likely that Zeno was convinced that the infinite sum $t_1 + t_2 + t_3 + \cdots + t_n + \cdots$ of positive numbers must always be infinite, and in this case there would be no discrepancy in the meaning of *always*. However, as we learn in calculus, the infinite sum $t = t_1 + t_2 + t_3 + \cdots + t_n + \cdots$, known as *series*, can be finite even if all the terms t_n are positive. For example

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

In fact, this is the case in the situation described in the Paradox. In other words, if *always* in the Paradox is understood in the standard way, as *forever,* the statement of the Paradox is simply false.

Next we will consider other two statements paradoxical in the same *counterintuitive* sense.

The first is known as *Riemann's Theorem.* For this consider the infinite sum

$$\begin{aligned} S &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots \\ &= \tfrac{1}{1} + \left(-\tfrac{1}{2}\right) + \tfrac{1}{3} + \left(-\tfrac{1}{4}\right) + \tfrac{1}{5} + \left(-\tfrac{1}{6}\right) + \cdots \end{aligned}$$

and note that S is finite. Riemann's Theorem tells that in this (or any other series converging *conditionally*)

> by rearranging the order of the addition one can arbitrary increase or decrease the value of the sum.

For example, taking two terms with odd indexes, one term with even, and so on, as in $a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + a_9 + a_{11} + a_6 + \cdots$ we get a sum

$$T = \frac{1}{1} + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \frac{1}{9} + \frac{1}{11} + \left(-\frac{1}{6}\right) + \cdots$$

which is strictly *greater* than S.

This clearly contradicts the intuition of most of us, which comes from the properties of finite sums, that the value of the sum does not depend of the order in which we add the terms.

[However, most of the budget administrators had long a sense that some kind of Riemann theorem must be true. This is the reason for which

they usually *take an infinite time*

to rearrange different items in the budget and, upon the completion, they try to convince all other people that

the total value of the budget has been increased

by all their rearrangements.]

Even more counterintuitive is the next theorem known as *Paradoxical Decomposition of the Ball* or *Banach-Tarski Paradox:*

> A solid ball B of radius one (so, of volume $\frac{4}{3}\pi$) can be split into five pieces and the pieces can be rearranged (by using only shifts and rotations) in such a way that two of the rearranged pieces form one ball B_1 of radius one, and the three other pieces form another identical ball B_2 of radius one. As a result of this operation the volume of the ball B has been doubled to $2 \cdot \frac{4}{3}\pi$, the volume of two balls B_1 and B_2.

This statement can induce some doubts in mathematics even between the strongest believers in science. How it is possible to double the volume by simple operations of "cutting" and rearranging? And, if it is really possible, why don't we double the US gold reserve by such a process?

The answer for the second question becomes clearer when we take a closer look at the original purpose for proving the theorem. It was done to show that we cannot associate a "reasonable" volume to every abstract subset of the three dimensional space. In particular, the pieces used in the decomposition

do not have a volume

in any reasonable sense of this term. Thus, there is no chance to cut any ball *in practice* to such weird pieces. This also explain the paradox of duplicating the volume: since in the process we leave the realm of objects that have a volume, all intuitive rules concerning the property of volume seize to bind. Thus Paradox is counterintuitive, but does not carry a contradiction.

These and other similar paradoxes convinced mathematicians that we cannot confidently rely on our intuitions when deciding whether something is true or false. To overcome this "luck of confidence" we have decided to adopt the following *axiomatic approach* to most mathematical theories:

- You choose some fundamental facts, called *axioms*, that are basic enough to be accepted without any doubts. They are considered to be true.
- The other facts are considered as true in our theory if they are logically deduced from the axioms. The logical deduction of a fact is called its *proof.* The properties that can be deduced that way are called *provable.*

There is nothing new in this approach to build *theories.* It was already used by Euclid (330–275 B.C.) in his book *Elements,* which gave an account of elementary geometry. It has been also *attempted* by many philosophers since then. However, as soon as we accept the axiomatic approach as our tool for searching for truth, we should realize the first limit in reaching this goal:

there is no absolute truth

that can be discovered that way. All we can get that way is the *conditional truth,* that is, the statements provable within a theory are true only as long as the axioms are true. It should be stressed here that as long as we *believe* that the axioms are true, we should also *believe* that our conclusions are true. But this requires at least some level of *faith.*

So, is this a weak point of axiomatic theories? Should we abandon this approach because of it?

The fact is that in all our reasoning

we must have some starting point(s).

There is no way to deduce something from nothing. In all human search some assumptions are made. However, in most of the cases

the assumptions are made *implicitly.*

This does not make the truth less conditional. It only makes more difficult to realize what part of the theory belongs to the base in which we need to believe, and which part is the consequence of these believes. Thus, if something goes wrong in *naïve (i.e., non-axiomatic) theories* — the experiments disagree with the observations — it is very difficult to sort the things out. This happened in physics twice in the last century causing the births of the *theory of relativity* and the *quantum theory* to be painful experiences.

[When Albert Einstein got his Noble prize in physics in 1921, well over a decade after he formulated his relativity theory, the Noble price committee stressed that the price was **not** for his work on the relativity theory!]

Coming back to axiomatic theories — there are two fundamental questions that can be asked about any such theory:

Not too big?: How can we be sure that the axioms are really true? Can we at least be sure that the axioms do not contradict each other?

Big enough?: Based on the given set of axioms, can we deduce about each statement (concerning our theory) whether is it true or false?

The first question was already of interest to Euclid. He was not convinced whether the following fact, the fifth of his postulates (axioms) of geometry, was intuitive enough.

> Given a straight line L and a point p not on L, then in the plane determined by L and p it is possible to draw *precisely one* line L' through p which never meets L.

Euclid apparently tried to deduce this fact from his other postulates. Not being able to do so, he added it as an axiom. Nevertheless, throughout the text of his *Elements* he was pointing out the theorems in which proofs the fifth postulate was used, creating an impression that they are “less trustworthy.”

Euclid’s fifth postulate is a perfect example how difficult sometimes may be to decide whether a property is convincing (true) enough to be accepted as an axiom. The postulate remained unchallenged for 2200 years, until first half of nineteenth century, when Lobatchewsky showed that Euclid was right in his reserve to this axiom — the postulate does not follow from the other axioms and it is legitimate to assume that it is false. Lobatchewsky proved this last fact by showing that, in a way, in the reality in which we live, *on the surface of the Globe,* the fifth postulate is false, while other axioms remain valid. This is the case, since the straight lines on the Globe — the curves (known as *geodesics*) on which any two points are connected by a segment with the shortest distance between them — are the big circles (the circles having the same radius that the Globe). And *every* two big circles on the Globe intersects.

This part of discussion should convince us that the choice of the axioms is not simply dictated by some absolute truth.

What is true or false may depend on the reality which we describe!

Although the fifth postulate remains true (at least I believe so) when we do not bound ourselves to the surface of the Earth but stay within the realm of the classic geometry, the same postulate is false when we adopt Lobatchewsky point of view of the geometry of the surface of Earth.

Does it mean that all theories people are building tell us nothing about reality? Are they telling us anything about the truth? The answer is that

all scientific facts are only *relatively* true.

They remain true only in the reality for which they have been design, in which the basic assumptions – axioms – remain valid. Thus, classical physics is true in a sense that their predictions remains (reasonable) accurate when we stay within the realm of everyday sizes, distances, and speeds. The same classical physics becomes false (its predictions become far from the observations) when we become experiments concerning extreme speeds or subatomic particles.

The above discussion should convince us not only that we should stick to the axiomatic theories (out of lack of better choice). It also indicates that, in general, there is no good answer to the question *How can we be sure that the axioms are really true?* since the fifth postulate is neither "really true" not "really false." (This depends on the reality which we would like to describe.) So, lets concentrate on two remaining questions on how well we can choose our axioms:

- Can we ensure that the axioms are *consistent*, that is, do not contradict each other?
- Based on the given set of axioms, can we deduce about each statement (concerning our theory) whether is it true or false?

Unfortunately, the answer to both questions is negative in a sense described by the following two theorems of Gödel.

> Assume that the theory is rich enough to allow us to talk about natural numbers, and that the axioms are chosen "reasonable" in a sense that given a sentence we can *effectively* decide whether it is an axiom or not. Assume also that the axioms do not contradict each other. Then
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> - We cannot prove the consistency of the theory within the theory itself. That is we can express the sentence ψ: *the theory is consistent* within the theory, however ψ cannot be deduced from the axioms.
> - There is a sentence φ (concerning our theory) which is *independent* of the axioms, that is, we can deduce from the axioms neither φ nor its negation.

Moreover, almost as interesting that the theorem itself is the fact that the sentence φ has the following very easy intuitive interpretation:

I (i.e., φ) cannot be deduced (proved) from our axioms.

Clearly this φ cannot be proved from the axioms. However its negation $\neg\varphi$:

φ can be deduced (proved) from our axioms

cannot be proved as well, since otherwise both φ and its negation $\neg\varphi$ would have the proofs implying that the axioms are self-contradicting.

Gödel's main achievement was not the discovery of the sentence φ, since it is a version of an ancient Greek's paradox stated below. His main work was in arguing that this intuitive self-reference sentence, which is a paradox of kind (2), can be expressed (coded) as a property of the natural numbers.

Liar Paradox: Imagine a land in which some inhabitants, Cretans, always lie and all other inhabitants, Athenians, always tell the truth. In this land a person says:

I am a liar.

Note that this person can be neither Cretan nor Athenian.

Indeed, Cretan cannot say this phrase, since it is true, and he/she never tells the truth. Athenians cannot say it, since he never lie and it would be a lie.

If you like to feel the depth of self-reference problem, as in Liar Paradox, consider the following:

Brain Boggler: Imagine that somewhere deep in West Virginia countryside there is a T-shape crossroad with roads going to Richmond, Morgantown, and Charleston, respectively. At the crossroad live two brothers, identical twins. Both always answer all questions perfectly precise, but one *always tells the truth,* the other *always lie.*

Driving from Richmond, and being in a hurry for this lecture, you approach to the crossroad and you realize that you do not know which way is to Morgantown. Luckily, one of the twins comes out from the house ready to help you. But which one it is? You do not know and you have a time to ask just **one** question, with an answer YES or NO, to find how to get to Morgantown on time.

What question should you ask?

The discussion above concentrated on limitations and difficulties in searching for the "real scientific truth" in general, and through axiomatic approach in particular. Is there anything good that can be said about it after all?

The bright side of Gödel's theorems is that

there will be always room for the interpretation in science, even in highly structured axiomatic theories. There will be always a need for some "human" input. We will not be (easily) replaced by computers. Our jobs, at least for a while, are safe!

There is even brighter side in seemingly very unpleasant fact that *the truth of the theoretical predictions depends on the truth of the axioms, and there is no such a thing as the "absolute truth" of the axiomatic system.* The reason is that

> any *consistent* axiomatic system, no matter how abstract and unbelievable, may lead to the theory which, in fact, describes some portion of our reality.

For example, when Lobatchewsky (in the first half of the nineteenth century) was developing his "abstract geometries" in different surfaces and "deformed three-dimensional spaces" (with "straight lines" identified with geodesics) it was just a pure play of thoughts. It did not seem to have any "real" connection with the reality. It was not until the general relativity theory was formulated when people realized that, most likely, we in fact live in such a "strange deformed three-dimensional space." Suddenly, purely speculative and abstract theory become useful in describing the reality!

This happened to many abstract mathematical theories: being for years, sometimes centuries, unuseful and speculative they suddenly become practical tools for other sciences.

My personal interests in this subject comes from the study of the theory of sets known as *set theory.* This theory is the most fundamental for all mathematics (with the exception of pure logic) in a sense that all mathematical theories can be treated as sub-theories of the set theory (can be *modeled* in it) and use, to some extend, the axioms of set theory. In particular, in recent years I was mostly preoccupied with studying the influence of the axioms of set theory on the theory of real functions, known as *real analysis,* which a theoretical base for *calculus.* This branch of research has become known as *set theoretic analysis.*

By Gödel's theorems the axioms of set theory have a similar flaw as all other "reasonable" theories — they cannot decide all statements that concern sets. To help you appreciate the difficulty in choosing "correct" axioms for set theory let us consider the following two principles:

AC: Let $\mathcal{C}$ be a collection (possibly infinite) of sets, each of which has at least one element. Assume also that no two different sets from $\mathcal{C}$ have a common element.

Then there exists a set S, called *selector*, which has exactly one element in common with each set from $\mathcal{C}$.

AD: Let I be the set of all numbers x with $0 \leq x \leq 1$ which are identified with their decimal expansions: $x = 0.\overline{x_1 x_2 x_3 x_4 \ldots}$. For every set A of numbers from I consider the following infinite game $G(A)$: player I chooses the first digit x_1 from $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and then player II chooses the second digit x_2; next player I chooses x_3 and player II picks x_4, and so on. At the end of the game the players end up with a number $x = 0.\overline{x_1 x_2 x_3 x_4 \ldots}$. We declare that player I wins if x belongs to A. Otherwise player II wins.

Then for every set A of numbers from I one of the players has a winning strategy in the game $G(A)$.

Both statements are quite intuitive.

The first tells you that if you take some objects and distribute them into different drawers — sets from the collection $\mathcal{C}$ — then we can take a *sampler* S containing exactly one example from each drawer. This seems pretty obvious.

The second principle tells you that in a game $G(A)$ in which both players have perfect information on the rules and all moves so far, one of the players should be able to guarantee himself a victory, assuming that he will play "perfect" game. (This is true for all finite games, like chess, checkers, or go, in which players have full information and control of their moves. It does not mean, however, that it is easy to find such a wining strategy.) This also should be easy to believe.

Do you believe that both these statements are true? If not, which one seem to be more trustworthy?

The first of this statement, AC, is known as the Axiom of Choice, and it is commonly accepted as one of the axioms of set theory, though it usually plays the role of the fifth Euclid principle — it is often singled out as the least believable.

The second statement, AD, is known as the Axiom of Determinacy.

It **contradicts** the Axiom of Choice!

However it is sometimes considered as its alternative.

I like to finish this lecture with three examples of theorems from the set theoretical analysis, my main line of recent research. For this consider the following fundamental fact which is taught at every course of multivariable calculus.

> Let $f(x,y)$ be a function with $0 \leq x, y, f(x,y) \leq 1$. If f is continuous then the following integrals, known as the *iterated integrals*, exist and are equal
>
> $$I_1 = \int_0^1 \left(\int_0^1 f(x,y)dx \right) dy \quad \text{and} \quad I_2 = \int_0^1 \left(\int_0^1 f(x,y)dy \right) dx.$$

If function f is not continuous, the integrals may or may not exist. However the following seem to be an intriguing question.

Question: If for some function f as above the iterated integrals I_1 and I_2 exist, must they be equal?

Surprisingly, assuming only the standard axioms of set theory we cannot decide what is the answer to this question. This means, and it has been proven by the methods of set theoretic analysis, that by assuming either of the answers to this question as an additional axiom of set theory, the obtained theory will remain contradiction free.

A big part of my research concerned different generalizations of continuities for the functions of one variable (from $\mathbb{R}$ to $\mathbb{R}$). For example recall the following basic property of of continuous functions f (from $\mathbb{R}$ to $\mathbb{R}$) known as the *Intermediate Value Property* and taught at every calculus course.

IVP: For every $a < b$ and every number y between $f(a)$ and $f(b)$ there exists c between a and b such that $f(c) = y$.

Thus, every continuous function has IVP. However, it is not difficult to find functions with IVP that are not continuous. (See e.g. $f(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.) But how much of continuity functions with IVP must have? For example, is the following statement true?

($*$) For every function f from $\mathbb{R}$ to $\mathbb{R}$ which has IVP there exists a subset A of $\mathbb{R}$ of the same size (cardinality) than $\mathbb{R}$ such that f considered only on A (i.e., restricted to A) is continuous.

Once again, the answer is that neither property $(*)$ nor its negation can be deduced from the usual axioms of set theory. This has been proved in my 1997 paper [1] (written with M. Balcerzak and T. Natkaniec).

I like to finish this talk with citing one of my earlier set theoretical analysis result related to the *Paradoxical Decomposition of the Ball* theorem. To state this I have to specify what we will understand by a three dimensional abstract volume.

> An *abstract volume* μ on the three dimensional space $\mathbb{R}^3$ is a function associating to every subset A of $\mathbb{R}^3$ from some fixed family $\mathcal{M}$ of subsets of $\mathbb{R}^3$ a number $\mu(A) \geq 0$ (possible infinity ∞) and such that
>
> 1. Every geometrically defined solid S belongs to $\mathcal{M}$ and $\mu(S)$ is equal to its standard volume.
> 2. If A is obtained from B by applying rigid motion and A and B belong to $\mathcal{M}$ then $\mu(A) = \mu(B)$.
> 3. If $A_1, A_2, A_3, \ldots$ is a sequence of sets from $\mathcal{M}$ no pair of which has common points and $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ is a union of all sets A_n then
>
> $$\mu(A) = \mu(A_1) + \mu(A_2) + \mu(A_3) + \cdots .$$

It is known that there exists an abstract volume (known as *Lebesgue measure*), and it follows from the Paradoxical Decomposition of the Ball theorem for any such abstract volume there are non-measurable sets, i.e., the sets (pieces used in the Paradoxical Decomposition) which do to belong to $\mathcal{M}$. So, there is no abstract volume measuring everything. But maybe at least there is the best abstract volume measuring all it possibly can (in a sense that no other sets can be added to $\mathcal{M}$)? This question was asked by Sierpiński in 1935. The answer was given in my 1985 paper [2], joint with Pelc, in which it is proved that

there is no best abstract volume,

that is, every abstract volume μ can be farther extended to measure more sets. The full story on the discussion on this topic can be found in my 1989 popular article [3] published in *Mathematical Intelligencer*.

Are there any answers to Brain Boggler:? An answer can be found in my web page:

http://www.math.wvu.edu/homepages/kcies/

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[3] K. Ciesielski, *How good is the Lebesgue measure?*, Math. Intelligencer **11**(2) (1989), 54–58.

[4] *Webster's New Word Dictionary of American English*, third college edition, Webster's New Word, New York. 1988.