Math 581: Topology 1

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Contents

1. Topological Spaces

1.1. Metric Spaces

Notation.

$a := b$ mean that a equals b by definition.

$\mathbb{N} := \{1, 2, \dots\}$ is the set of positive integers.

$\mathbb{Z}$ is the set of integers.

$\mathbb{R}$ is the set of real numbers.

$\mathbb{Q}$ is the set of rational numbers $\square$

1.1.1. Convergence of sequences of real numbers.

A sequence of real numbers $(a_n)_{n=1}^{\infty}$ *converges* to a real number a if, for every positive real number ε, there exists $k \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ whenever $n \geq k$.

1.1.2. Continuity of real functions.

A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $a \in \mathbb{R}$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

1.1.3. Definition of a metric space.

A *metric space* is a set X together with a function $d : X \times X \to \mathbb{R}$, called a *metric* on X, that satisfies the following conditions:

1. $d(x, y) \geq 0$ with equality if and only if $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, y) + d(y, z) \geq d(x, z)$.

Condition 1. is called *positivity*, 2. is called *symmetry* and 3. is called *triangle inequality*.

Remark.

Note that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) := |x - y|$ is a metric on $\mathbb{R}$.

1.1.4. Example (Euclidean spaces).

Let $n \in \mathbb{N}$ and $X := \mathbb{R}^n$. For

$$x := (x_1, x_2, \ldots, x_n) \in X,$$

let

$$\|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}$$

and for $x, y \in X$, let $d(x, y) := \|x - y\|$. Then d is a metric on X.

Proof. It is clear that d is positive and symmetric. We prove the triangle inequality. If $1 \leq i < j \leq n$, then

$$(x_i y_j - x_j y_i)^2 \geq 0$$

so

$$2 x_i y_i x_j y_j \leq x_i^2 y_j^2 + x_j^2 y_i^2.$$

Thus

$$\begin{aligned}\left(\sum_{i=1}^{n} x_i y_i\right)^2 &= \sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{1\le i<j\le n} 2x_i y_i x_j y_j \\ &\le \sum_{i=1}^{n} x_i^2 y_i^2 + \sum_{1\le i<j\le n} \left(x_i^2 y_j^2 + x_j^2 y_i^2\right) \\ &= \sum_{1\le i,j\le n} x_i^2 y_j^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).\end{aligned}$$

Hence

$$\begin{aligned}\|x+y\|^2 = \sum_{n=1}^{n} (x_i + y_i)^2 &= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} x_i y_i \\ &\le \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} \\ &= (\|x\| + \|y\|)^2,\end{aligned}$$

which implies that $\|x+y\| \le \|x\| + \|y\|$. Thus, for $x, y, z \in X$, we have

$$d(x,y) + d(y,z) = \|x-y\| + \|y-z\| \ge \|x-z\| = \|a\|\, d(x,z),$$

so the triangle inequality holds. □

1.1.5. Example (Hilbert space).

Let $X := \ell_2$ be the set of all infinite sequences $(x_i)_{i=1}^{\infty}$ of real numbers with

$$\sum_{i=1}^{\infty} x_i^2 < \infty.$$

For

$$x := (x_1, x_2, \dots) \in X,$$

let

$$\|x\| := \sqrt{\sum_{i=1}^{\infty} x_i^2}$$

and for $x, y \in X$, let $d(x,y) := \|x-y\|$. Then d is a metric on X.

Proof. First, we verify that the values of d are finite. If $x, y \in X$, then

$$\|x-y\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

For each $n \in \mathbb{N}$ we have

$$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2} \leq \|x\| + \|y\|,$$

so $\|x-y\| \leq \|x\| + \|y\| < \infty$.

It is clear that d is positive and symmetric. It satisfies the triangle inequality since

$$\begin{aligned} d(x,y) + d(y,z) &= \|x-y\| + \|y-z\| \\ &= \|x-y\| + \|z-y\| \\ &\geq \|(x-y) - (z-y)\| \\ &= \|x-z\| = d(x,z). \end{aligned}$$

□

1.1.6. Example (integration metric).

Let $X := \mathscr{C}(I)$ be the set of all continuous real-valued functions on the interval $I := [0,1]$. Given $f \in X$, let

$$\|f\| := \int_0^1 |f| \, dx$$

and for $f, g \in X$, let $d(f,g) := \|f-g\|$. Then d is a metric on X.

1.1.7. Example (supremum metric).

Let Y be any set and let $X := \mathscr{B}(Y)$ be the set of all real-valued bounded functions on Y. Given $f \in X$, let

$$\|f\| := \sup \{|f(y)| : y \in Y\}$$

and for $f, g \in X$ let $d(f,g) := \|f - g\|$. Then d is a metric on X. It is called the *supremum* metric.

1.1.8. Definition of continuity.

Let (X,d) and (Y,d') be metric spaces and $f : X \to Y$. Then f is *continuous at* $x \in X$ provided for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x' \in X$ with $d(x,x') < \delta$ we have

$$d'(f(x), f(x')) < \varepsilon.$$

The function f is *continuous* if it is continuous at each $x \in X$.

1.1.9. Definition of convergence of sequences.

Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X,d). The sequence *converges* to $x \in X$ provided for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq k$.

Remark. We are going to describe continuity of functions and convergence of sequences using the concept of an open set. □

1.1.10. Open balls.

Let (X,d) be a metric space, $x \in X$ and $r > 0$. An *open ball* with center x and radius r is the set

$$B(x,r) := \{y \in X : d(x,y) < r\}.$$

1.1.11. Open sets.

Let (X, d) be a metric space and $U \subseteq X$. We say that U is *open* if for each $x \in U$ there is $r > 0$ such that $B(x, r) \subseteq U$.

Remark.

Each open ball is an open set.

Proof. Let $U := B(x, r)$ be an open ball in X and let $y \in U$. Then $d(x, y) < r$. Let $r' := r - d(x, y)$. If $z \in B(y, r')$, then $d(y, z) < r'$ so

$$d(x, z) \leq d(x, y) + d(y, z) < (r - r') + r' = r,$$

which implies that $z \in U$. Thus $B(y, r') \subseteq U$. □

1.1.12. Theorem (properties of metric spaces).

Let (X, d) be a metric space. The following conditions hold:

1. X is open and $\varnothing$ is open.
2. The union of any family of open sets is open.
3. The intersection of any nonempty finite family of open sets is open.

Proof. X is open since for any $x \in X$ we have $B(x, 1) \subseteq X$. The empty set $\varnothing$ is open since there are no $x \in \varnothing$.

Assume that $\mathscr{A}$ is any family of open sets. Let $x \in \bigcup \mathscr{A}$. Then there is $U \in \mathscr{A}$ with $x \in U$ so there is $r > 0$ with $B(x, r) \subseteq U$. Then $B(x, r) \subseteq \bigcup \mathscr{A}$. It follows that $\bigcup \mathscr{A}$ is open.

Assume that $\mathscr{A}$ is a nonempty finite family of open sets. Let $x \in \bigcap \mathscr{A}$. Then $x \in U$ for every $U \in \mathscr{A}$, so for every $U \in \mathscr{A}$ there is $r_U > 0$ with $B(x, r_U) \subseteq U$. Since $\mathscr{A}$ is finite,

$$r := \inf \{r_U : U \in \mathscr{A}\} = \min \{r_U : U \in \mathscr{A}\} > 0.$$

Then $B(x, r) \subseteq U$ for every $U \in \mathscr{A}$ so $B(x, r) \subseteq \bigcap \mathscr{A}$. Thus $\bigcap \mathscr{A}$ is open. □

1.1.13. Theorem (continuity for metric spaces).

Let (X, d) and (Y, d') be metric space. A function $f : X \to Y$ is continuous if and only if $f^{-1}[U]$ is open in X whenever U is open in Y.

Proof. Assume that f is continuous. Let $U \subseteq Y$ be open and $x \in f^{-1}[U]$. Then $f(x) \in U$ so there is $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subseteq U$. Since f is continuous, there exists $\delta > 0$ such that

$$d'(f(x), f(x')) < \varepsilon$$

whenever $d(x, x') < \delta$. If $x' \in B(x, \delta)$, then $d(x, x') < \delta$ so $d'(f(x), f(x')) < \varepsilon$ and

$$f(x') \in B(f(x), \varepsilon) \subseteq U,$$

implying that $x' \in f^{-1}[U]$. Thus $B(x, \delta) \subseteq f^{-1}[U]$ so $f^{-1}[U]$ is open.

Now assume that $f^{-1}[U]$ is open in X whenever U is open in Y. Let $x \in X$. We will show that f is continuous at x. Let $\varepsilon > 0$ and $U = B(f(x), \varepsilon)$. Then U is open in Y so $f^{-1}[U]$ is open in X. Since $x \in f^{-1}[U]$, there is $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}[U]$. If $x' \in X$ with $d(x, x') < \delta$, then $x' \in f^{-1}[U]$ so $f(x') \in U$ and $d'(f(x), f(x')) < \varepsilon$ as required. □

1.1.14. Homework 1 (due 1/14).

Problem 1.

Given a set X, define $d(x, y) := 0$ if $x = y$ and $d(x, y) := 1$ if $x \neq y$. Prove that d is a metric.

Solution. We have $d(x, y) \geq 0$ for each $x, y \in X$ with equality only when $x = y$ so positivity holds. Since $d(x, y) = d(y, x)$ for every $x, y \in X$, symmetry holds. It remains to verify the triangle inequality. Suppose, for a contradiction, that the triangle inequality fails so there are $x, y, z \in X$ with

$$d(x, y) + d(y, z) < d(x, z).$$

Then $d(x, z) = 1$ and $d(x, y) = d(y, z) = 0$ so $x = y$ and $y = z$. Thus $x = z$ and we have a contradiction. □

Problem 2.

Let (X, d) be a metric space. Define

$$d_1(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

and

$$d_2(x, y) := \min\left(1, d(x, y)\right).$$

Prove that d_1 and d_2 are metrics on X.

Solution. Since $d(x, y) \geq 0$, it follows that $d_1(x, y) \geq 0$ for every $x, y \in X$. If $d_1(x, y) = 0$, then $d(x, y) = 0$ so d_1 satisfies positivity. Since d satisfies symmetry, it follows that d_1 satisfies symmetry.

Now, we verify the triangle inequality for d_1. Let $x, y, z \in X$. We have

$$\begin{aligned} d_1(x, y) + d_1(y, z) &= \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\ &= 2 - \frac{1}{1 + d(x, y)} - \frac{1}{1 + d(y, z)}. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{1}{1 + d(x, y)} + \frac{1}{1 + d(y, z)} &\leq \frac{2 + d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= 1 + \frac{1}{1 + d(x, y) + d(y, z)}. \end{aligned}$$

Since d satisfies the triangle inequality, it follows that $d(x, y) + d(y, z) \geq d(x, z)$ so

$$\frac{1}{1 + d(x, y)} + \frac{1}{1 + d(y, z)} \leq 1 + \frac{1}{1 + d(x, z)}.$$

Thus

$$\begin{aligned} d_1(x, y) + d_1(y, z) &\geq 2 - \left(1 + \frac{1}{1 + d(x, z)}\right) \\ &= 1 - \frac{1}{1 + d(x, z)} = \frac{d(x, z)}{1 + d(x, z)} = d_1(x, z). \end{aligned}$$

Thus d_1 is a metric.

Now, we verify that d_2 is a metric. It is clear that d_2 satisfies positivity and symmetry. We prove that d_2 satisfies the triangle inequality. Let $x, y, z \in X$. Suppose, for a contradiction, that

$$d_2(x, y) + d_2(y, z) < d_2(x, z).$$

Since $d_2(x, z) \leq 1$, it follows that

$$d_2(x, y) + d_2(y, z) < 1,$$

so $d_2(x, y) < 1$, which implies that $d_2(x, y) = d(x, y)$. Similarly $d_2(y, z) = d(y, z)$. Since $d_2(x, y) < d(x, y)$, we conclude that

$$d(x, y) + d(y, z) < d(x, z),$$

which is a contradiction. □

Problem 3.

Prove that what we defined as the "supremum metric" in Example 1.1.7 is a metric.

Solution. If $f, g \in X$

$$d(f, g) := \sup \{|f(y) - g(y)| : y \in Y\}.$$

Since the absolute value is never negative, we have $d(f, g) \geq 0$ for any $f, g \in X$. If $d(f, g) = 0$, then $|f(y) - g(y)| = 0$ for every $y \in Y$ so $f = g$. Thus positivity holds. Since

$$|f(y) - g(y)| = |g(y) - f(y)|,$$

the symmetry holds. It remains to verify the triangle inequality. Let $f, g, h \in X$. Then

$$|f(z) - g(z)| + |g(z) - h(z)| \geq |f(z) - h(z)|$$

for any $z \in Y$ so

$$d(f,g) + d(g,h) = \sup\{|f(y) - g(y)| : y \in Y\} \\ + \sup\{|g(y) - h(y)| : y \in Y\} \geq |f(z) - h(z)|$$

for any $z \in Y$. Thus

$$d(f,g) + d(g,h) \geq \sup\{|g(y) - h(y)| : y \in Y\} = d(f,h).$$

□

Problem 4.

Let (X,d) be a metric space and $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove that the sequence $(x_n)_{n=1}^{\infty}$ converges to $x \in X$ if and only if for every open $U \subseteq X$ with $x \in U$ there exists $k \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq k$.

Solution. Let $x \in X$. Assume that $(x_n)_{n=1}^{\infty}$ converges to x. Let U be open in X with $x \in U$. There is $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$. Since $(x_n)_{n=1}^{\infty}$ converges to x, there is $k \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for every $n \geq k$. Then $x_n \in B(x,\varepsilon)$ so $x_n \in U$ for every $n \geq k$.

Now assume that for every open $U \subseteq X$ with $x \in U$ there exists $k \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq k$. Let $\varepsilon > 0$ be arbitrary. Let $U := B(x,\varepsilon)$. Then U is open and $x \in U$ so there is $k \in \mathbb{N}$ with $x_n \in U$ for every $n \geq k$. Thus $d(x_n, x) < \varepsilon$ for every $n \geq k$, which implies that $(x_n)_{n=1}^{\infty}$ converges to x. □

1.2. Topologies

1.2.1. Example of a convergence not induced by a metric.

Let X be the set of real-valued functions on the interval $[0,1]$. Consider the following question. Is there a metric d on X such that a sequence $(f_n)_{n=1}^{\infty}$ in X

converges to $f \in X$ in the metric space (X, d) if and only if $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$ for every $x \in X$?

The answer is no!

Proof. Nested Interval Property (Theorem 1.4.1. in Abbott's book) says:

For each $n \in \mathbb{N}$ let $I_n := [a_n, b_n]$ be a closed interval such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots.$$

Then

$$\bigcap_{i=1}^{\infty} I_i \neq \varnothing.$$

Suppose, for a contradiction, that such a metric d on X exists. For each $n \in \mathbb{N}$, let $f_n \in X$ be defined by

$$f_n(x) := \begin{cases} 1 & \text{if } x \in \left(0, \frac{1}{n}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f_n)_{n=1}^{\infty}$ converges pointwise to the constant function f such that $f(x) := 0$ for each $x \in [0, 1]$. Then there is $k_1 \in \mathbb{N}$ such that $d(f_{k_1}, f) < 1$. Let $a_1 := 0$, $b_1 := \dfrac{1}{k_1}$ and $g_1 := f_{k_1}$.Analogous argument shows that there are a_2, b_2 with $a_1 < a_2 < b_2 < b_1$ and $d(g_2, f) < \dfrac{1}{2}$, where $g_2 \in X$ is defined by:

$$g_2(x) := \begin{cases} 1 & \text{if } x \in (a_2, b_2) \\ 0 & \text{otherwise,} \end{cases}$$

By induction, for each $n \geq 2$, we get a_n, b_n such that $a_{n-1} < a_n < b_n < b_{n+1}$ and $d(g_n, f) < \dfrac{1}{n}$, where $g_n \in X$ is defined by:

$$g_n(x) := \begin{cases} 1 & \text{if } x \in (a_n, b_n) \\ 0 & \text{otherwise.} \end{cases}$$

Then $(g_n)_{n=1}^{\infty}$ converges to f in the metric space (X, d). However, the Nested Interval Property implies that

$$C := \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \varnothing.$$

If $c \in C$, then $c \in (a_n, b_n)$ for each $n \in \mathbb{N}$ so $g_n(c) = 1$ for each $n \in \mathbb{N}$. Thus $(g_n(c))_{n=1}^{\infty}$ does not converge to $f(c) = 0$, which is a contradiction. □

1.2.2. Definition of topology.

A *topological structure* (or just *topology*) on a set X is a family $\mathscr{T}$ of subsets (called *open sets*) of X such that the following conditions hold:

1. X is open and $\varnothing$ is open.
2. The union of any family of open sets is open.
3. The intersection of any nonempty finite family of open sets is open.

A *topological space* is a set X together with a topology on X.

Examples.

1. The *discrete topology* on X is the family of all subsets of X.
2. The *trivial topology* on X is the family $\{X, \varnothing\}$.
3. The *Sierpiński space* is the set $X = \{1, 2\}$ with the topology $\{X, \varnothing, \{1\}\}$.
4. For any metric space X, the family of open sets is a topology on X.
5. Let X be an infinite set. The family of cofinite subsets of X (whose complements are finite) is a topology on X. It is called the *cofinite* topology.
6. Let X be an uncountable set. The family of cocountable subsets of X (with countable complements) is a topology on X. It is called the *cocountable* topology on X.

1.2.3. Comparing topologies on the same set.

Let $\mathscr{T}$ and $\mathscr{T}'$ be topologies on the same set X. If $\mathscr{T} \subseteq \mathscr{T}'$, then we say that $\mathscr{T}$ is *coarser*, *smaller* or *weaker* than $\mathscr{T}'$ and that $\mathscr{T}'$ is *finer*, *larger* or *stronger* than $\mathscr{T}$.

Remark.

Note that the trivial topology on a set X is smaller than any topology on X and the discrete topology on X is larger than any topology on X.

1.2.4. Closed sets.

A subset C of a topological space X is *closed* if $X \smallsetminus C$ is open.

1.2.5. Proposition (properties of closed sets).

Let X be a topological space.

1. The sets X and $\varnothing$ are closed.
2. The intersection of any nonempty family of closed sets is closed.
3. The union of any finite family of closed sets is closed.

Proof. The set X is closed since $\varnothing$ is open and $\varnothing$ is closed since X is open.

Let $\mathscr{C}$ be a nonempty family of closed sets. Then

$$\mathscr{A} := \{X \smallsetminus C : C \in \mathscr{C}\}$$

is a family of open sets so $\bigcup \mathscr{A}$ is open. Since

$$\bigcap \mathscr{C} = X \smallsetminus \bigcup \mathscr{A},$$

it follows that $\bigcap \mathscr{C}$ is closed.

Let $\mathscr{C}$ be a finite family of closed sets. If $\mathscr{C} = \varnothing$, then $\bigcup \mathscr{C} = \varnothing$ is closed. If $\mathscr{C} \neq \varnothing$, then

$$\mathscr{A} = \{X \setminus C : C \in \mathscr{C}\}$$

is a nonempty family of open sets so $\bigcap \mathscr{A}$ is open. Since

$$\bigcup \mathscr{C} = X \setminus \bigcap \mathscr{A},$$

it follows that $\bigcup \mathscr{C}$ is closed. □

Remark.

An infinite union of closed sets does not have to be closed.

Example.

The closed interval $\left[\frac{1}{n}, 2\right]$ is closed in $\mathbb{R}$ for each $n \in \mathbb{N}$, but the union

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right] = (0, 2]$$

is not closed.

Remark.

In a discrete space every set is both closed and open.

Example.

In $\mathbb{Z}$ with the cofinite topology finite sets are closed, but not open. Cofinite sets are open, but not closed. Then set $\mathbb{N}$ is neither closed nor open.

1.2.6. Homework 2 (due 1/21)

Problem 1.

Let $X := \{1,2,3\}$ and $\mathscr{T}_1 := \{\varnothing, X, \{1\}, \{1,2\}\}$ and $\mathscr{T}_2 := \{\varnothing, X, \{3\}, \{2,3\}\}$ be topologies on X.

- Prove that $\mathscr{T}_1 \cup \mathscr{T}_2$ is not a topology on X.
- Find the smallest topology on X containing $\mathscr{T}_1 \cup \mathscr{T}_2$.
- Find the largest topology on X contained in $\mathscr{T}_1 \cap \mathscr{T}_2$.

Solution.

- $\mathscr{T}_1 \cup \mathscr{T}_2$ is not a topology on X since $\{1\}, \{3\} \in \mathscr{T}_1 \cup \mathscr{T}_2$ but $\{1,3\} \notin \mathscr{T}_1 \cup \mathscr{T}_2$.
- The smallest topology on X containing $\mathscr{T}_1 \cup \mathscr{T}_2$ is the discrete topology.
- The largest topology on X contained in $\mathscr{T}_1 \cap \mathscr{T}_2$ is the trivial topology.

$\square$

Problem 2.

Let X be an infinite set and $x_0 \in X$. Let

$$\mathscr{T} := \{G \subseteq X : X \setminus G \text{ is finite or } x_0 \notin G\}.$$

- Prove that $\mathscr{T}$ is a topology on X.
- Let $x \in X$. Prove that $\{x\}$ is both open and closed if and only if $x \neq x_0$.

Solution. We have $\varnothing \in \mathscr{T}$ since $x_0 \notin \varnothing$ and $X \in \mathscr{T}$ since $X \setminus X$ is empty hence finite.

Let $\mathscr{A} \subseteq \mathscr{T}$. If for every $A \in \mathscr{A}$ we have $x_0 \notin A$, then $x_0 \notin \bigcup \mathscr{A}$ so $\bigcup \mathscr{A} \in \mathscr{T}$. Otherwise, there is $A_0 \in \mathscr{A}$ such that $x_0 \in A_0$. Then $X \setminus A_0$ is finite. Since $X \setminus \bigcup \mathscr{A} \subseteq X \setminus A_0$, it follows that $X \setminus \bigcup \mathscr{A}$ is finite so $\bigcup \mathscr{A} \in \mathscr{T}$.

Let $\mathscr{A} \subseteq \mathscr{T}$ be finite and nonempty. If there is $A_0 \in \mathscr{A}$ such that $x_0 \notin A_0$, then $x_0 \notin \bigcap \mathscr{A}$ so $\bigcap \mathscr{A} \in \mathscr{T}$. Otherwise, we have $x_0 \in A$ for every $A \in \mathscr{A}$ so $X \setminus A$ is finite for every $A \in \mathscr{A}$. Then

$$X \setminus \bigcap \mathscr{A} = \bigcup_{A \in \mathscr{A}} (X \setminus A)$$

is finite as a finite union of finite sets. Thus $\bigcap \mathscr{A} \in \mathscr{T}$.

We have proved that $\mathscr{T}$ is a topology on X.

Let $x \in X$. Assume that $\{x\}$ is both open and closed. Since $\{x\}$ is open, it follows the either $X \setminus \{x\}$ is finite or $x_0 \notin \{x\}$. Since X is infinite, it follows that $X \setminus \{x\}$ is infinite so $x_0 \notin \{x\}$ and $x \neq x_0$.

Now assume that $x \neq x_0$. Then $x_0 \notin \{x\}$ so $\{x\}$ is open. Let $A := X \setminus \{x\}$. Then $X \setminus A = \{x\}$ is finite so A is open. It follows that $\{x\}$ is closed. □

Problem 3.

Let $X := \mathscr{B}([0,1])$ with the supremum metric (see Example 1.1.7). Show that

$$C := \{f \in X : f \text{ is continuous}\}$$

is closed in X.

Solution. We will show that $X \setminus C$ is open in X. Let $g \in X \setminus C$. We need $r > 0$ such that the ball $B(g,r) \subseteq X \setminus C$.

Since g is not continuous, there is $x \in [0,1]$ such that g is not continuous at x. Thus there is $\varepsilon > 0$ such that for every $\delta > 0$ there is $y \in [0,1]$ with $|x - y| < \delta$ and $|g(x) - g(y)| \geq \varepsilon$. Let $r := \varepsilon/3$. If $f \in B(g,r)$, then $|f(z) - g(z)| < \varepsilon/3$ for every $z \in [0,1]$. We show that $f \in X \setminus C$ by showing that f is not continuous at x. Actually, we will show that for every $\delta > 0$ there is $y \in [0,1]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| > \varepsilon/3$.

Let $\delta > 0$. There is $y \in [0,1]$ with $|x - y| < \delta$ and $|g(x) - g(y)| \geq \varepsilon$. Then

$$\begin{aligned} \varepsilon \leq |g(x) - g(y)| &\leq |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| \\ &< \varepsilon/3 + |f(x) - f(y)| + \varepsilon/3 = 2\varepsilon/3 + |f(x) - f(y)|, \end{aligned}$$

which implies that $|f(x) - f(y)| > \varepsilon/3$. $\square$

Problem 4.

Consider $X := \mathbb{R}^2$ with the standard Euclidean metric d. Give an example of nonempty disjoint closed subsets $A, B \subseteq X$ such that

$$\inf\{d(x, y) : x \in A,\ y \in B\} = 0.$$

Solution. Let

$$A := \{\langle a, b\rangle : a, b \in \mathbb{R} \text{ with } ab = 1\}$$

and $B := \{\langle 0, b\rangle : b \in \mathbb{R}\}$. Then A and B are nonempty disjoint closed subsets of X. For any $\varepsilon > 0$ we have $x := \langle \varepsilon, 1/\varepsilon\rangle \in A$ and $b := \langle 0, 1/\varepsilon\rangle \in B$ with $d(x, y) = \varepsilon$, which implies that

$$\inf\{d(x, y) : x \in A,\ y \in B\} = 0.$$

$\square$

1.2.7. Neighborhoods.

Let X be a topological space and $x \in X$. A set $N \subseteq X$ is a *neighborhood* of x (*nbhd* for short) if there exists an open set U such that $x \in U \subseteq N$.

Example.

The interval $[0, 2)$ is a nbhd of 1 in $\mathbb{R}$ that is not open.

Remark.

If U is an open set, then it is a nbhd of each $x \in U$. In particular, X is a nbhd of each $x \in X$.

1.2.8. Proposition (properties of nbhds).

Let X be a topological space and, for each $x \in X$, let $\mathscr{N}_x$ be the family of all nbhds of x. Then the following conditions hold for each $x \in X$:

1. $\mathscr{N}_x \neq \varnothing$;
2. $x \in N$ for every $N \in \mathscr{N}_x$;
3. if $N_1, N_2 \in \mathcal{N}_x$, then $N_1 \cap N_2 \in \mathscr{N}_x$;
4. if $N \in \mathscr{N}_x$ and $N \subseteq M \subseteq X$, then $M \in \mathscr{N}_x$;
5. if $N \in \mathscr{N}_x$, then $\{y \in N : N \in \mathscr{N}_y\} \in \mathscr{N}_x$.

Proof. 1. holds since $X \in \mathscr{N}_x$.

2. holds since if $N \in \mathscr{N}_x$, then there is open U with $x \in U \subseteq N$, which implies that $x \in N$.

3. holds since if $N_1, N_2 \in \mathscr{N}_x$, then there are open U_1, U_2 with $x \in U_1 \subseteq N_1$ and $x \in U_2 \subseteq N_2$. Let $U := U_1 \cap U_2$. Then U is open and

$$x \in U \subseteq N_1 \cap N_2.$$

4. holds since if N, M are as assumed, then there is open U with $x \in U \subseteq N$. Since $N \subseteq M$, this implies that $U \subseteq M$ so $M \in \mathscr{N}_x$.

5. holds since $N \in \mathscr{N}_x$ implies that there is open U with $x \in U \subseteq N$. Since $U \in \mathcal{N}_y$ for every $y \in U$, we have $N \in \mathscr{N}_y$ for every $y \in U$. Thus

$$U \subseteq \mathring{N} := \{y \in N : N \in \mathscr{N}_y\}.$$

Since $U \in \mathscr{N}_x$, it follows that $\mathring{N} \in \mathscr{N}_x$. □

Example.

Let $X := \{1, 2, 3\}$ with $\mathscr{N}_1 := \{\{1, 2\}, X\}$ and $\mathscr{N}_2 := \mathscr{N}_3 := \{X\}$. Then conditions 1.–4. of Proposition 1.2.8 hold, but 5. fails since $N := \{1, 2\} \in \mathscr{N}_1$, but $\mathring{N} = \{1\} \notin \mathscr{N}_1$.

If

$$\mathscr{T} := \{U \subseteq X : U \in \mathscr{N}_x \text{ for every } x \in U\},$$

then $\mathscr{T} = \{X, \varnothing\}$ is the trivial topology on X. Then N is not a nbhd of 1 relative to $\mathscr{T}$.

1.2.9. Proposition (topology from nbhds).

Let X be a set and for each $x \in X$ let $\mathscr{N}_x$ be a family of subsets of X such that the conditions 1.–4. of Proposition 1.2.8 hold. Then

$$\mathscr{T} := \{U \subseteq X : U \in \mathscr{N}_x \text{ for every } x \in U\}$$

is a topology on X such that each nbhd of $x \in X$ relative to $\mathscr{T}$ belongs to $\mathscr{N}_x$. If 5. is also satisfied, then $\mathscr{N}_x$ is equal to the family of all nbhds of x relative to $\mathscr{T}$ and $\mathscr{T}$ is the unique topology having that property.

Proof. It is clear that $\varnothing \in \mathscr{T}$. Now we show that $X \in \mathscr{T}$. Given $x \in X$ we have $\mathscr{N}_x \neq \varnothing$ by 1. so there is $N \in \mathscr{N}_x$ which implies that $X \in \mathscr{N}_x$ by 4. Thus $X \in \mathscr{N}_x$ for every $x \in X$, which implies that $X \in \mathscr{T}$.

Let $\mathscr{A}$ be a family of members of $\mathscr{T}$. To show that $\bigcup \mathscr{A} \in \mathscr{T}$ we need to show that $\bigcup \mathscr{A} \in \mathscr{N}_x$ for every $x \in \bigcup \mathscr{A}$. Let $x \in \bigcup \mathscr{A}$. Then there is $U \in \mathscr{A}$ with $x \in U$. Since $U \in \mathscr{T}$, we have $U \in \mathscr{N}_y$ for each $y \in U$ so, in particular, $U \in \mathscr{N}_x$. Then 4. implies that $\bigcup \mathscr{A} \in \mathscr{N}_x$.

Let $\mathscr{A}$ be a nonempty finite family of members of $\mathscr{T}$. To show that $\bigcap \mathscr{A} \in \mathscr{T}$ we need to show that $\bigcap \mathscr{A} \in \mathscr{N}_x$ for every $x \in \bigcap \mathscr{A}$. Let $x \in \bigcap \mathscr{A}$. Then $x \in U$ for every $U \in \mathscr{A}$. Since $\mathscr{A} \subseteq \mathscr{T}$, we have $U \in \mathscr{N}_x$ for every $U \in \mathscr{A}$. Since $\mathscr{A}$ is finite, applying 3. and induction we conclude that $\bigcap \mathscr{A} \in \mathscr{N}_x$.

Let N be a nbhd of $x \in X$ relative to $\mathscr{T}$. Then there is $U \in \mathscr{T}$ with $x \in U \subseteq N$. The definition of $\mathscr{T}$ implies that $U \in \mathscr{N}_x$. Thus $N \in \mathscr{N}_x$ by 4.

Now assume 5. as well. Let $x \in X$ and $N \in \mathscr{N}_x$. To show that N is a nbhd of x relative to $\mathscr{T}$, it suffices to show that

$$\mathring{N} := \{y \in N : N \in \mathscr{N}_y\} \in \mathscr{T}.$$

If $z \in \mathring{N}$, then 5. implies that $\mathring{N} \in \mathscr{N}_z$ so $\mathring{N} \in \mathscr{N}_z$ for every $z \in \mathring{N}$ and consequently $\mathring{N} \in \mathscr{T}$ as required.

Suppose that $\mathscr{T}'$ is any topology having the property that $\mathscr{N}_x$ is the family of nbhds of x relative to $\mathscr{T}'$. To show that $\mathscr{T}' = \mathscr{T}$ it suffices to show that $U \in \mathscr{T}'$ if and only if $U \in \mathscr{N}_x$ for every $x \in U$. Assume that $U \in \mathscr{T}'$. Then U is a nbhd of every $x \in U$ by the definition of a nbhd. Assume that $U \in \mathscr{N}_x$ for every $x \in U$. Then for each $x \in U$, there is $V_x \in \mathscr{T}'$ such that $x \in V_x \subseteq U$. Since

$$U = \bigcup_{x \in U} V_x,$$

it follows that $U \in \mathscr{T}'$. □

1.2.10. Homework 3 (due 1/28).

Problem 1.

In a metric space (X, d), for any real number $r \geq 0$, the closed r-ball at $x \in X$ is the set $\{y \in X : d(x, y) \leq r\}$. Show that a closed ball is always closed in the metric topology.

Solution. Let $r \geq 0$ and

$$C := \{y \in X : d(x, y) \leq r\}.$$

We will show that $X \setminus C$ is open in X. Let $z \in X \setminus C$. Then $d(x, z) > r$ so $\varepsilon := d(x, z) - r > 0$. To show that $X \setminus C$ is open in X it suffices to show that $B(z, \varepsilon) \subseteq X \setminus C$.

Let $y \in B(z, \varepsilon)$. Then $d(y, z) < \varepsilon$ so

$$r + \varepsilon = d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon,$$

which implies that $d(x, y) > r$ so $y \in X \setminus C$ as required. □

Problem 2.

What is the topology determined by the metric on X given by $d(x, y) := 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$?

Solution. The resulting topology is the discrete topology. For any $x \in X$, the set $\{x\}$ is open in X since $B(x, 1) \subseteq \{x\}$. Since any subset of X is a union of singletons, any subset of X is open so the obtained topology is discrete. □

Problem 3.

Let X be a set with at least two elements. Prove that there are no metric on X that induces the trivial topology on X.

Solution. Let $x, y \in X$ with $x \neq y$. Suppose, for a contradiction, that d is a metric on X that induces the trivial topology. Let $d(x, y) = r > 0$. Then $U := B(x, r)$ is open in X with $x \in U$ and $y \notin U$. Thus $U \neq \varnothing$ and $U \neq X$. This is a contradiction since $\varnothing$ and X are the only open sets in the trivial topology. □

Problem 4.

Let (X, d) be a metric space and $C \subseteq X$ be closed. Prove that there is a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X such that $C = \bigcap_{n\in\mathbb{N}} U_n$.

Solution. For each $n \in \mathbb{N}$, let

$$U_n := \bigcup_{x\in C} B\left(x, \frac{1}{n}\right).$$

Then U_n is open in X as a union of a family of open sets. It remains to show that

$$C = \bigcap_{n\in\mathbb{N}} U_n.$$

Since $x \in U_n$ for every $x \in C$ and every $n \in \mathbb{N}$, it follows that $C \subseteq U_n$ for every $n \in \mathbb{N}$ so

$$C \subseteq \bigcap_{n \in \mathbb{N}} U_n.$$

Now let $y \in \bigcap_{n \in \mathbb{N}} U_n$. We aim at showing that $y \in C$. Suppose, for a contradiction, that $y \in X \smallsetminus C$. Since $X \smallsetminus C$ is open there is $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq X \smallsetminus C$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \varepsilon$. Since $y \in U_n$, there is $x \in C$ with $d(x, y) < \frac{1}{n}$. Then $d(x, y) < \varepsilon$ so $x \in B(y, \varepsilon)$, implying that $B(y, \varepsilon) \cap C \neq \varnothing$, which is a contradiction. □

1.3. Derived Concepts

1.3.1. Interior

Let X be a topological space and $A \subseteq X$. The *interior* of A is denoted by A° or by $\operatorname{int}(A)$ and is defined by

$$A^\circ := \bigcup \{U \subseteq A : U \text{ is open}\}.$$

If $x \in A^\circ$, then x is an *interior point* of A.

Remarks.

Note that A° is open and it is the largest open subset of A. Moreover, x is an interior point of A if and only if A is a nbhd of x. It is also clear that A is open if and only if $A = A^\circ$.

Examples.

In $\mathbb{R}$ we have $\mathbb{Q}^\circ = \varnothing$. If A is the closed interval $[a, b]$, then A° is the open interval (a, b).

1.3.2. Proposition (properties of interior).

Let X be a topological space and $A, B \subseteq X$.

1. $(A^\circ)^\circ = A^\circ$,
2. $A \subseteq B$ implies that $A^\circ \subseteq B^\circ$,
3. $A^\circ \cap B^\circ = (A \cap B)^\circ$,
4. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

Example.

If $X := \mathbb{R}$, $A := [0,1]$ and $B := [1,2]$, then $1 \notin A^\circ \cup B^\circ$, but $1 \in (A \cup B)^\circ$.

1.3.3. Closure.

Let X be a topological space and $A \subseteq X$. The closure of A is denoted by $\operatorname{cl}(A)$ or $\overline{A}$. It is defined by

$$\overline{A} := \bigcap \{A \subseteq C : C \text{ is closed}\}.$$

If $x \in \overline{A}$, then x is an *adherent point* of A.

Remarks.

Note that $\overline{A}$ is the smallest closed set containing A and that A is closed if and only if $\overline{A} = A$.

Examples.

If $X = \mathbb{R}$, then $\overline{\mathbb{Q}} = \mathbb{R}$. If A is the open interval (a,b) with $a < b$, then $\overline{A}$ is the closed interval $[a,b]$.

1.3.4. Proposition (properties of closure).

Let X be a topological space and $A, B \subseteq X$.

1. $\overline{\overline{A}} = \overline{A}$,
2. $A \subseteq B$ implies that $\overline{A} \subseteq \overline{B}$,
3. $\overline{A} \cup \overline{B} = \overline{A \cup B}$,
4. $\overline{A} \cap \overline{B} \supseteq \overline{A \cap B}$.

Example.

If $X := \mathbb{R}$, $A := (0,1)$ and $B := (1,2)$, then $1 \in \overline{A} \cap \overline{B}$, but $1 \notin \overline{A \cup B}$.

1.3.5. Theorem (characterization of closure).

Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if $U \cap A \neq \varnothing$ for every open nbhd U of x.

Proof. Assume that $x \in \overline{A}$. Let U be an open nbhd of x and suppose, for a contradiction, that $U \cap A = \varnothing$. Then $A \subseteq X \smallsetminus U$ and $X \smallsetminus U$ is closed. Thus $x \in \overline{A} \subseteq X \smallsetminus U$, which contradicts $x \in U$.

Now assume that $U \cap A \neq \varnothing$ for every open nbhd U of x. Let C be closed with $A \subseteq C$. Then $X \smallsetminus C$ is open and disjoint with A so $x \notin X \smallsetminus C$. Thus $x \in C$. Since x belongs to every closed set containing A, it follows that $x \in \overline{A}$. □

1.3.6. Limit points.

Let X be a topological space and $A \subseteq X$. A point $x \in X$ is a *limit point* (*cluster point*) of A if every open nbhd U of x contains at least one point of $A \smallsetminus \{x\}$. The set A' of all limit points of A is called the *derived set* of A.

Example.

In $\mathbb{R}$ if $A := (0,1) \cup \{2\}$, then $A' = [0,1]$.

1.3.7. Theorem (closure and derived set).

Let X be a topological space and $A \subseteq X$. Then $\overline{A} = A \cup A'$.

Proof. If $x \in A$, then $x \in \overline{A}$. If $x \in A'$, then every open nbhd U of x contains at least one point of $A \smallsetminus \{x\}$ so $A \cap U \neq \varnothing$, which implies that $x \in \overline{A}$.

Now assume that $x \in \overline{A} \smallsetminus A$. We show that $x \in A'$. Let U be an open nbhd of x. Since $x \in \overline{A}$, we have $U \cap A \neq \varnothing$. Since $A \smallsetminus \{x\} = A$, we have

$$U \cap (A \smallsetminus \{x\}) \neq \varnothing$$

as required. □

Corollary.

A set is closed if and only if it contains all its limit points.

Proof. A is closed if and only if $\overline{A} = A$, which holds if and only if $A = A \cup A'$, which is equivalent to $A' \subseteq A$. □

1.3.8. Boundary.

Let X be a topological space and $A \subseteq X$. The *boundary* (also called the *frontier*) of A is denoted by ∂A and is defined by

$$\partial A := \overline{A} \cap \overline{X \smallsetminus A}.$$

Example.

In $\mathbb{R}$ if $A = [0,1]$, then $\partial A = \{0,1\}$.

1.3.9. Theorem (closure and boundary).

Let X be a topological space and $A \subseteq X$. Then $\overline{A} = A \cup \partial A$.

Proof. We have $A \subseteq \overline{A}$ and the definition of ∂A implies that $\partial A \subseteq \overline{A}$. Thus $\overline{A} \supseteq A \cup \partial A$.

Now assume that $x \in \overline{A} \smallsetminus A$. Then $x \in X \smallsetminus A$ so $x \in \overline{X \smallsetminus A}$. Thus $x \in \partial A$. □

Corollary.

A set is closed if and only if it contains it's boundary.

Proof. A is closed if and only if $\overline{A} = A$, which is equivalent to $A = A \cup \partial A$ and to $\partial A \subseteq A$. □

1.3.10. Isolated points.

Let X be a topological space and $A \subseteq X$. If $x \in A \smallsetminus A'$, then x is an isolated point of A.

1.3.11. Perfect sets.

Let X be a topological space and $A \subseteq X$. We say that A is *perfect* if A is closed and has no isolated points.

1.3.12. Example (the Cantor set).

Let

$$J_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$$

$$J_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$$

and, in general, for each $n \in \mathbb{N}$, let

$$J_n = \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \setminus \bigcup_{j=1}^{n-1} J_j.$$

The set

$$C := [0,1] \setminus \bigcup_{n=1}^{\infty} J_n$$

is called the *Cantor set*. It consists of those $x \in [0,1]$ that have a triadic expansion using only the digits 0 and 2.

The Cantor set is perfect since it is closed and has no isolated points. If $x \in C$ has a triadic expansion $0.a_1 a_2 \ldots$, and U is any open interval containing x, then we can choose $n \in \mathbb{N}$ large enough, so that changing the digit a_n to $2 - a_n$ produces a different point in $U \cap C$.

1.3.13. Dense sets.

Let X be a topological space and $A \subseteq X$. We say that A is *dense* if $\overline{A} = X$.

Examples.

$\mathbb{Q}$ is dense in $\mathbb{R}$. If X is an infinite set with cofinite topology, then any infinite subset of X is dense.

1.3.14. Homework 4 (due 2/4).

Problem 1.

Let $X := \{a, b, c\}$ with the topology $\{\varnothing, X, \{a\}, \{a, b\}\}$. Find the derived sets of $\{a\}$, $\{b\}$, $\{c\}$ and $\{a, c\}$.

Solution.

$$\{a\}' = \{b, c\}$$
$$\{b\}' = \{c\}$$
$$\{c\}' = \varnothing$$
$$\{a, c\}' = \{b, c\}$$

□

Problem 2.

Let U be open in a topological space X. Prove that

$$\overline{U} = \overline{\operatorname{int}\left(\overline{U}\right)}.$$

Solution. U is open and $U \subseteq \overline{U}$ so $U \subseteq \operatorname{int}\left(\overline{U}\right)$, which implies that $\overline{U} \subseteq \overline{\operatorname{int}\left(\overline{U}\right)}$. Since $\operatorname{int}\left(\overline{U}\right) \subseteq \overline{U}$ and since $\overline{U}$ is closed, it follows that $\overline{\operatorname{int}\left(\overline{U}\right)} \subseteq \overline{U}$. Thus $\overline{U} = \overline{\operatorname{int}\left(\overline{U}\right)}$. □

Problem 3.

Let X be a topological space and $G \subseteq X$. Prove that G is open if and only if

$$\overline{G \cap \overline{A}} = \overline{G \cap A}$$

for every $A \subseteq X$.

Solution. Assume that G is open. Since $A \subseteq \overline{A}$, it follows that $G \cap A \subseteq G \cap \overline{A}$, which implies that $\overline{G \cap A} \subseteq \overline{G \cap \overline{A}}$.

Let $x \in \overline{G \cap \overline{A}}$. If U is an open nbhd of x, then $U \cap \left(G \cap \overline{A}\right) \neq \varnothing$. Let $y \in U \cap \left(G \cap \overline{A}\right)$. Then $U \cap G$ is an open nbhd of y and $y \in \overline{A}$ so $U \cap G \cap A \neq \varnothing$. Since any open nbhd of x has a nonempty intersection with $G \cap A$, it follows that $x \in \overline{G \cap A}$. Thus $\overline{G \cap \overline{A}} \subseteq \overline{G \cap A}$. Therefore $\overline{G \cap \overline{A}} = \overline{G \cap A}$.

Now assume that $\overline{G \cap \overline{A}} = \overline{G \cap A}$ for every $A \subseteq X$. We aim to show that G is open. Suppose, for a contradiction, that G is not open. Then $A := X \setminus G$ is not closed, so $\overline{A} \cap G \neq \varnothing$. Then $\overline{G \cap A} = \varnothing$, but $\overline{G \cap \overline{A}} \neq \varnothing$, which is a contradiction. □

Problem 4.

Let X be an infinite set with the cofinite topology. Prove that if $A \subseteq X$ is infinite, then every point in A is a limit point of A and that if A is finite, then it has no limit points.

Solution. Assume that $A \subseteq X$ is infinite. If $x \in A$ and U is an open nbhd of x, then $X \setminus U$ is finite so $U \cap A$ is infinite and hence contains an element of A distinct from x. Thus x is a limit point of A.

Assume that A is finite. If $x \in X$ then $U := (X \setminus A) \cup \{x\}$ is an open nbhd of x such that $U \cap A \subseteq \{x\}$. Thus x is not a limit point of A. □

1.4. Bases

1.4.1. Proposition (family of topologies).

Let X be a set and $\mathbb{A}$ be a nonempty family of topologies on X. Then $\bigcap \mathbb{A}$ is a topology on X.

Proof. Both X and $\varnothing$ belong to $\mathscr{T}$ for every $\mathscr{T} \in \mathbb{A}$. Thus $X, \varnothing \in \bigcap \mathbb{A}$.

Assume that $\mathscr{A} \subseteq \bigcap \mathbb{A}$. Then $\mathscr{A} \subseteq \mathscr{T}$ for every $\mathscr{T} \in \mathbb{A}$ so $\bigcup \mathscr{A} \in \mathscr{T}$ for every $\mathscr{T} \in \mathbb{A}$. Thus $\bigcup \mathscr{A} \in \bigcap \mathbb{A}$.

Assume that $\mathscr{A} \subseteq \bigcap \mathbb{A}$ is nonempty and finite. Then $\mathscr{A} \in \mathscr{T}$ for every $\mathscr{T} \in \mathbb{A}$ so $\bigcap \mathscr{A} \in \mathscr{T}$ for every $\mathscr{T} \in \mathbb{A}$. Thus $\bigcap \mathscr{A} \in \bigcap \mathbb{A}$. □

1.4.2. Subbases.

Let X be a set and $\mathscr{S}$ be any family of subsets of X. Define

$$\mathscr{T}(\mathscr{S}) := \bigcap \mathbb{A},$$

where $\mathbb{A}$ is the family of all topologies $\mathscr{T}$ on X such that $\mathscr{S} \subseteq \mathscr{T}$. The set $\mathscr{S}$ is called the *subbasis* of the topological space $(X, \mathscr{T}(\mathscr{S}))$.

Remark.

Note that any family of subsets of X is a subbasis of a unique topology on X. For a given topology on X there are usually many possible subbases.

1.4.3. Proposition (topology from subbasis).

Let X be a set and $\mathscr{S}$ be a family of subsets of X. The topology $\mathscr{T}(\mathscr{S})$ consists of $X, \varnothing$ and all unions of all possible intersections of nonempty finite subfamilies of $\mathscr{S}$.

Proof. It is clear that $X, \varnothing \in \mathscr{T}(\mathscr{S})$ since $X, \varnothing$ belong to any topology on X. Moreover, any topology on X containing $\mathscr{S}$ contains all unions of all possible intersections of nonempty finite subfamilies of $\mathscr{S}$.

To complete the proof, it suffices to show that the family $\mathscr{T}$ of all unions of all possible intersections of nonempty finite subfamilies of $\mathscr{S}$ together with X and $\varnothing$ is a topology on X. It is clear that $X, \varnothing \in \mathscr{T}$.

Let $\mathscr{A} \subseteq \mathscr{T}$. If $X \in \mathscr{A}$, then $\bigcup \mathscr{A} = X \in \mathscr{T}$. Otherwise, $\bigcup \mathscr{A} = \bigcup \mathscr{A}'$, where $\mathscr{A}' = \mathscr{A} \setminus \{\varnothing\}$. Every member of $\mathscr{A}'$ is a union of intersection of nonempty finite subfamilies of $\mathscr{S}$, which implies that $\bigcup \mathscr{A}'$ is such a union. Thus $\bigcup \mathscr{A} \in \mathscr{T}$.

Let $\mathscr{A} \subseteq \mathscr{T}$ be finite and nonempty. If $\varnothing \in \mathscr{A}$, then $\bigcap \mathscr{A} = \varnothing \in \mathscr{T}$. Otherwise, let $\mathscr{A}' = \mathscr{A} \setminus \{X\}$. If $\mathscr{A}' = \varnothing$,then $\bigcap \mathscr{A} = X \in \mathscr{T}$. If $\mathscr{A}' \neq \varnothing$, then each member of $\mathscr{A}'$ is a union of intersections of nonempty finite subfamilies of $\mathscr{S}$. Let $\mathscr{A}' = \{A_1, A_2, \ldots, A_n\}$ with

$$A_i = \bigcup_{j \in J_i} A_{i,j},$$

where $A_{i,j}$ is the intersection of some nonempty finite subfamily of $\mathscr{S}$ and J_i is some set for each $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}\bigcap \mathscr{A}' &= \left(\bigcup_{j \in J_1} A_{1,j}\right) \cap \left(\bigcup_{j \in J_2} A_{2,j}\right) \cap \cdots \cap \left(\bigcup_{j \in J_n} A_{n,j}\right) \\ &= \bigcup_{f \in J} \left(A_{1,f(1)} \cap A_{2,f(2)} \cap \cdots \cap A_{n,f(n)}\right),\end{aligned}$$

where J is the set of all functions f on $\{1, 2, \ldots, n\}$ with $f(i) \in J_i$ for every i. Thus $\bigcap \mathscr{A}$ is a union of intersections of nonempty finite subfamilies of $\mathscr{S}$ and so $\bigcap \mathscr{A} \in \mathscr{T}$. □

1.4.4. Linear order

Let X be a set and $\leq$ be a binary relation on X. We say that $<$ is a *linear order* on X if

1. $\leq$ is *reflexive* ($x \leq x$ for each $x \in X$).
2. $\leq$ is *transitive* ($x \leq y$ and $y \leq z$ implies that $x \leq z$ for each $x, y, z \in X$).
3. $\leq$ is *antisymmetric* ($x \leq y$ and $y \leq x$ implies that $x = y$ for every $x, y \in X$).
4. $\leq$ is *total* ($x \leq y$ or $y \leq x$ for every $x, y \in X$).

Example.

The standard order on $\mathbb{R}$ is a linear order.

1.4.5. Order topology

Let X be a set with a linear order $\leq$. For $x \in X$, let

$$(-\infty, x) = \{y \in X : y \leq x \text{ and } y \neq x\}$$

and

$$(x, \infty) = \{y \in X : x \leq y \text{ and } y \neq x\}.$$

The *order topology* on X induced by $\leq$ is the topology $\mathscr{T}(\mathscr{S})$, where

$$\mathscr{S} = \{(-\infty, x) : x \in X\} \cup \{(x, \infty) : x \in X\}.$$

Example.

The standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ induced by the standard order.

1.4.6. Homework 5 (due 2/11).

Problem 1.

Consider $\mathbb{N}$ with the standard order. Prove that the resulting order topology on $\mathbb{N}$ is discrete.

Solution. Let $n \in \mathbb{N}$. If $n = 1$, then $\{n\} = (-\infty, 2)$ is open in the order topology. If $n \geq 2$, then

$$\{n\} = (-\infty, n + 1) \cap (n - 1, \infty)$$

so $\{n\}$ is open as well. Since every singleton $\{n\}$ is open, the topology is discrete. $\square$

Problem 2.

Consider the set $X := \{1,2\} \times \mathbb{N}$ with the dictionary order, that is such that $\langle a,b\rangle \leq \langle c,d\rangle$ when $a < c$ or ($a = c$ and $b \leq d$). Prove that the resulting order topology on X is not discrete.

Solution. We show that $A := \{\langle 2,1\rangle\}$ is not open. Suppose, for a contradiction, that A is open. Then $A = \bigcup \mathscr{K}$ for some family $\mathscr{K}$ consisting of intersections of finite nonempty subfamilies of the subbasis

$$\mathscr{S} = \{(-\infty, x) : x \in X\} \cup \{(x, \infty) : x \in X\}.$$

Let $K \in \mathscr{K}$ be such that $A \subseteq K$. Then K must contain $\langle 2,1\rangle$ so

$$\begin{aligned} K = (-\infty, \langle 2, k_1\rangle) \cap \cdots \cap (-\infty, \langle 2, k_s\rangle) \\ \cap (\langle 1, k_{s+1}\rangle, \infty) \cap \cdots \cap (\langle 1, k_t\rangle, \infty) \end{aligned}$$

for some $s, t \in \mathbb{N}$ with $0 \leq s \leq t$. Let $\ell := \max\{k_{s+1}, \ldots, k_t\} + 1$. Then $\langle 1, \ell\rangle \in K$ so $\langle 1, \ell\rangle \in A$, which is a contradiction. □

Problem 3.

Describe the topology on the plane for which the family of all straight lines is a subbasis.

Solution. The intersection of two lines that are not parallel is a singleton. Any singleton on the plane can be represented as the intersection of two lines. Thus each singleton is open and so the topology is discrete. □

Problem 4.

For each $q \in \mathbb{Q}$, let $A_q := \{x \in \mathbb{R} : x > q\}$ and $B_q := \{x \in \mathbb{R} : x < q\}$. Prove that the set

$$\mathscr{S} := \{A_q : q \in \mathbb{Q}\} \cup \{B_q : q \in \mathbb{Q}\}$$

is a subbasis for the standard topology on $\mathbb{R}$.

Solution. Let $\mathscr{T} := \mathscr{T}(\mathscr{S})$ and $\mathscr{T}'$ be the standard topology on $\mathbb{R}$. Since $\mathscr{S} \subseteq \mathscr{T}'$, it follows that $\mathscr{T} \subseteq \mathscr{T}'$. It remains to show that $\mathscr{T}' \subseteq \mathscr{T}$.

Let $U \in \mathscr{T}'$. For each $x \in U$ there are rational p, q with $p < x < q$ and $J_x := (p, q) \subseteq U$. Then

$$U = \bigcup_{x \in U} J_x$$

and $J_x = A_p \cap B_q \in \mathscr{T}$ for every $x \in U$. Thus $U \in \mathscr{T}$. □

1.4.7. Well-ordered sets.

A set X is *well-ordered* by $\leq$ if $\leq$ is a linear order on X and for every nonempty $A \subseteq X$ there is $a \in A$ such that $a \leq b$ for every $b \in A$.

1.4.8. Well-ordering principle.

The axioms of set theory imply that every set can be well-ordered.

1.4.9. The well-ordered set $[0, \Omega]$ as a topological space.

Let X be any uncountable set and $\leq$ be a well-ordering of X. If the set

$$A = \{x \in X : \{y \in X : y \leq x\} \text{ is uncountable}\}$$

is nonempty, let Ω be the smallest element of A and

$$[0, \Omega] := \{x \in X : x \leq \Omega\}.$$

Otherwise, let

$$[0, \Omega] := X \cup \{\Omega\}$$

with $\leq$ extended to $[0, \Omega]$ by declaring that $x \leq \Omega$ for every $x \in [0, \Omega]$.

We will consider $[0, \Omega]$ as a topological space with the order topology.

1.4.10. Theorem (the space $[0, \Omega]$).

The set $[0, \Omega]$ is an uncountable well-ordered set such that for every $x \in [0, \Omega]$ that is strictly smaller than Ω, the set

$$\{y \in [0, \Omega] : y \leq x\}$$

is countable. Moreover, if X is any well-ordered set with the largest element Ω' such that for every $x \in X$ that is strictly smaller than Ω', the set

$$\{y \in X : y \leq x\}$$

is countable, then there is a bijection $\varphi : X \to [0, \Omega]$ such that for every $x, y \in X$ with $x \leq y$, we have $\varphi(x) \leq \varphi(y)$.

Proof. The set $[0, \Omega]$ has the required properties directly from the definition. The proof of the existence of the bijection φ is omitted. □

1.4.11. Bases.

Let X be a topological space. A *basis* for the topology on X is a family $\mathscr{B}$ of open subsets of X such that for every open set U and $x \in U$ there is $B \in \mathscr{B}$ with $x \in B \subseteq U$.

Remarks.

$\mathscr{B}$ is a basis for the topology $\mathscr{T}$ on X if and only if $\mathscr{B} \subseteq \mathscr{T}$ and every open set is a union of members of $\mathscr{B}$. Any basis for the topology on X is also a subbasis.

Examples.

In a discrete space X the family of all singletons $\{x\}$ is a basis. In a metric space X the collection of all open balls $B(x, r)$ for $x \in X$ and $r > 0$ is a basis.

Remark.

Let $\mathscr{S}$ be a subbasis of the topology on X. Then the family $\mathscr{B}$ consisting of X and all intersections of finite nonempty subfamilies of $\mathscr{S}$ is a basis for the topology on X.

Remark.

Let X be the topological space having the order topology induced by a linear order $\leq$. If for $a, b \in X$ we define

$$(a, b) := \{x \in X \setminus \{a, b\} : a \leq x \leq b\},$$

then

$$\mathscr{B} := \{(a, b) : a, b \in X\} \cup \{(-\infty, a) : a \in X\} \cup \{, (a, \infty) : a \in X\}$$

is a basis for the topology on X.

Example.

Consider the topological space $[0, \Omega]$. Let S be the set of all *successor elements* in $[0, \Omega]$, that is let $x \in S$ if the set

$$\{y \in [0, \Omega] : y < x\}$$

has the largest element. Let $S' := S \cup \{0\}$, where 0 is the smallest element of $[0, \Omega]$ and $L := [0, \Omega] \setminus S'$. Define

$$\mathscr{B} := \{\{a\} : a \in S'\} \cup \{(a, b] : a < b,\, b \in L\}.$$

Then $\mathscr{B}$ is a basis for the topology on $[0, \Omega]$.

1.4.12. Theorem (basis for a topology).

Let X be a set and $\mathscr{B}$ be a collection of subsets of X. Then $\mathscr{B}$ is a basis for some topology on X if and only if the following conditions hold:

1. $\bigcup \mathscr{B} = X$ and
2. for every $B_1, B_2 \in \mathscr{B}$ and every $x \in B_1 \cap B_2$ there is $B \in \mathscr{B}$ with $x \in B \subseteq B_1 \cap B_2$.

Proof. Assume that $\mathscr{B}$ is a basis for a topology $\mathscr{T}$ on X. Since $X \in \mathscr{T}$, it follows that 1. holds. To prove 2., assume that $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$. Since $B_1, B_2 \in \mathscr{T}$, it follows that $B_1 \cap B_2 \in \mathscr{T}$ so there is $B \in \mathscr{B}$ with $x \in B \subseteq B_1 \cap B_2$. Thus 2. holds.

Now assume that 1. and 2. hold. Let $\mathscr{T}$ be the family of all unions of subfamilies of $\mathscr{B}$. Then 1. implies that $X \in \mathscr{T}$ and $\varnothing$ is the union of the empty family so $\varnothing \in \mathscr{T}$. The family $\mathscr{T}$ is closed under taking unions since the union of unions of subfamilies of $\mathscr{B}$ is also a union of subfamilies of $\mathscr{B}$.

Let $U, V \in \mathscr{T}$. We show that $U \cap V \in \mathscr{T}$. Assume

$$U = \bigcup_{\alpha \in A} B_\alpha$$

and

$$V = \bigcup_{\alpha \in C} D_\alpha,$$

where A, C are some sets and $B_\alpha, D_\beta \in \mathscr{B}$ for each $\alpha \in A$ and $\beta \in C$. We have

$$U \cap V = \bigcup_{(\alpha,\beta) \in A \times C} B_\alpha \cap D_\beta.$$

For each $(\alpha, \beta) \in A \times C$ and each $x \in B_\alpha \cap D_\beta$ let $G_{\alpha,\beta,x} \in \mathscr{B}$ be such that

$$x \in G_{\alpha,\beta,x} \subseteq B_\alpha \cap D_\beta.$$

Then

$$U \cap V = \bigcup_{(\alpha,\beta) \in A \times C} \ \bigcup_{x \in B_\alpha \cap D_\beta} G_{\alpha,\beta,x}$$

so $U \cap V \in \mathscr{T}$.

We have proved that $\mathscr{T}$ is a topology on X. Clearly, $\mathscr{B} \subseteq \mathscr{T}$. To show that $\mathscr{B}$ is a basis for $\mathscr{T}$, let $U \in \mathscr{T}$ and $x \in U$. There is $B \in \mathscr{B}$ with $x \in B \subseteq U$. □

1.4.13. Homework 6 (due 2/18).

Problem 1.

Let $A := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Let $\mathscr{B}_1$ be the collection of open intervals in $\mathbb{R}$ and $\mathscr{B}_2$ be the collection of all subsets of $\mathbb{R}$ that are of the form $(a, b) \setminus A$ for $a, b \in \mathbb{R}$ with $a < b$. Prove that $\mathscr{B} := \mathscr{B}_1 \cup \mathscr{B}_2$ is a basis for a topology on $\mathbb{R}$ and that the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ does not converge to 0 in this topology.

Solution. For each $x \in \mathbb{R}$ we have $x \in (x-1, x+1) \in \mathscr{B}_1$. Thus $\bigcup \mathscr{B} = \mathbb{R}$. Let $B_1, B_2 \in \mathscr{B}$ with $x \in B_1 \cap B_2$. We need to find $B \in \mathscr{B}$ with $x \in B$. There are a_1, a_2, b_1 and $b_2$such that $B_1 = (a_1, b_1)$ or $B_1 = (a_1, b_1) \setminus A$ and $B_2 = (a_2, b_2)$ or $B_2 = (a_2, b_2) \setminus A$. If $x \in A$, then $B_1 = (a_1, b_1)$ and $B_2 = (a_2, b_2)$ so $B = B_1 \cap B_2$ satisfies the requirements. If $x \notin A$, then

$$B = ((a_1, b_1) \cap (a_2, b_2)) \setminus A$$

satisfies the requirements. Thus $\mathscr{B}$ is a basis for the topology on $\mathbb{R}$.

Now we show that $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ does not converge to 0 in this topology. Suppose, for a contradiction, that it does converge to 0. Then for every nbhd U of 0 there is $k \in \mathbb{N}$ with $\frac{1}{n} \in U$ for every $n \geq k$. In particular, this holds when $U = (-1, 1) \setminus A$. However, for such U we have $\frac{1}{n} \notin U$ for all $n \in \mathbb{N}$ so we have a contradiction. □

Problem 2.

Let $\mathscr{B} := \{(x, \infty) : x \in \mathbb{R}\}$. Prove that $\mathscr{B}$ is a basis of a topology on $\mathbb{R}$ and find the closures of $A := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $B := \mathbb{N}$ in this topology.

solution. We have $\bigcap \mathscr{B} = \mathbb{R}$ since for every $x \in \mathbb{R}$ we have $x \in (x-1, \infty)$. If $B_1, B_2 \in \mathscr{B}$, then there are $b_1, b_2 \in \mathbb{R}$ such that $B_1 = (b_1, \infty)$ and $B_2 = (b_2, \infty)$ and let $x \in B_1 \cap B_2$. Let $b = \max\{b_1, b_2\}$. Then $(b, \infty) \in \mathscr{B}$ and $x \in (b, \infty) \subseteq B_1 \cap B_2$ as required. Thus $\mathscr{B}$ is a basis for a topology on $\mathbb{R}$.

The closure of A is this topology is $(-\infty, 1]$ and the closure of B is $\mathbb{R}$. □

Problem 3.

Consider $\mathbb{R}$ with the topology generated by the basis $\mathscr{B} := \{[a, b) : a, b \in \mathbb{Q}\}$. Find the boundary, closure and interior of the subsets $\left(0, \sqrt{2}\right)$ and $\left(\sqrt{3}, 4\right)$ of $\mathbb{R}$.

Solution. The set $\left(0, \sqrt{2}\right)$ is equal to it's interior. It's closure is $\left[0, \sqrt{2}\right]$ and the boundary is $\left\{0, \sqrt{2}\right\}$.

The set $\left(\sqrt{3}, 4\right)$ is also equal to it's interior. It's closure is $\left[\sqrt{3}, 4\right)$ and the boundary is $\left\{\sqrt{3}\right\}$. □

Problem 4.

Let $\mathscr{B}$ be a basis of the topological space X and $A \subseteq X$. Prove that $x \in \overline{A}$ if and only if $B \cap A \neq \varnothing$ for every $B \in \mathscr{B}$ such that $x \in B$.

Solution. Assume that $x \in \overline{A}$. Let $B \in \mathscr{B}$ with $x \in B$. Since B is an open nbhd of x, it follows that $B \cap A \neq \varnothing$.

Now assume that $B \cap A \neq \varnothing$ for evey $B \in \mathscr{B}$ such that $x \in B$. Let U be any open nbhd of x. Then there is $B \in \mathscr{B}$ with $x \in B \subseteq U$. Since $B \cap A \neq \varnothing$, it follows that $U \cap A \neq \varnothing$. Thus $x \in \overline{A}$. □

1.4.14. Proposition (comparing topologies).

Let $\mathscr{T}$ and $\mathscr{T}'$ be topologies on the set X generated by bases $\mathscr{B}$ and $\mathscr{B}'$, respectively. Then $\mathscr{T}'$ is finer than $\mathscr{T}$ if and only if for every $B \in \mathscr{B}$ and every $x \in B$ there is $B' \in \mathscr{B}'$ with $x \in B' \subseteq B$.

Proof. Assume that $\mathscr{T} \subseteq \mathscr{T}'$. If $B \in \mathscr{B}$ and $x \in \mathscr{B}$, then $B \in \mathscr{T}$ so $B \in \mathscr{T}'$ and there is $B' \in \mathscr{B}'$ with $x \in B' \subseteq B$.

Assume that every $B \in \mathscr{B}$ and every $x \in B$ there is $B' \in \mathscr{B}'$ with $x \in B' \subseteq B$. Let $U \in \mathscr{T}$. For every $x \in U$, there is $B_x \in \mathscr{B}$ with $x \in B_x \subseteq U$. By assumption, there is $B'_x \subseteq B_x$ with $x \in B'_x$. Thus

$$U = \bigcup_{x \in U} B'_x \in \mathscr{T}'.$$

Therefore $\mathscr{T} \subseteq \mathscr{T}'$. □

1.4.15. Equivalent metrics.

Two metrics on the same set X are *equivalent* if they induce the same topology on X.

1.4.16. Proposition (equivalent metrics).

The metrics d and d' on a set X are equivalent if and only if for each $x \in X$ and each $\varepsilon > 0$ there are $\delta_1, \delta_2 > 0$ such that

$$B_d(x, \delta_1) \subseteq B_{d'}(x, \varepsilon)$$

and

$$B_{d'}(x, \delta_2) \subseteq B_d(x, \varepsilon).$$

Proof. Let $\mathscr{T}$ and $\mathscr{T}'$ be the topologies induced by d and d', respectively. Assume that d and d' are equivalent. Then $\mathscr{T} = \mathscr{T}'$. Let $x \in X$ and $\varepsilon > 0$. Since $B_{d'}(x, \varepsilon) \in \mathscr{T}'$, it follows that $B_{d'}(x, \varepsilon) \in \mathscr{T}$ so there is $\delta_1 > 0$ with

$$B_d(x, \delta_1) \subseteq B_{d'}(x, \varepsilon).$$

Similarly, there is $\delta_2 > 0$ with

$$B_{d'}(x, \delta_2) \subseteq B_d(x, \varepsilon).$$

Now assume that for each $x \in X$ and each $\varepsilon > 0$ there are $\delta_1, \delta_2 > 0$ such that

$$B_d(x, \delta_1) \subseteq B_{d'}(x, \varepsilon)$$

and

$$B_{d'}(x, \delta_2) \subseteq B_d(x, \varepsilon).$$

Let $U \in \mathscr{T}$. For each $x \in U$ there is $\varepsilon_x > 0$ with $B_d(x, \varepsilon_x) \subseteq U$. For each $x \in U$, let δ_x be such that

$$B_{d'}(x, \delta_x) \subseteq B_d(x, \varepsilon).$$

Then

$$U = \bigcup_{x \in U} B_{d'}(x, \delta_x) \in \mathscr{T}'.$$

Thus $\mathscr{T} \subseteq \mathscr{T}'$. Similarly $\mathscr{T}' \subseteq \mathscr{T}$. □

1.4.17. Corollary (bounded metric)

Let (X, d) be a metric space. For each $\lambda > 0$, there is a metric d_λ that is equivalent to d such that the diameter of X in d_λ is at most λ.

Proof. Define

$$d_\lambda(x, y) := \min\{\lambda, d(x, y)\}$$

for every $x, y \in X$. Then d_λ is a metric on X. Indeed, the positivity and symmetry of d_λ are clear and the triangle inequality holds since otherwise there are $x, y, z \in X$ with

$$d_\lambda(x, y) + d_\lambda(y, z) < d_\lambda(x, z)$$

and, since $d_\lambda(x, z) \leq d(x, z)$, this implies that that

$$d_\lambda(x, y) + d_\lambda(y, z) < d(x, z)$$

so at least one of $d_\lambda(x, y)$, $d_\lambda(y, z)$ must be equal λ and consequently $d_\lambda(x, z) > \lambda$, which is a contradiction. The diameter of X in d_λ, which is equal to

$$\sup\{d_\lambda(x, y) : x, y \in X\}$$

is at most λ since $d_\lambda(x,y) \leq \lambda$ for every $x, y \in X$.

It remains to show that the metrics d and d_λ are equivalent. Let $x \in X$ and $\varepsilon > 0$. Taking $\delta_1 := \varepsilon$ and $\delta_2 = \min\{\varepsilon, \lambda\}$, we get

$$B_d(x, \delta_1) \subseteq B_{d_\lambda}(x, \varepsilon),$$

since $d(x,y) < \delta_1 = \varepsilon$ implies that $d_\lambda(x,y) < \varepsilon$, and

$$B_{d_\lambda}(x, \delta_2) \subseteq B_d(x, \varepsilon),$$

since $d_\lambda(x,y) < \delta_2$ implies that $d_\lambda(x,y) < \lambda$ so $d_\lambda(x,y) = d(x,y)$ and so $d(x,y) < \varepsilon$. It follows that d and d_λ are equivalent. □

1.4.18. Local basis.

Let X be a topological space and $x \in X$. A *nbhd basis* (*local basis*) at x is a collection $\mathscr{B}_x$ of nbhds of x such that each nbhd of x contains a member of $\mathscr{B}_x$.

Examples.

The family of all open nbhds of x is a nbhd basis at x. In a discrete space, the family consisting of the singleton $\{x\}$ is a nbhd basis at x. In a metrics space the set $\{B(x,\varepsilon) : \varepsilon > 0\}$ is a nbhd basis at x.

1.4.19. Proposition (properties of nbhd basis).

Let X be a topological space and, for each $x \in X$, let $\mathscr{B}_x$ be a nbhd basis at x. Then the following conditions hold for every $x \in X$:

1. $\mathscr{B}_x \neq \varnothing$;
2. $x \in B$ for every $B \in \mathscr{B}_x$;
3. for every $B_1, B_2 \in \mathscr{B}_x$ there is $B \in \mathscr{B}_x$ with $B \subseteq B_1 \cap B_2$;

4. for each $B \in \mathscr{B}_x$ there is $B' \in \mathscr{B}_x$ such that B contains a member of $\mathscr{B}_y$ for every $y \in B'$.

Proof. Conditions 1.–3. follow easily from conditions 1.–3. of Proposition 1.2.8. We show that 4. holds. Let $B \in \mathscr{B}_x$. Since B is a nbhd of x, condition 5. of Proposition 1.2.8 implies that if

$$\mathring{B} := \{y \in B : B \text{ is a nbhd of } y\}$$

is a nbhd of x. By the definition of a nbhd basis, there is $B' \in \mathscr{B}_x$ with $B' \subseteq \mathring{B}$. Then for every $y \in B'$, the set B is a nbhd of y so contains a member of $\mathscr{B}_y$. □

1.4.20. Theorem (topology from nbhd basis).

Let X be a set and, for each $x \in X$, let $\mathscr{B}_x$ be a family of subsets of X satisfying conditions 1.–4. of Proposition 1.4.19. Then there exists a unique topology on X such that $\mathscr{B}_x$ is a nbhd basis at x for every $x \in X$.

Proof. First note that if $\mathscr{T}$ is a topology on X such that for each $x \in X$, the family$\mathscr{B}_x$ is a nbhd basis at x relative to $\mathscr{T}$, then a subset U of X is in $\mathscr{T}$ if and only if U contains a member of $\mathscr{B}_x$ for each $x \in U$. Thus such a topology $\mathscr{T}$ is unique provided it exists.

Define

$$\mathscr{T} := \{U \subseteq X : \text{for every } x \in U \text{ there is } B \in \mathscr{B}_x \text{ with } B \subseteq U\}.$$

We verify that $\mathscr{T}$ is a topology on X. We have $\varnothing \in \mathscr{T}$ since there are no $x \in \varnothing$. We have $X \in \mathscr{T}$ since for every $x \in X$ the set $\mathscr{B}_x$ is nonempty.

Let $\mathscr{A} \subseteq \mathscr{T}$ be arbitrary. We show that $\bigcup \mathscr{A} \in \mathscr{T}$. Let $x \in \bigcup \mathscr{A}$. Then $x \in U$ for some $U \in \mathscr{A}$ so there is $B \in \mathscr{B}_x$ with $B \subseteq U$ hence $B \subseteq \bigcup \mathscr{A}$.

Now let $U, V \in \mathscr{T}$. We show that $U \cap V \in \mathscr{T}$. Let $x \in U \cap V$. Then there are $B, D \in \mathscr{B}_x$ with $B \subseteq U$ and $D \subseteq V$. Let $G \in \mathscr{B}_x$ with $G \subseteq B \cap D$. Then $G \subseteq U \cap V$.

It remains to show that $\mathscr{B}_x$ is a nbhd basis at x for every $x \in X$ relative to $\mathscr{T}$. Let $x \in X$. First we check that any $B \in \mathscr{B}_x$ is a nbhd of x.

Given $B \in \mathscr{B}_x$, let

$$U := \{y \in B : \text{there is } D \in \mathscr{B}_y \text{ with } D \subseteq B\}.$$

Since $x \in U$, it suffices to verify that $U \in \mathscr{T}$. If $y \in U$, and $D \in \mathscr{B}_y$ with $D \subseteq B$, then 4. implies that there is $D' \in \mathscr{B}_y$ such that D contains a member of $\mathscr{B}_z$ for every $z \in D'$. Then B contains a member of $\mathscr{B}_z$ for every $z \in D'$, which implies that $D' \subseteq U$. Thus for every $y \in U$ there is $D' \in \mathscr{B}_y$ with $D' \subseteq U$. Hence $U \in \mathscr{T}$.

If N be a nbhd of x, then there is $U \in \mathscr{T}$ with $x \in U \subseteq N$ so there is $B \in \mathscr{B}_x$ with $B \subseteq U$. Thus $B \subseteq N$. □

1.4.21. Homework 7 (due 2/25).

Problem 1.

Let $\mathscr{S}$ be a subbasis for the topology of a space X and $D \subseteq X$ be such that $U \cap D \neq \varnothing$ for each $U \in \mathscr{S}$. Does it follow that D is dense in X? Give a proof or a counterexample.

Solution. No. Here is a counterexample. Let $X := \mathbb{R}$,

$$\mathscr{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$

and $D := \mathbb{Z}$. Then $\mathscr{S}$ is a subbasis for the standard topology on X and $U \cap D \neq \varnothing$ for each $U \in \mathscr{S}$. However, D is not dense in X. □

Problem 2.

Let (X, d) be a metric space. Show that the metric d', defined by

$$d'(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

is equivalent to d.

Solution. Let $x \in X$ and $\varepsilon > 0$. We find $\delta_1, \delta_2 > 0$ such that

$$B_d(x, \delta_1) \subseteq B_{d'}(x, \varepsilon) \qquad \text{and} \qquad B_{d'}(x, \delta_2) \subseteq B_d(x, \varepsilon).$$

Define $\delta_1 := \varepsilon$. If $y \in B_d(x, \delta_1)$, then $d(x,y) < \delta_1 = \varepsilon$ so $d'(x,y) \leq d(x,y) < \varepsilon$. Thus $y \in B_{d'}(x, \varepsilon)$ as required.
Define $\delta_2 := \dfrac{\varepsilon}{1+\varepsilon}$. If $y \in B_{d'}(x, \delta_2)$, then $d'(x,y) < \delta_2$ so

$$\frac{d(x,y)}{1+d(x,y)} < \frac{\varepsilon}{1+\varepsilon}$$
$$(1+\varepsilon)\, d(x,y) < \varepsilon\,(1 + d(x,y))$$
$$d(x,y) < \varepsilon$$

and $y \in B_d(x, \varepsilon)$ as required. □

Problem 3.

Let d and d' be metrics defined on the set $\mathscr{C}(I)$ of all continuous function $f : [0,1] \to \mathbb{R}$ defined by

$$d(f,g) := \int_0^1 |f(t) - g(t)|\, dt$$
$$d'(f,g) := \sup\{|f(t) - g(t)| : t \in [0,1]\}.$$

Prove that d and d' are not equivalent.

Solution. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) := \begin{cases} 1 - nx & x \in \left[0, \frac{1}{n}\right]; \\ 0 & x \in \left[\frac{1}{n}, 1\right]; \end{cases}$$

and let $A = \{f_n : n \in \mathbb{N}\} \subseteq \mathscr{C}(I)$. Let $f : [0,1] \to \mathbb{R}$ be the constant function with $f(x) := 0$ for every $x \in [0,1]$. Then f is in the closure of A when $\mathscr{C}(I)$ has the topology induced by the metric d, but f is not in the closure of A when $\mathscr{C}(I)$ has the topology induced by the metric d'. Thus these two topologies are not the same, which means that d and d' are not equivalent. □

Problem 4.

Consider $\mathbb{R}$ with the standard topology. Prove that for each $x \in \mathbb{R}$ the collection

$$\mathscr{B}_x := \{(x-r, x+r) : r \in \mathbb{Q},\, r > 0\}$$

is a nbhd basis at x.

Solution. Each member of $\mathscr{B}_x$ is open and contains x so it is a nbhd of x. Let N be any nbhd of x. There is open U with $x \in U \subseteq N$. There is an open interval $(a, b) \subseteq U$ with $x \in (a, b)$. Let $r \in \mathbb{Q}$ be such that

$$0 < r < \min\{x-a, b-x\}.$$

Then $x \in (x-r, x+r) \subseteq U$ and $(x-r, x+r) \in \mathscr{B}_x$. Therefore, $\mathscr{B}_x$ is a nbhd basis at x. □

1.5. Subspaces

1.5.1. Proposition (subspace topology).

Let $(X, \mathscr{T})$ be a topological space and $Y \subseteq X$. Let

$$\mathscr{T}' := \{U \cap Y : U \in \mathscr{T}\}.$$

Then $\mathscr{T}'$ is a topology on Y.

Proof. Since $\varnothing \in \mathscr{T}$, we have

$$\varnothing = \varnothing \cap Y \in \mathscr{T}'.$$

Since $X \in \mathscr{T}$ and $Y = X \cap Y$, it follows that $Y \in \mathscr{T}'$.

Let $\mathscr{A} \subseteq \mathscr{T}'$. Then for each $U \in \mathscr{A}$, there is $V_U \in \mathscr{T}$ with $U = Y \cap V_U$. Since

$$\bigcup_{U \in \mathscr{A}} V_U \in \mathscr{T}$$

and

$$\bigcup \mathscr{A} = \bigcup_{U \in \mathscr{A}} (Y \cap V_U) = Y \cap \left(\bigcup_{U \in \mathscr{A}} V_U \right),$$

it follows that $\bigcup \mathscr{A} \in \mathscr{T}'$.

Let $U, V \in \mathscr{T}'$. There are $U', V' \in \mathscr{T}$ with $U = Y \cap U'$ and $V = Y \cap V'$. Then

$$U \cap V = Y \cap (U' \cap V')$$

and $U' \cap V' \in \mathscr{T}$ so $U \cap V \in \mathscr{T}'$. □

1.5.2. Relative topology.

Let $(X, \mathscr{T})$ be a topological space and $Y \subseteq X$. The topology

$$\mathscr{T}' := \{U \cap Y : U \in \mathscr{T}\}$$

is called the *relative topology* (or *subspace topology*) on Y.

Remark.

Let $(Y, \mathscr{T}')$ be a subspace of a topological space $(X, \mathscr{T})$. If $Z \subseteq Y$ then the subspace topology on Z induced by $\mathscr{T}$ is the same as the subspace topology induced by $\mathscr{T}'$.

1.5.3. Proposition (closed sets in subspaces).

Let X be a topological space and $Y \subseteq X$. Consider Y as a topological space with the subspace topology. Then $C \subseteq Y$ is closed in Y if and only if there is a closed C' in X with $C = C' \cap Y$.

Proof. Assume that C is closed in Y. Then $Y \setminus C$ is open in Y so there is open U in X with

$$Y \setminus C = U \cap Y.$$

Let $C' := X \setminus U$. Then C' is closed in X and $C = C' \cap Y$.

Now assume that there is a closed C' in X with $C = C' \cap Y$. Then $X \setminus C'$ is open in X so

$$Y \setminus C = (X \setminus C') \cap Y$$

is open in Y implying that C is closed in Y. □

1.5.4. Proposition (relative metric induces relative topology).

Let (X, d) be a metric space and $Y \subseteq X$. Then the restriction d' of d to $Y \times Y$ is a metric that induces the relative topology on Y.

Proof. Let $\mathscr{T}$ be the topology on X induced by d, let $\mathscr{T}'$ be the corresponding subspace topology on Y. We show that $\mathscr{T}'$ is induced by the metric d'. Let $U \in \mathscr{T}'$ and $x \in U$. There is $V \in \mathscr{T}$ with $U = V \cap Y$. Since $x \in V$, there is $\varepsilon > 0$ with $B_d(x, \varepsilon) \subseteq V$. Then

$$B_{d'}(x, \varepsilon) = B_d(x, \varepsilon) \cap Y \subseteq U,$$

implying that U is open in the metric space (Y, d').

Now assume that U is open in the metric space (Y, d'). For each $x \in U$ there is $\varepsilon_x > 0$ with

$$B_{d'}(x, \varepsilon_x) \subseteq U.$$

Since

$$B_{d'}(x, \varepsilon_x) = B_d(x, \varepsilon_x) \cap Y,$$

it is open in the subspace topology on Y. Thus

$$U = \bigcup_{x \in U} B_{d'}(x, \varepsilon_x) \in \mathscr{T}'$$

as required. □

1.5.5. Proposition (closed and open subspaces).

Let X be a topological space and $Y \subseteq X$ be a subspace of X. If Y is open in X, then for any $A \subseteq Y$, the set A is open in Y if and only if it is open in X. If Y is closed in X, then any $A \subseteq Y$ is closed in Y if and only if it is closed in X.

Proof. Assume that Y is open in X. Let $A \subseteq Y$. If A is open in X, then $A = A \cap Y$ is open in Y. If A is open in Y, then $A = U \cap Y$ for some $U \subseteq X$ that is open in X. Then A is open in X.

If Y is closed in X, the proof is similar. □

1.5.6. Proposition (relative subbasis, basis and nbhd basis).

Let X be a topological space and $Y \subseteq X$ be a subspace.

1. If $\mathscr{S}$ is a subbasis of the topology on X then

$$\mathscr{S}' := \{S \cap Y : S \in \mathscr{S}\}$$

   is a subbasis for the topology on Y.

2. If $\mathscr{B}$ is a basis of the topology on X then

$$\mathscr{B}' := \{B \cap Y : B \in \mathscr{B}\}$$

   is a basis for the topology on Y.

3. If $\mathscr{B}_x$ is a nbhd basis at $x \in X$ in X and if $x \in Y$, then

$$\mathscr{B}'_x := \{B \cap Y : B \in \mathscr{B}_x\}$$

   is a nbhd basis at x in Y.

1.5.7. Proposition (relative closure and derived set).

Let X be a topological space and $Y \subseteq X$. For $A \subseteq Y$, let $\overline{A}_X$ and A'_X denote the closure and the derived set of A, respectively, relative to the topology on X and let $\overline{A}_Y$ and A'_Y denote the closure and the derived set of A, respectively, relative to the topology on Y. Then

$$\overline{A}_Y = \overline{A}_X \cap Y$$

and

$$A'_Y = A'_X \cap Y,$$

for every $A \subseteq Y$.

Proof. Let $A \subseteq Y$. Assume that $x \in \overline{A}_Y$. Then $x \in Y$. To show that $x \in \overline{A}_X$, let U be an open nbhd of x in X. Then $U' := U \cap Y$ is an open nbhd of x in Y so $U' \cap A \neq \varnothing$. It follows that $U \cap A \neq \varnothing$ as required.

Assume that $x \in \overline{A}_X \cap Y$. Let U be an open nbhd of x in Y. Then there is open U' in X with $U = U' \cap Y$. Since $x \in \overline{A}_X$, we have $U' \cap A \neq \varnothing$. Since $A \subseteq Y$, it follows that $U \cap A \neq \varnothing$. Thus $x \in \overline{A}_Y$.

The equality for derived sets is proved in a similar way. □

1.5.8. Proposition (relative interior and boundary).

Let X be a topological space and $Y \subseteq X$. For $A \subseteq Y$, let A°_X and ∂A_X denote the interior and the boundary of A, respectively, relative to the topology on X and let A°_Y and ∂A_Y denote the interior and the boundary of A, respectively, relative to the topology on Y. Then

$$A^\circ_Y \supseteq A^\circ_X \cap Y$$

and

$$\partial A_Y \subseteq \partial A_X \cap Y,$$

for every $A \subseteq Y$.

Proof. Let $A \subseteq Y$. The inclusion $A^\circ_Y \supseteq A^\circ_X \cap Y$ holds since if $x \in A^\circ_X \cap Y$, then there is an open U in X with $x \in U \subseteq A$. Since U is open in Y, it follows that $x \in A^\circ_Y$.

Let $x \in \partial A_Y$. Then $x \in \overline{A}_Y$ and $x \in Y$. Since

$$\overline{Y \setminus A_Y} = \overline{Y \setminus A}_X \cap Y \subseteq \overline{X \setminus A}_X \cap Y$$

and $x \in \overline{X \setminus A_X}$, it follows that $x \in \overline{Y \setminus A_Y}$. □

Example.

Let $X = \mathbb{R}$ and $Y = \{0\}$ with $A = \{0\}$. Then

$$A_Y^\circ = \{0\} \neq A_X^\circ \cap Y = \varnothing$$

and

$$\partial A_Y = \varnothing \neq \partial A_X \cap Y = \{0\}.$$

1.5.9. Proposition (relative linear order).

Let X be a topological space with an order topology induced by a linear order $\leq$ on X and let $Y \subseteq X$. Let $\mathscr{T}$ be the subspace topology on Y and $\mathscr{T}'$ be the order topology on Y induced by the restriction $\leq'$ of $\leq$ to $Y \times Y$. Then $\mathscr{T}' \subseteq \mathscr{T}$. If moreover Y is an interval, then equality holds.

Proof. Let $V \in \mathscr{T}'$. If $x \in V$ then there are $a, b \in Y \cup \{-\infty, \infty\}$ with

$$x \in V_x = (a, b)_Y \subseteq V,$$

where

$$(a, b)_Y := \{y \in Y : a < y < b\}.$$

Let

$$U_x := (a, b)_X := \{y \in X : a < y < b\}$$

and $U = \bigcup_{x \in V} U_x$. Then U is open in X. Since $V_x = U_x \cap Y$ for each $x \in V$, we have

$$V = \bigcup_{x \in V} V_x = \left(\bigcup_{x \in V} U_x \right) \cap Y = U \cap Y,$$

so $V \in \mathscr{T}$ as required.

Assume that Y is an interval and $V \in \mathscr{T}$. Let U be open in X with $V = U \cap Y$. If $x \in V$, then there are $a, b \in X \cup \{-\infty, \infty\}$ with $x \in (a, b)_X \subseteq U$. Define $a' := a$ if $a \in Y$ and $a' := -\infty$ otherwise. Let $b' := b$ if $b \in Y$ and $b := \infty$ otherwise. Since Y is an interval, it follows that

$$V_x := (a', b')_Y = (a, b)_X \cap Y \subseteq V,$$

so

$$V = \bigcup_{x \in V} V_x \in \mathscr{T}'$$

as required. □

Example.

Let

$$X := \{0\} \cup (1, 2) \subseteq \mathbb{R}.$$

Then $\{0\}$ is open in subspace topology on X induced from the topology on $\mathbb{R}$. However, if X is equipped with the order topology induced by restricting the linear order on $\mathbb{R}$ to X, then $\{0\}$ is not open.

1.5.10. Homework 8 (due 3/3)

Problem 1.

A subset Y of a topological space is called *discrete* if the relative topology on Y is discrete. Prove that every subset of a discrete space is discrete. Prove that the subset $\{1/n : n \in \mathbb{N}\}$ of the real line $\mathbb{R}$ with the standard topology is discrete and the subset $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ is not discrete.

Solution. Let X be a discrete topological space and $Y \subseteq X$. If $A \subseteq Y$, then A is open in X and $A = A \cap Y$ so A is open in Y. Since every subset of Y is open in Y, the relative topology on Y is discrete.

Let $A := \{1/n : n \in \mathbb{N}\} \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$, let $U_n := \left(\dfrac{1}{n+1}, \dfrac{1}{n-1}\right)$ provided $n \geq 2$ and $U_1 := \left(\dfrac{1}{2}, 2\right)$. For each $n \in \mathbb{N}$ the set U_n is open in $\mathbb{R}$ and $U_n \cap \mathbb{R} = \left\{\dfrac{1}{n}\right\}$. Thus $\left\{\dfrac{1}{n}\right\}$ is open in A for each $n \in \mathbb{N}$, which implies that the relative topology on A is discrete.

Let $B := \{0\} \cup \{1/n : n \in \mathbb{N}\} \subseteq \mathbb{R}$. To prove that the relative topology on B is not discrete, we show that the set $\{0\}$ is not open in the relative topology on B. Suppose, for a contradiction that $\{0\}$ is open in B. Then there is open U in $\mathbb{R}$ with $U \cap B = \{0\}$. Since U is open in $\mathbb{R}$, there are is an open interval $(a, b) \subseteq \mathbb{R}$ with $x \in (a, b) \subseteq U$. Since $b > 0$, there is $n \in \mathbb{N}$ with $\dfrac{1}{n} < b$. Thus $\dfrac{1}{n} \in U \cap B$, which is a contradiction. □

Problem 2.

Let $a, b \in \mathbb{R} \setminus \mathbb{Q}$ with $a < b$. Prove that $[a, b] \cap \mathbb{Q}$ is both open and closed in the relative topology on $\mathbb{Q}$.

Solution. Since $a, b \notin \mathbb{Q}$, we have $[a, b] \cap \mathbb{Q} = (a, b) \cap \mathbb{Q}$. Since (a, b) is open in $\mathbb{R}$, it follows that $(a, b) \cap \mathbb{Q}$ is open in the relative topology on $\mathbb{Q}$. Thus $[a, b] \cap \mathbb{Q}$ is open in the relative topology on $\mathbb{Q}$.

Since $[a, b]$ is closed in $\mathbb{R}$, it follows that $[a, b] \cap \mathbb{Q}$ is closed in the relative topology on $\mathbb{Q}$. □

Problem 3.

Let X be a topological space such that every finite subspace of X has the trivial relative topology. Prove that the topology on X is trivial.

Solution. Suppose, for a contradiction, that the topology of X is non-trivial. Let $U \subseteq X$ be open in X with $U \notin \{\varnothing, X\}$. Let $x \in U$ and $y \in X \setminus U$. Then $\{x\} = U \cap \{x, y\}$ so $\{x\}$ is open in the relative topology on $\{x, y\}$, which means that the relative topology on $\{x, y\}$ is not trivial. This is a contradiction. □

Problem 4.

Let X be a topological space such that every finite subspace of X has the discrete topology. Does it follow that X has the discrete topology? Give a proof or a counterexample.

Solution. No. Here is a counterexample. Let X be $\mathbb{R}$ with the standard topology. If $A \subseteq \mathbb{R}$ is finite, then the relative topology on A is discrete. However the topology on $\mathbb{R}$ is not discrete. $\square$

2. Continuity and the Product Topology

2.1. Continuous Functions

2.1.1. Definition of a continuous function.

Let X and Y be topological spaces and $f : X \to Y$. We say that f is *continuous* if $f^{-1}[U]$ is open in X for every open $U \subseteq Y$.

Examples.

If X is discrete, then any function with domain X into any topological space Y is continuous.

If Y has the trivial topology then any function $f : X \to Y$ for any topological space X is discrete.

If X and Y are any topological spaces and $f : X \to Y$ is constant, then it is continuous.

If X is a topological space and $f : X \to X$ is the identity function ($f(x) = x$ for each $x \in X$), then f is continuous.

2.1.2. Theorem (characterization of continuity).

Let X and Y be topological spaces and $f : X \to Y$. The following conditions are equivalent:

1. f is continuous.
2. $f^{-1}[C]$ is closed in X for any closed $C \subseteq Y$.
3. $f\left[\overline{A}\right] \subseteq \overline{f[A]}$ for any $A \subseteq X$.
4. $\overline{f^{-1}[B]} \subseteq f^{-1}\left[\overline{B}\right]$ for any $B \subseteq Y$.

Proof. Assume 1. and let $C \subseteq Y$ be closed. Then $Y \setminus C$ is open in Y so $f^{-1}[Y \setminus C]$ is open in X. Since

$$f^{-1}[C] = X \setminus f^{-1}[Y \setminus C],$$

it follows that $f^{-1}[C]$ is closed in X. Thus 2. holds. Similarly, 2. implies 1.

Now we show that 1. implies 3. Let $A \subseteq X$ and $y \in f\left[\overline{A}\right]$. Then $y = f(x)$ for some $x \in \overline{A}$. Let U be an open nbhd of y in Y. Then $f^{-1}[U]$ is an open nbhd of x so

$$f^{-1}[U] \cap A \neq \varnothing.$$

Thus there is $z \in A$ with $f(z) \in U$ so

$$f[A] \cap U \neq \varnothing.$$

It follows that $y \in \overline{f[A]}$ and so 3. holds.

Now we show that 3. implies 4. Let $B \subseteq Y$. With $A := f^{-1}[B]$, 3. implies that

$$f\left[\overline{f^{-1}[B]}\right] \subseteq \overline{f[f^{-1}[B]]}.$$

Since $f\left[f^{-1}[B]\right] \subseteq B$, it follows that $f\left[\overline{f^{-1}[B]}\right] \subseteq \overline{B}$, so

$$\overline{f^{-1}[B]} \subseteq f^{-1}\left[\overline{B}\right].$$

It remains to show that 4. implies 2. Let $C \subseteq Y$ be closed. Then 4. implies that

$$\overline{f^{-1}[C]} \subseteq f^{-1}\left[\overline{C}\right] = f^{-1}[C].$$

Since $f^{-1}[C] \subseteq \overline{f^{-1}[C]}$, it follows that $f^{-1}[C]$ is closed. Thus 2. holds. $\square$

2.1.3. Theorem (continuity and basis).

Let X and Y be topological spaces and $f : X \to Y$. Let $\mathscr{B}$ be a basis and $\mathscr{S}$ be a subbasis for the topology on Y. The following conditions are equivalent:

1. f is continuous.
2. $f^{-1}[B]$ is open in X for every $B \in \mathscr{B}$.
3. $f^{-1}[S]$ is open in X for every $S \in \mathscr{S}$.

Proof. Since every member of $\mathscr{B}$ and every member of $\mathscr{S}$ is open in Y, 1. implies both 2. and 3. If 2. holds, and U is open in Y, then $U = \bigcup \mathscr{A}$ for some $\mathscr{A} \subseteq \mathscr{B}$. Then

$$f^{-1}[U] = \bigcup_{B \in \mathscr{A}} f^{-1}[B]$$

is open in X so 1. holds.

Now assume 3. Let U be open in Y. If $U = Y$, then $f^{-1}[U] = X$ is open in X. If $U = \varnothing$, then $f^{-1}[U] = \varnothing$ is open in X. Otherwise, $U = \bigcup \mathscr{A}$ for some family $\mathscr{A}$ consisting of intersections of finite nonempty subfamilies of $\mathscr{S}$. Since

$$f^{-1}[U] = \bigcup_{A \in \mathscr{A}} f^{-1}[A],$$

it suffices to show that $f^{-1}[A]$ is open for every $A \in \mathscr{A}$. If

$$A := S_1 \cap S_2 \cap \cdots \cap S_k,$$

then

$$f^{-1}[A] = f^{-1}[S_1] \cap f^{-1}[S_2] \cap \cdots \cap f^{-1}[S_k],$$

which is an intersection of finitely many open sets in X. Thus $f^{-1}[A]$ is open in X. □

2.1.4. Theorem (composition of continuous functions).

Let X, Y, Z be topological space and $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Y$ is continuous.

Proof. Let U be open in Z. Then

$$(g \circ f)^{-1}[U] = g^{-1}\big[f^{-1}[U]\big].$$

Since f is continuous, $f^{-1}[U]$ is open in Y and since g is continuous, $g^{-1}\big[f^{-1}[U]\big]$ is open in X. Thus $g \circ f$ is continuous. □

2.1.5. Theorem (characterization of subspace topology).

Let X be a topological space and Y be a subset of X. The subspace topology on Y is the smallest topology on Y for which the embedding $j : Y \to X$ (with $j(y) := y$ for each $y \in Y$) is continuous.

Proof. If U is open in X, then $j^{-1}[U] = U \cap Y$ is open in the subspace topology on Y. Thus j is continuous. Assume that $\mathscr{T}$ is any topology on Y for which j is continuous. Then

$$U \cap Y = j^{-1}[U] \in \mathscr{T}$$

for any open $U \subseteq X$ so $\mathscr{T}$ is larger than the subspace topology on Y. □

2.1.6. Localized continuity.

Let X and Y be topological spaces and $f : X \to Y$. Then f is *continuous at* $x \in X$ provided $f^{-1}[U]$ is a nbhd of x for any nbhd U of $f(x)$.

2.1.7. Theorem (localized continuity).

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if f is continuous at each $x \in X$.

Proof. Assume that f is continuous and $x \in X$. If U is a nbhd of $f(x)$, then there is open U' in Y with $f(x) \in U' \subseteq U$. Since f is continuous, $f^{-1}[U']$ is open in X and

$$x \in f^{-1}[U'] \subseteq f^{-1}[U].$$

Thus $f^{-1}[U]$ is a nbhd of x in X.

Now assume that f is continuous at each $x \in X$. Let U be open in Y. For each $x \in f^{-1}[U]$, the set U is a nbhd of $f(x)$ so $f^{-1}[U]$ is nbhd of x. Thus $f^{-1}[U]$ is open in X. □

2.1.8. Theorem (Gluing Lemma).

Let X and Y be topological spaces with

$$X = \bigcup_{i=1}^{n} X_i,$$

where each X_i is open in X. If $f_i : X_i \to Y$ is continuous for each $i = 1, 2, \ldots, n$ and

$$f := \bigcup_{i=1}^{n} f_i$$

is a function, then $f : X \to Y$ is continuous. The same conclusion holds if we assume that each X_i is closed in X.

Proof. Let U be open in Y. Then

$$f^{-1}[U] = \bigcup_{i=1}^{n} f_i^{-1}[U]$$

and $f_i^{-1}[U]$ is open in X_i for each $i = 1, 2, \ldots, n$. Since X_i is open in X, it follows that $f_i^{-1}[U]$ is open in X for each $i = 1, 2, \ldots, n$. Thus $f^{-1}[U]$ is open in X.

Assuming that each X_i is closed in X, we use a similar argument starting with a closed subset of Y. □

Remark.

Let X and Y be topological spaces with $X = \bigcup_{i \in A} X_i$, where each X_i is open in X. If $f_i : X_i \to Y$ is continuous for each $i \in A$ and $f := \bigcup_{i \in A} f_i$ is a function, then $f : X \to Y$ is continuous.

Example.

Let $X := \mathbb{R}$ with $X_r := \{r\}$ for each $r \in \mathbb{R}$. Then each X_r is closed in X. If $f_r : X_r \to \mathbb{R}$ is defined by $f_r(r) := 1$ for $r \in \mathbb{Q}$ and $f_r(r) := 0$ for $r \in \mathbb{R} \setminus \mathbb{Q}$, then f_r is continuous for each $r \in \mathbb{R}$, but the functions $f := \bigcup_{i \in A} f_i$ is not continuous.

2.1.9. Locally finite family.

Let X be a topological space and $\mathscr{A}$ be a family of subsets of X. We say that $\mathscr{A}$ is *locally finite* when each $x \in X$ has a nbhd U such that

$$\{A \in \mathscr{A} : A \cap U \neq \varnothing\}$$

is finite.

2.1.10. Proposition (closure and locally finite family).

Let X be a topological space and $\mathscr{A}$ be a locally finite family of subsets of X. Then $\bigcup_{A \in \mathscr{A}} \overline{A}$ is closed.

Proof. Let

$$x \in \overline{\bigcup_{A \in \mathscr{A}} \overline{A}}.$$

There is an open nbhd U of x such that

$$\mathscr{A}' := \{A \in \mathscr{A} : A \cap U \neq \varnothing\}$$

is finite. Then $U \cap \overline{A} = \varnothing$ for any $A \in \mathscr{A} \setminus \mathscr{A}'$. Suppose, for a contradiction, that

$$x \notin \bigcup_{A \in \mathscr{A}} \overline{A}.$$

For each $A \in \mathscr{A}'$, let $U_A := X \setminus \overline{A}$. Then U_A is an open nbhd of x with $U_A \cap \overline{A} = \varnothing$. If

$$V := U \cap \bigcap_{A \in \mathscr{A}'} U_A,$$

then V is a nbhd of x such that

$$V \cap \bigcup_{A \in \mathscr{A}} \overline{A} = \varnothing,$$

which is a contradiction. □

Remark.

In particular, the union of a locally finite family of closed sets is closed.

2.1.11. Corollary (closure and locally finite family).

Let X and Y be topological spaces with $X = \bigcup_{i \in A} X_i$, where each X_i is closed in X and $\{X_i : i \in A\}$ is locally finite. If $f_i : X_i \to Y$ is continuous for each $i \in A$ and $f := \bigcup_{i \in A} f_i$ is a function, then $f : X \to Y$ is continuous.

Proof. Let C be closed in Y. Then $f_i^{-1}[C]$ is closed in X_i for each $i \in A$ so it is closed in X. Since $\{X_i : i \in A\}$ is locally finite, it follows that $\left\{f_i^{-1}[C] : i \in A\right\}$ is locally finite. Since

$$f^{-1}[C] = \bigcup_{i \in A} f_i^{-1}[C],$$

it follows that $f^{-1}[C]$ is closed. □

2.1.12. Homeomorphism.

Let X and Y be topological space. If $f : X \to Y$ is a bijection such that both f and f^{-1} are continuous, then f is a *homeomorphism*. If there exists a homeomorphism $X \to Y$, then we say that X and Y are *homeomorphic*.

Example.

The Euclidean space $\mathbb{R}^n$ is homeomorphic to the open ball $B(0,1)$ in $\mathbb{R}^n$. The map $f : \mathbb{R}^n \to B(0,1)$ defined by

$$f(x) := \frac{x}{1 + \|x\|}$$

is a homeomorphism.

2.1.13. Open and closed functions.

Let X and Y be topological spaces and $f : X \to Y$. We say that f is open if $f[U]$ is open in Y for every open U in X. We say that f is closed if $f[C]$ is closed in Y for every closed C in X.

Examples.

The inclusion function $f : (0,1) \to \mathbb{R}$ is continuous and open, but not closed.

The inclusion function $f : [0,1] \to \mathbb{R}$ is continuous and closed, but not open.

The function $f : [0, 2\pi) \to \mathbb{R}^2$ defined by $f(x) = \langle \cos x, \sin x \rangle$ is continuous, but it is neither open nor closed.

2.1.14. Theorem (characterization of homeomorphisms).

Let X and Y be topological spaces and $f : X \to Y$ be a bijection. The following conditions are equivalent:

1. f is a homeomorphism.
2. f is continuous and open.
3. f is continuous and closed.

Proof. Let $g := f^{-1}$. Assume 1. Then $g : Y \to X$ is continuous. If U is open in X, then $f[U] = g^{-1}[U]$ is open in X. Thus 2. hold. Similarly, we show that 3. holds.

Now assume that 2. holds. If U is open in X, then $g^{-1}[U] = f[U]$ is open in Y so 1. holds. Similarly, we show that 3. implies 1. $\square$

2.1.15. Proposition (characterization of closed functions).

Let X and Y be topological spaces and $f : X \to Y$. Then f is closed if and only if $\overline{f[A]} \subseteq f\left[\overline{A}\right]$ for every $A \subseteq X$.

Proof. Assume that f is closed. Let $A \subseteq X$. Since $f\left[\overline{A}\right]$ is closed and contains $f[A]$, it follows that $\overline{f[A]} \subseteq f\left[\overline{A}\right]$.

Now assume that $\overline{f[A]} \subseteq f\left[\overline{A}\right]$ for every $A \subseteq X$. Let C be closed in X. Then

$$\overline{f[C]} \subseteq f\left[\overline{C}\right] = f[C].$$

Since $f[C] \subseteq \overline{f[C]}$, it follows that equality holds so $f[C]$ is closed. $\square$

Corollary.

A function $f : X \to Y$ is continuous and closed if and only if $f\left[\overline{A}\right] = \overline{f[A]}$ for any $A \subseteq X$.

2.1.16. Theorem (characterization of open functions).

Let X and Y be topological spaces with $\mathscr{B}$ being a basis for the topology on X and $f : X \to Y$. Then f is open if and only if $f[B]$ is open in Y for every $B \in \mathscr{B}$.

Proof. Assume that f is open. Since the members of $\mathscr{B}$ are open, the image $f[B]$ is open in Y for every $B \in \mathscr{B}$.

Now assume that $f[B]$ is open in Y for every $B \in \mathscr{B}$. Let U be open in X. Then $U = \bigcup \mathscr{A}$ for some $\mathscr{A} \subseteq \mathscr{B}$. Since

$$f[U] = \bigcup_{A \in \mathscr{A}} f[A]$$

and $f[A]$ is open in Y for every $A \in \mathscr{A}$, it follows that $f[U]$ is open in Y. Thus f is open. $\square$

2.1.17. Topological embedding.

Let X and Y be topological spaces and $f : X \to Y$. The function f is a *topological embedding* if it is a homeomorphism onto $f[X]$.

Remark.

f is a topological embedding provided it is injective, continuous and it's inverse as a function $f[X] \to X$ is continuous.

Example.

The function $f : [0, 2\pi) \to \mathbb{R}^2$ defined by $f(x) = \langle \cos x, \sin x \rangle$ is injective and continuous, but it is not a topological embedding.

2.1.18. Homework 9 (due 4/7)

Problem 1.

Let X be an uncountable set with the cofinite (or cocountable) topology. Show that every continuous function $X \to \mathbb{R}$ is constant.

Problem 2.

Give an example of topological spaces X, Y a function $f : X \to Y$ and a subspace $A \subseteq X$ such that $f\restriction A$ is continuous, although f is not continuous at any point of A.

Problem 3.

Let X be a partially ordered set. Define a topology on X be declaring $U \subseteq X$ to be open if it satisfies the condition: if $y \leq x$ and $x \in U$, then $y \in U$. Show that a function $f : X \to X$ is continuous if and only if it is order preserving (i.e., $x \leq x'$ implies that $f(x) \leq f(x')$).

Problem 4.

Prove that the addition function $\mathbb{R}^2 \to \mathbb{R}$ is open, but is not closed.

2.2. Product Spaces

2.2.1. Proposition (basis for product topology)

Let X and Y be topological spaces and

$$\mathscr{B} := \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

Then $\mathscr{B}$ is a basis for a topology on $X \times Y$.

Proof. According to Theorem 1.4.12, need to verify that $\bigcup \mathscr{B} = X \times Y$ and for every $B_1, B_2 \in \mathscr{B}$ and every $z \in B_1 \cap B_2$ there is $B \in \mathscr{B}$ with $z \in B \subseteq B_1 \cap B_2$. Since $X \times Y \in \mathscr{B}$, it follows that $\bigcup \mathscr{B} = X \times Y$.

Let $B_1, B_2 \in \mathscr{B}$ with $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$. Then $\langle x, y\rangle \in B_1 \cap B_2$ if and only if $x \in U_1 \cap U_2$ and $y \in V_1 \cap V_2$ so

$$B_1 \cap B_2 = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathscr{B}.$$

Thus we can take $B = B_1 \cap B_2$ for any $z \in B_1 \cap B_2$. $\square$

2.2.2. Product topology.

Let X and Y be topological spaces. The topology induced by the basis

$$\mathscr{B} := \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

is called the *product topology* on $X \times Y$ and the obtained topological space is called the *product space*.

The functions $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ defined by $p_X(x, y) := x$ and $p_Y(x, y) := y$ are called *projections*.

2.2.3. Proposition (characterization of product topology).

Let X and Y be topological spaces and $Z := X \times Y$ be the product space. The projections $p_X : Z \to X$ and $p_Y : Z \to Y$ are continuous and open. Moreover, the product topology on $X \times Y$ is the smallest topology for which both p_X and p_Y are continuous.

Proof. Let

$$\mathscr{B} := \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

Since $p_X^{-1}[U \times V] = U$ is open in X for every $U \times V \in \mathscr{B}$, it follows that p_X is continuous. Similarly, p_Y is continuous. Theorem 2.1.16 implies that both p_X and p_Y are open.

Assume that $\mathscr{T}$ is any topology on $X \times Y$ for which both p_X and p_Y are continuous. If U is open in X and V is open in Y, then

$$U \times V = (U \times Y) \cap (X \times V) = p_X^{-1}[U] \cap p_Y^{-1}[V]$$

belongs to $\mathscr{T}$. Thus $\mathscr{B} \subseteq \mathscr{T}$, which implies that $\mathscr{T}$ is finer than the product topology on $X \times Y$. $\square$

Example.

Let $X = Y := \mathbb{R}$ and

$$C := \{\langle x, y\rangle \in X \times Y : xy = 1\}.$$

Then C is closed in $X \times Y$, but $p_X[C] = X \setminus \{0\}$ is not closed in X.

Remark.

If X and Y are topological spaces and

$$\mathscr{S} := \left\{p_X^{-1}[U] : U \text{ is open in } X\right\} \cup \left\{p_Y^{-1}[V] : V \text{ is open in } Y\right\},$$

then $\mathscr{S}$ is a subbasis for the product topology on $X \times Y$.

2.2.4. Proposition (basis for product topology from bases).

Let $\mathscr{B}$ and $\mathscr{D}$ be bases for the topologies on X and Y, respectively. Then

$$\mathscr{E} := \{B \times D : B \in \mathscr{B},\, D \in \mathscr{D}\}$$

is a basis for the product topology on $X \times Y$.

Proof. It is clear that the members of $\mathscr{E}$ are open in the product topology on $X \times Y$. Let W be open in the product topology on $X \times Y$ and $\langle x, y\rangle \in W$. There is an open U in X and an open V in Y with

$$\langle x, y\rangle \in U \times V \subseteq W.$$

Then $x \in B \subseteq U$ and $y \in D \subseteq V$ for some $B \in \mathscr{B}$ and $D \in \mathscr{D}$ so

$$\langle x, y\rangle \in B \times D \subseteq U \times V.$$

Thus $B \times D \subseteq W$ as required. □

2.2.5. Proposition (product topology on $\mathbb{R}^2$).

The standard topology on $\mathbb{R}^2$ is the product topology on $\mathbb{R} \times \mathbb{R}$.

Proof. Let

$$\mathscr{B} := \{B(\langle x, y\rangle, r) : \langle x, y\rangle \in \mathbb{R} \times \mathbb{R},\ r > 0\}$$

and

$$\mathscr{D} := \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R},\ a < b,\ c < d\}.$$

Then $\mathscr{B}$ is a basis of the standard topology on $\mathbb{R}^2$ and $\mathscr{D}$ is a basis for the product topology on $\mathbb{R} \times \mathbb{R}$. Let $B \in \mathscr{B}$ and $\langle x, y\rangle \in B$. There are open intervals (a, b) and (c, d) with

$$\langle x, y\rangle \in (a, b) \times (c, d) \subseteq B.$$

Since $(a, b) \times (c, d) \in \mathscr{D}$, Proposition 1.4.14 implies that the product topology on $\mathbb{R} \times \mathbb{R}$ is finer than the standard topology. Similarly, the standard topology is finer than the product topology. □

Remark.

If X, Y and Z are topological spaces then $X \times Y$ is homeomorphic with $Y \times X$ and $(X \times Y) \times Z$ is homeomorphic with $X \times (Y \times Z)$.

2.2.6. Theorem (product and subspace topologies commute).

Let X and Y be topological spaces with $A \subseteq X$ and $B \subseteq Y$. Let $\mathscr{T}$ be the subspace topology on $A \times B$ inherited from the product topology on $X \times Y$ and $\mathscr{T}'$ be the product topology on $A \times B$, where A has the subspace topology inherited from X and B has the subspace topology inherited from Y. Then $\mathscr{T} = \mathscr{T}'$.

Proof. The family

$$\mathscr{B} := \{(A \times B) \cap (U \times V) : U \text{ open in } X \text{ and } V \text{ open in } Y\}$$

is a basis for $\mathscr{T}$ and

$$\mathscr{B}' := \{(A \cap U) \times (B \cap V) : U \text{ open in } X \text{ and } V \text{ open in } Y\}$$

is a basis for $\mathscr{T}'$. Since

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$$

for any $U \subseteq X$ and $V \subseteq Y$, it follows that $\mathscr{B} = \mathscr{B}'$ so $\mathscr{T} = \mathscr{T}'$. $\square$

2.2.7. Theorem (continuity into products).

Let X, Y and Z be topological spaces and $f : Z \to X \times Y$. Then f is continuous if and only if both compositions $p_X \circ f$ and $p_Y \circ f$ are continuous.

Proof. If f is continuous, then both $p_X \circ f$ and $p_Y \circ f$ are continuous since compositions of continuous functions is continuous.

Assume that both $p_X \circ f$ and $p_Y \circ f$ are continuous. To show that f is continuous it suffices to prove that $f^{-1}[U \times V]$ is open in Z for any open $U \subseteq X$ and any open $V \subseteq Y$. Assume that U is open in X and V is open in Y. Note that $z \in f^{-1}[U \times V]$ if and only if $(p_X \circ f)(z) \in U$ and $(p_Y \circ f)(z) \in V$ so

$$f^{-1}[U \times V] = (p_X \circ f)^{-1}[U] \cap (p_Y \circ f)^{-1}[V].$$

Since both $(p_X \circ f)^{-1}[U]$ and $(p_Y \circ f)^{-1}[V]$ are open in Z, it follows that $f^{-1}[U \times V]$ is open in Z, as required. $\square$

Corollary.

Let X, X', Y and Y' be topological spaces and $f : X \to X'$ and $g : Y \to Y'$ be continuous. Let

$$f \times g : X \times Y \to X' \times Y'$$

be defined by

$$(f \times g)(x, y) := (f(x), g(y))$$

for any $x \in X$ and $y \in Y$. Then $f \times g$ is continuous.

Proof. It suffices to show that both $p_{X'} \circ (f \times g)$ and $p_{Y'} \circ (f \times g)$ are continuous. Since

$$p_{X'} \circ (f \times g) = f \circ p_X,$$

and since both f and p_X are continuous, it follows that $p_{X'} \circ (f \times g)$ is continuous. Similarly, $p_{Y'} \circ (f \times g)$ is continuous. □

Corollary.

Let X, Y and Z be topological spaces and $f : Z \to X$ and $g : Z \to Y$ be continuous. Let

$$(f, g) : Z \to X \times Y$$

be defined by

$$(f, g)(z) := (f(z), g(z))$$

for any $z \in Z$. Then (f, g) is continuous.

Proof. Since $p_X \circ (f, g) = f$ and $p_Y \circ (f, g) = g$ are continuous, it follows that (f, g) is continuous. □

2.2.8. Infinite Cartesian products.

Let X_α be a set for any $\alpha \in A$. The *Cartesian product* $X := \prod_{\alpha \in A} X_\alpha$ is the set of all functions f with domain A such that $f(\alpha) \in X_\alpha$ for any $\alpha \in A$. Such a function f will be denoted by $(x_\alpha)_{\alpha \in A}$, where x_α is the value of f at α.

For each $\alpha \in A$, let $p_\alpha : X \to X_\alpha$ be the projection defined by

$$p_\alpha\Big((x_\beta)_{\beta \in A}\Big) := x_\alpha.$$

2.2.9. Box topology.

Let X_α be a topological space for each $\alpha \in A$ and $X := \prod_{\alpha \in A} X_\alpha$. The collection

$$\mathscr{B} := \left\{ \prod_{\alpha \in A} U_\alpha : U_\alpha \text{ open in } X_\alpha \text{ for every } \alpha \in A \right\}$$

can be proved to be a basis for a topology on X. This topology is called the *box topology* on X.

Example.

Let $X_n := \mathbb{R}$ for every $n \in \mathbb{N}$ and $X := \prod_{n\in\mathbb{N}} X_n$ with the box topology. Let $f_n : \mathbb{R} \to X_n$ be the identity function and $f : \mathbb{R} \to X$ be defined by

$$f(x) = (x, x, x, \dots).$$

Then $f_n = p_n \circ f$ is continuous for every $n \in \mathbb{N}$. However, f is not continuous.

Proof. Let $U_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$ for each $n \in \mathbb{N}$. Then $U := \prod_{n\in\mathbb{N}} U_n$ is open in the box topology on X, but $f^{-1}[U] = \{0\}$ is not open in $\mathbb{R}$. □

2.2.10. Product topology.

Let X_α be a topological space for each $\alpha \in A$ and let $X := \prod_{\alpha\in A} X_\alpha$. The *product topology* (or *Tychonoff topology*) on X is induced by the subbasis

$$\mathscr{S} := \left\{p_\alpha^{-1}[U_\alpha] : U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in A\right\}.$$

2.2.11. Proposition (characterization of infinite products).

Let X_α be a topological space for each $\alpha \in A$ and let $X := \prod_{\alpha\in A} X_\alpha$. The product topology on X is the smallest topology on X for which each projection p_α is continuous.

Proof. If X has the product topology, then each p_α is continuous since each member of the subbasis

$$\mathscr{S} := \left\{p_\alpha^{-1}[U_\alpha] : U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in A\right\}$$

is open in the product topology on X.

Assume that $\mathscr{T}$ is a topology on X such that $p_\alpha : X \to X_\alpha$ is continuous for each $\alpha \in A$. Then $\mathscr{S} \subseteq \mathscr{T}$ so $\mathscr{T}$ is finer than the product topology on X. □

Remark.

The box topology on X is finer than the product topology and when A is finite, these topologies are identical.

2.2.12. Proposition (subbasis for infinite product).

Let X_α be a topological space for each $\alpha \in A$ and let $X := \prod_{\alpha\in A} X_\alpha$. If $\mathscr{S}_\alpha$ is a subbasis for the topology on X_α for every $\alpha \in A$, then

$$\mathscr{S} := \left\{p_\alpha^{-1}[S_\alpha] : S_\alpha \in \mathscr{S}_\alpha \text{ for every } \alpha \in A\right\}$$

is a subbasis for the product topology on X.

Proof. Let

$$\mathscr{S}' := \left\{p_\alpha^{-1}[U_\alpha] : U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in A\right\},$$

with $\mathscr{T}$ being the topology induced by $\mathscr{S}$ and $\mathscr{T}'$ being the product topology (induced by $\mathscr{S}'$). Since $\mathscr{S} \subseteq \mathscr{S}'$, it follows that $\mathscr{T} \subseteq \mathscr{T}'$. To show that $\mathscr{T}' \subseteq \mathscr{T}$, it suffices to show that $\mathscr{S}' \subseteq \mathscr{T}$.

Assume that $S \in \mathscr{S}'$. Then $S = p_\alpha^{-1}(U_\alpha)$ for some $\alpha \in A$ and U_α open in X_α. If $U_\alpha = X_\alpha$, then $S = X \in \mathscr{T}$. If $U_\alpha = \varnothing$, then $S = \varnothing \in \mathscr{T}$. Otherwise, U_α is a union of a family $\mathscr{A}$ of nonempty finite intersections of members of $\mathscr{S}_\alpha$ so

$$S = p_\alpha^{-1}\left[\bigcup \mathscr{A}\right] = \bigcup_{A\in\mathscr{A}} p_\alpha^{-1}[A].$$

For $A \in \mathscr{A}$, if

$$A = S_1 \cap S_2 \cap \cdots \cap S_n,$$

where $S_1, \ldots, S_n \in \mathscr{S}$, then

$$p_\alpha^{-1}[A] = p_\alpha^{-1}[S_1] \cap p_\alpha^{-1}[S_2] \cap \cdots \cap p_\alpha^{-1}[S_n] \in \mathscr{T}.$$

Since $p_\alpha^{-1}[A] \in \mathscr{T}$ for every $A \in \mathscr{A}$, it follows that $S \in \mathscr{T}$, as required. □

2.2.13. Proposition (basis for infinite product).

Let X_α be a topological space for each $\alpha \in A$ and let $X := \prod_{\alpha \in A} X_\alpha$. Assume that $\mathscr{B}_\alpha$ is a subbasis for the topology on X_α for every $\alpha \in A$, and let $\mathscr{B}$ consist of products $\prod_{\alpha \in A} B_\alpha$ such that there is a finite $A' \subseteq A$ with $B_\alpha \in \mathscr{B}_\alpha$ for $\alpha \in A'$ and $B_\alpha = X_\alpha$ for $\alpha \in A \setminus A'$. Then $\mathscr{B}$ is a basis for the product topology on X.

Proof. Since $\mathscr{B}_\alpha$ is a subbasis for the topology on X_α for every $\alpha \in A$, Proposition 2.2.12 implies that the family

$$\mathscr{S} := \left\{ p_\alpha^{-1}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha \text{ for every } \alpha \in A \right\}$$

is a subbasis for the product topology on X. Since $\mathscr{B}$ consists of X and all intersections of finite nonempty subfamilies of $\mathscr{S}$, it follows that $\mathscr{B}$ is a basis for the topology on X. □

2.2.14. Proposition (infinite products and subspaces commute).

Let X_α be a topological space for each $\alpha \in A$ and let $X := \prod_{\alpha \in A} X_\alpha$. For each $\alpha \in A$ let Y_α be a subspace of X_α and let $Y := \prod_{\alpha \in A} Y_\alpha$. Then the product topology on Y coincides with the subspace topology inherited from X.

Proof. Let $\mathscr{T}$ be the product topology on Y and $\mathscr{T}'$ be the subspace topology. Let $\mathscr{S}_\alpha$ be a subbasis for the topology on X_α for all $\alpha \in A$. Then

$$\mathscr{S} := \left\{ p_\alpha^{-1}[S_\alpha] : S_\alpha \in \mathscr{B}_\alpha \text{ for every } \alpha \in A \right\}$$

is a subbasis for the product topology on X so

$$\mathscr{S}' := \left\{ p_\alpha^{-1}[S_\alpha] \cap Y : S_\alpha \in \mathscr{B}_\alpha \text{ for every } \alpha \in A \right\}$$

is a subbasis for $\mathscr{T}$. Since

$$p_\alpha^{-1}[S_\alpha] \cap Y = \left(p_\alpha \restriction Y\right)^{-1} [S_\alpha \cap Y_\alpha]$$

and since

$$\mathscr{S}'_\alpha := \{S_\alpha \cap Y_\alpha : \alpha \in A\}$$

is a subbasis for the topology on X_α for each $\alpha \in A$, it follows the $\mathscr{S}'$ is a subbasis for $\mathscr{T}'$. □

2.2.15. Theorem (infinite products and continuity).

Let X_α be a topological space for each $\alpha \in A$ and let $X := \prod_{\alpha \in A} X_\alpha$. Let $f : Y \to X$ for some topological space Y. Then f is continuous if and only if $p_\alpha \circ f : Y \to X_\alpha$ is continuous is continuous for each $\alpha \in A$. Moreover, the product topology on X is the unique topology with such a property.

Proof. If f is continuous, then it is clear that $p_\alpha \circ f$ is continuous for each $\alpha \in A$. Assume that $p_\alpha \circ f$ is continuous for each $\alpha \in A$. Since

$$\mathscr{S} := \left\{p_\alpha^{-1}[U_\alpha] : U_\alpha \text{ is open in } X_\alpha \text{ for each } \alpha \in A\right\}$$

is a subbasis for the product topology on X, and since

$$f^{-1}\left[p_\alpha^{-1}[U_\alpha]\right] = (p_\alpha \circ f)^{-1}\,[U_\alpha]$$

is open in Y for each $\alpha \in A$, it follows that f is continuous.

Suppose that $\mathscr{T}$ is any topology on X such that for any topological space Y and any $f : Y \to X$, the continuity of $p_\alpha \circ f$ for each $\alpha \in A$ is equivalent to the continuity of f. Taking $Y := X$ with the same topology $\mathscr{T}$ and f to be the identity function (which is continuous), we conclude that p_α is continuous for each $\alpha \in A$. This implies that $\mathscr{T}$ is finer than the product topology on X by Proposition 2.2.11.

Taking $Y := X$ with the product topology (and X with topology $\mathscr{T}$) and $f : Y \to X$ to be the identity function, we have $p_\alpha \circ f$ continuous for each $\alpha \in A$ so f is continuous. It follows that any member of $\mathscr{T}$ is open in Y, thus $\mathscr{T}$ is coarser than the product topology on X. □

2.2.16. Theorem (countable products are metrizable).

Let (X_n, d_n) be a metric space for each $n \in \mathbb{N}$ and let $X := \prod_{n\in\mathbb{N}} X_n$. Consider X_n to be the topological space with the topology induced by d_n. Then there exists a metric d on X that induces the product topology on X.

Proof. For each $n \in \mathbb{N}$, let $\lambda_n > 0$ with $\lim_{n\to\infty} \lambda_n = 0$ and let d'_n be a metric on X_n such that the diameter of X_n in d'_n is at most λ_n. Such a metric d'_n exists by Corollary 1.4.17. For $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$ in X, define

$$d(x, y) = \sup \{d'_n(x_n, y_n) : n \in \mathbb{N}\}.$$

It is clear that d is positive and symmetric. We verify the triangle inequality. Let $x := (x_n)_{n\in\mathbb{N}}$, $y := (y_n)_{n\in\mathbb{N}}$ and $z := (z_n)_{n\in\mathbb{N}}$ be in X. Then

$$d'_n(x_n, y_n) + d'_n(y_n, z_n) \geq d'_n(x_n, z_n)$$

for each $n \in \mathbb{N}$ so

$$d(x, y) + d(y, z) \geq d'_n(x_n, z_n)$$

for each $n \in \mathbb{N}$. Thus

$$d(x, y) + d(y, z) \geq d(x, z),$$

as required. Thus d is a metric on X.

Now we show that d induces the product topology on X. Let $n \in \mathbb{N}$ and U_n be open in X_n. For each

$$x = (x_k)_{k\in\mathbb{N}} \in p_n^{-1}[U_n]$$

there is $r_x > 0$ such that

$$x \in B_{d'_n}(x_n, r_x) \subseteq U_n.$$

If

$$y = (y_k)_{k\in\mathbb{N}} \in B_d(x, r_x),$$

then $d(x, y) < r_x$ so $d(x_k, y_k) < r_x$ for each $k \in \mathbb{N}$ and, in particular, $d(x_n, y_n) < r_x$ so $y_n \in U_n$ and consequently $y \in p_n^{-1}[U_n]$. Thus

$$B_d(x, r_x) \subseteq p_n^{-1}[U_n],$$

which implies that $p_n^{-1}[U_n]$ is open in the topology induced by d. Since

$$\mathscr{S} := \left\{p_n^{-1}[U_n] : U_n \text{ is open in } X_n \text{ for each } n \in \mathbb{N}\right\}$$

is a subbasis for the product topology on X, it follows that the topology induced by d is finer than the product topology.

Let U be open in the topology induced by d. We will show that U is open in the product topology. Let $x := (x_k)_{k\in\mathbb{N}} \in U$. There is $r > 0$ such that $B_d(x, r) \subseteq U$. Let $n \in \mathbb{N}$ be such that $\lambda_k < r/2$ for each $k > n$. For each $k = 1, 2, \ldots, n$ let

$$U_k := B_{d'_k}(x_k, r)$$

and for $k > n$, let $U_k := X_k$. Then

$$U' := \prod_{k=1}^{\infty} U_k$$

is open in the product topology and

$$x \in U' \subseteq B_d(x, r) \subseteq U,$$

which implies that U is open in the product topology. □

2.2.17. Metrizable spaces.

A topological space X is *metrizable* if there exists a metric d on X that induces the given topology.

Remark.

We have proved that the product of countably many metrizable spaces is metrizable.

Example.

Let A be an uncountable set and $X_\alpha := \mathbb{R}$ for each $\alpha \in A$. Then $X := \prod_{\alpha \in A} X_\alpha$ is not metrizable.

Proof. Suppose, for a contradiction, that d is a metric on X that induces the product topology. For each $n \in \mathbb{N}$ let B_n be the open ball $B_d(0, 1/n)$, where $0 \in X$ is the constant function with value 0, and let $A_n \subseteq A$ be finite such that

$$0 \in U_n := \prod_{\alpha \in A} U_{n,\alpha} \subseteq B_n,$$

where $U_{n,\alpha}$ is an open interval in $\mathbb{R}$ for all $\alpha \in A_n$ and $U_{n,\alpha} = \mathbb{R}$ for all $\alpha \in A \setminus A_n$. Since $\bigcup_{n \in \mathbb{N}} A_n$ is countable and A is not, there is

$$\beta \in A \setminus \bigcup_{n \in \mathbb{N}} A_n.$$

Let

$$x := (x_\alpha)_{\alpha \in A} \in X,$$

with $x_\beta := 1$ and $x_\alpha := 0$ for $\alpha \in A \setminus \{\beta\}$. Then $x \in U_n$ for each $n \in \mathbb{N}$. Since $x \neq 0$ and since

$$\bigcap_{n \in \mathbb{N}} U_n \subseteq \bigcap_{n \in \mathbb{N}} B_n = \{0\},$$

we have a contradiction. □

2.2.18. Exercises.

3. Connectedness

3.1. Connected Spaces

3.1.1. Separation.

Let X be a topological space. A *separation* of X is a pair $\{A, B\}$ of nonempty disjoint open subsets of X with $A \cup B = X$.

Remark.

If $\{A, B\}$ is a separation of X, then both A and B are closed.

3.1.2. Definition of connected spaces.

A topological space X is *connected* if it has no separation, otherwise is it is *disconnected*.

Examples.

The Sierpiński space is connected. The infinite space with cofinite topology is connected. Any trivial space is connected. A discrete space with more than one point is disconnected.

3.1.3. Theorem (connectedness and functions into discrete).

A topological space X is connected if and only if any continuous function from X to a discrete space is constant.

Proof. Assume that X is connected and Y is a discrete space. Suppose, for a contradiction, that $f : X \to Y$ is continuous and not constant. Then there are $x, x' \in X$ with $f(x) \neq f(x')$. Since Y is discrete, the sets $U := \{f(x)\}$ and $V := Y \setminus \{f(x)\}$ are open. Then $\left\{f^{-1}[U], f^{-1}[V]\right\}$ is a separation of X and we get a contradiction.

Now assume that any continuous function from X to a discrete space is constant. Suppose, for a contradiction, that $\{A, B\}$ is a separation of X. Let $Y := \{0, 1\}$ have the discrete topology. Define $f(x) := 0$ for $x \in A$ and $f(x) := 1$ for $x \in B$. Then f is continuous, but not constant, which is a contradiction. □

3.1.4. Connected subsets.

A subset Y of a topological space X is *connected* if it is connected as a topological space with the subspace topology.

Remark.

A subset Y of a topological space X is connected if and only if there are no open subsets $A, B \subseteq X$ such that

1. $Y \subseteq A \cup B$,
2. $A \cap Y \neq \varnothing \neq B \cap Y$, and
3. $A \cap B \cap Y = \varnothing$.

3.1.5. Theorem (connected subsets of $\mathbb{R}$).

A subset Y of $\mathbb{R}$ is connected if and only if Y is an interval.

Proof. Assume that Y is not an interval. Then there are $a, b \in Y$ and $c \in (a, b) \setminus Y$. With $A := (-\infty, c)$ and $B := (c, \infty)$, the pair $\{A \cap Y, B \cap Y\}$ is a separation of Y so Y is disconnected.

Now assume that Y is disconnected. Let A, B be open in $\mathbb{R}$ and such that $\{A \cap Y, B \cap Y\}$ is a separation of Y. Let $a \in A \cap Y$ and $b \in B \cap Y$. Without loss of generality, we can assume that $a < b$. Let

$$c := \sup \{x \in A : x < b\}.$$

Since B is open in $\mathbb{R}$ and since $b \in B$, it follows that $c \notin B$. Since A is open, it follows that $c \notin A$. Thus $c \notin Y$ and since $a, b \in Y$ and $a < c < b$, it follows that Y is not an interval. □

3.1.6. Separated subsets.

Let X be a topological space and $A, B \subseteq X$. We say that A and B are *separated* if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$.

3.1.7. Proposition (connectedness and separated subsets).

Let X be a topological space and $Y \subseteq X$. Then Y is connected if and only if Y is not a union of two nonempty separated subsets of X.

Proof. Assume that Y is connected. Suppose, for a contradiction, that $Y = A \cup B$, where A, B are nonempty and separated subsets of X. Let $U := X \setminus \overline{B}$ and $V := X \setminus \overline{A}$. Then $\{U \cap Y, V \cap Y\}$ is a separation of Y, which is a contradiction.

Assume that Y is disconnected. Let U, V be open in X and such that $\{A, B\}$ is a separation of Y, where $A := U \cap Y$ and $B := V \cap Y$. Proposition 1.5.7 implies that the closure of A in Y is equal to $\overline{A} \cap Y$. Since A is closed in Y, it follows that $\overline{A} \cap Y = A$ so $\overline{A} \cap B = \varnothing$. Similarly, $A \cap \overline{B} = \varnothing$ so A and B are separated in X. Thus Y is a union of two nonempty separated subsets A and B of X. □

3.1.8. Theorem (continuous preserve connectedness).

Let X and Y be topological spaces and $f : X \to Y$. If X is connected, then Y is connected.

Proof. If Y is disconnected, then there is a separation $\{A, B\}$ of Y. If follows that $\{f^{-1}[A], f^{-1}[B]\}$ is a separation of X. □

3.1.9. Corollary (Generalized Intermediate Value Theorem).

Let X be a connected topological space, $f : X \to \mathbb{R}$ be continuous and $a < b < c$ be such that a and c are values of f. Then b is also a value of f.

Proof. Since $f[X]$ is a connected subset of $\mathbb{R}$, it is an interval. □

Corollary.

If $f : [0, 1] \to [0, 1]$, then $f(t) = t$ for some $t \in [0, 1]$.

Proof. Suppose, for a contradiction, that such t does not exist. Then $f(0) > 0$ and $f(1) < 1$. Define $g : [0, 1] \to \mathbb{R}$ be $g(x) := f(x) - x$. Then g is continuous, $g(0) > 0$ and $g(1) < 0$ so there is $t \in (0, 1)$ with $g(t) = 0$. Then $f(t) = t$, which is a contradiction. □

3.1.10. Theorem (union of connected sets).

Let X be a topological space and $\mathscr{A}$ be a family of connected subsets of X such that $A \cap A' \neq \varnothing$ for any $A, A' \in \mathscr{A}$. Then $\bigcup \mathscr{A}$ is a connected subset of X.

Proof. Suppose, for a contradiction, that $\bigcup \mathscr{A}$ is disconnected. Let $\{U, V\}$ be a separation of $\bigcup \mathscr{A}$. Then there are $A, A' \in \mathscr{A}$ with $U \cap A \neq \varnothing$ and $V \cap A' \neq \varnothing$. Let $x \in A \cap A'$. If $x \in V$, then $\{U \cap A, V \cap A\}$ is a separation of A, which is a contradiction. Similarly, we get a contradiction when $x \in U$. □

Remark.

Let X be a topological space such that for any $x, y \in X$ there is a connected $A \subseteq X$ with $x, y \in A$. Then X is connected.

Proof. Suppose, for a contradiction, that X is disconnected. Let $\{A, B\}$ be a separation of X. Let $a \in A$ and $b \in B$. If C is a connected subset of X with $a, b \in C$, then $\{A \cap C, B \cap C\}$ is a separation of C, which is a contradiction. $\square$

Remark.

Let X be a topological space, C be a connected subset of X and $\mathscr{A}$ be a family of connected subsets of X such that $C \cap Y \neq \varnothing$ for any $Y \in \mathscr{A}$. Then $C \cup \bigcup \mathscr{A}$ is connected.

Proof. Suppose, for a contradiction, that $C \cup \bigcup \mathscr{A}$ is disconnected. Let $\{A, B\}$ be a separation of $C \cup \bigcup \mathscr{A}$. Since C is connected, either $A \cap C = \varnothing$ or $B \cap C = \varnothing$. Assume $A \cap C = \varnothing$. Let $a \in A$ and $Y \in \mathscr{A}$ be such that $a \in Y$. Since $C \subseteq B$ and $C \cap Y \neq \varnothing$, it follows that $B \cap Y \neq \varnothing$ so $\{A \cap Y, B \cap Y\}$ is a separation of Y, which is a contradiction. $\square$

3.1.11. Lemma (connectedness of closure).

Let X be a topological space and $A \subseteq X$ be connected. If $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof. Let D be a discrete space and $f : B \to D$ be continuous. Then $f{\restriction}A$ is constant. Let d be the value of f on A. Since B is the closure of A in B, it follows that

$$f[B] \subseteq \overline{f[A]} = \overline{\{d\}} = \{d\}\,,$$

so f is constant. Thus B is connected. $\square$

3.1.12. Theorem (product of connected spaces).

Let X_α be a topological space for every $\alpha \in A$. Then $X := \prod_{\alpha\in A} X_\alpha$ is connected if and only if X_α is connected for each $\alpha \in A$.

Proof. If X is connected, then each X_α is connected since $p_\alpha : X \to X_\alpha$ is continuous for each $\alpha \in A$.

Assume that X_α is connected for each $\alpha \in A$. Let $y = (y_\alpha)_{\alpha\in A} \in X$ be fixed and let $\mathscr{A}$ be the family of all connected subsets of X containing y. Then $\bigcup\mathscr{A}$ is connected so $\overline{\bigcup\mathscr{A}}$ is connected. We will show that $\overline{\bigcup\mathscr{A}} = X$. Let $x = (x_\alpha)_{\alpha\in A} \in X$. To show that $x \in \overline{\bigcup\mathscr{A}}$ we will take any basic open nbhd B of x and prove that there is $Y \in \mathscr{A}$ such that $B \cap Y \neq \varnothing$, that is, that there is a connected subset Y of X with $y \in Y$ and $Y \cap B \neq \varnothing$.

Let $A' \subseteq A$ be finite and

$$B := \prod_{\alpha\in A} B_\alpha,$$

where B_α is an open nbhd of x_α for $\alpha \in A'$ and $B_\alpha = X_\alpha$ for $\alpha \in A \smallsetminus A'$. Assume that

$$A' := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Let

$$Y_1 := \left\{(z_\alpha)_{\alpha\in A} : z_\alpha = y_\alpha \text{ for every } \alpha \in A \smallsetminus \{\alpha_1\}\right\}.$$

Then Y_1 is homeomorphic to X_{α_1} and $y \in Y_1$. Let

$$Y_2 := \left\{(z_\alpha)_{\alpha\in A} : z_{\alpha_1} = x_{\alpha_1},\ z_\alpha = y_\alpha \text{ for every } \alpha \in A \smallsetminus \{\alpha_1, \alpha_2\}\right\}.$$

Then Y_2 is homeomorphic to X_{α_2} and $Y_1 \cap Y_2 \neq \varnothing$. Thus $Y_1 \cup Y_2$ is connected. By induction, for $k \in \{2, 3, \ldots, n\}$ let

$$\begin{aligned} Y_k := \big\{(z_\alpha)_{\alpha\in A} : z_{\alpha_i} = x_{\alpha_i} &\text{ for } i = 1, \ldots, k-1 \\ &\text{and } z_\alpha = y_\alpha \text{ for } \alpha \in A \smallsetminus \{\alpha_1, \ldots, \alpha_k\}\big\}. \end{aligned}$$

Then Y_k is homeomorphic to X_{α_k} and $Y_{k-1} \cap Y_k \neq \varnothing$. Thus

$$(Y_1 \cup \cdots \cup Y_{k-1}) \cap Y_k \neq \varnothing$$

so $Y_1 \cup \cdots \cup Y_k$ is connected. In particular,

$$Y := Y_1 \cup Y_2 \cup \cdots \cup Y_n$$

is connected and $y \in Y$ since $y \in Y_1$. Moreover, $Y \cap B \neq \varnothing$ since $x \in Y_n$. □

3.1.13. Exercises.

3.2. Connected Components

3.2.1. Definition of components.

Let X be a topological space and $x \in X$. The component of x in X, denoted $C(x)$ is the union of all connected subsets of X that contain x.

Remark.

The components of X are connected subsets.

3.2.2. Proposition (properties of components).

Let X be a topological space.

1. The set of components of X is a partition of X.
2. Each component is closed.
3. Each connected subset of X is contained in a component of X.

Proof. To prove 1. we show that if $x, y \in X$, then either $C(x) = C(y)$ or $C(x) \cap C(y) = \varnothing$. Let $x, y \in X$. Suppose that $C(x) \cap C(y) \neq \varnothing$ then $C(x) \cup C(y)$ is connected and contains x so

$$C(x) \cup C(y) \subseteq C(x)$$

and consequently

$$C(x) \cup C(y) = C(x).$$

Similarly,

$$C(x) \cup C(y) = C(y).$$

Thus $C(x) = C(y)$ as required.

Let $x \in X$. Since $C(x)$ is connected, it follows that $\overline{C(x)}$ is connected. Thus $\overline{C(x)} \subseteq C(x)$ so $\overline{C(x)} = C(x)$ and $C(x)$ is closed.

If A is a connected subset of X and $A \neq \varnothing$, then $A \subseteq C(x)$, where $x \in A$. $\square$

Example.

Let $\mathbb{Q}$ have the subspace topology. Then no subset of $\mathbb{Q}$ with at least two points is connected ($\mathbb{Q}$ contains no nontrivial intervals). Thus singletons are the components of $\mathbb{Q}$. They are not open in $\mathbb{Q}$.

Example.

Let C be the Cantor set. The components of C are singletons, since C contains no nontrivial interval.

3.2.3. Totally disconnected space.

A topological space X is *totally disconnected* if the components of X are singletons.

Examples.

The set of rational numbers $\mathbb{Q}$, the Cantor set C or the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers are all totally disconnected as topological spaces with the subspace topology inherited from $\mathbb{R}$.

3.2.4. Quasi-components.

Let X be a topological space and $K \subseteq X$. We say that K is a *quasi-component* of X if:

1. for each separation $\{A, B\}$ of X, either $K \subseteq A$ or $K \subseteq B$, and
2. for any $L \subseteq X$ with $K \subsetneqq L$ there is a separation $\{A, B\}$ of X with $L \cap A \neq \varnothing$ and $L \cap B \neq \varnothing$.

Example.

Let

$$J := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \times [0,1] \cup \{0\} \times \left([0,1] \setminus \left\{ \frac{1}{2} \right\} \right)$$

with the subspace topology inherited from $\mathbb{R}^2$. Then

$$J_1 = \{0\} \times \left[0, \frac{1}{2} \right) \qquad \text{and} \qquad J_2 := \{0\} \times \left(\frac{1}{2}, 1 \right]$$

are components of J, but they are not quasi-components. $J_1 \cup J_2$ is a quasi-component.

3.2.5. Proposition (properties of quasi-components).

Let X be a space.

1. Each point belongs to a unique quasi-component of X.
2. The quasi-component containing a point x is the intersection of all clopen subsets of X that contain x.
3. Each component of X is contained in a quasi-component of X.

Proof. 1. For $x \in X$, let $K(x)$ be the set of all $y \in X$ such that there are no separation $\{A, B\}$ of X with $x \in A$ and $y \in B$. Then $K(x)$ is a quasi-component of X containing x. Suppose that K and K' are quasi-components of X with $x \in K \cap K'$. If $\{A, B\}$ is a separation of X, then either $x \in A$ or $x \in B$. If $x \in A$, then $K \subseteq A$ and $K' \subseteq A$ so $K \cup K' \subseteq A$. If $x \in B$, then $K \cup K' \subseteq B$. Thus $K \cup K'$ can't be a proper superset of neither K nor K'. It follows that

$$K = K \cup K' = K'.$$

2. Let K be a quasi-component of X with $x \in K$. If $A \subseteq X$ is clopen with $x \in A$, then either $A = X$ or $\{A, X \setminus A\}$ is a separation of X. In either case, $K \subseteq A$. Thus if $\mathscr{A}$ is the family of all clopen subsets of X that contain x, then $K \subseteq \bigcap \mathscr{A}$. Let $\{A, B\}$ be any separation of X. If $x \in A$, then $A \in \mathscr{A}$ so $\bigcap \mathscr{A} \subseteq A$. If $x \in B$, then $\bigcap \mathscr{A} \subseteq B$. Thus $K = \bigcap \mathscr{A}$.

3. Let C be a component of X. Let $x \in C$ and K be the quasi-component of X with $x \in K$. Suppose, for a contradiction, that there is $y \in C \setminus K$. Then there is a separation $\{A, B\}$ of X with $x \in A$ and $y \in B$. This implies that $\{C \cap A, C \cap B\}$ is a separation of C, which is a contradiction. □

Remark.

Any component of X that is clopen is a quasi-component. In particular, if there are only finitely many components, they are clopen so they are quasi-components.

3.2.6. Exercises.

3.3. Path-connected Spaces

3.3.1. Paths.

Let X be a topological space. A *path* in X is a continuous function $f : [0, 1] \to X$, where $[0, 1]$ is the closed interval with the subspace topology inherited from $\mathbb{R}$. If f is a path in X with $x := f(0)$ and $y := f(1)$, then we say that f is a *path from x to y*.

3.3.2. Definition of path-connectivity.

A topological space X is *path-connected* if for every $x, y \in X$ there is a path in X from x to y.

Examples.

The trivial space is path-connected. The Sierpiński space is path-connected. The Euclidean space $\mathbb{R}^n$ is path connected for each $n \in \mathbb{N}$.

3.3.3. Lemma.

Let X be a topological space and $x \in X$. Then X is path-connected if and only if for every $y \in X$ there is a path in X from x to y.

Proof. If X is path-connected, then for every $y \in X$ there is a path in X from x to y. Assume that for every $y \in X$ there is a path in X from x to y. Let $y, z \in X$. We show that there is a path in X from y to z. Let f be a path in X from x to y and g be a path in X from x to z. Define $h : [0, 1] \to X$ by

$$h(x) := \begin{cases} f(2x) & \text{if } 0 \le x \le \frac{1}{2}; \\ g\big(2\left(\frac{1}{2} - x\right)\big) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Then h is a path in X from y to z. □

Corollary.

The union of path-connected subspaces of X containing a given point x is path-connected.

Proof. If y belongs to such a union Y, then y belongs to a subspace Z of X that is path-connected and contains x. Consequently, there is a path f in Z from x to y. Then f is a path in Y from x to y. □

3.3.4. Theorem (path-connected are connected)

Every path-connected space is connected.

Proof. Let X be path-connected and suppose, for a contradiction, that $\{A, B\}$ is a separation of X. Let $x \in A$ and $y \in B$ and f be a path in X from x to y. Then $\left\{f^{-1}[A], f^{-1}[B]\right\}$ is a separation of $[0,1]$, which is a contradiction since $[0,1]$ is connected. $\square$

Remark.

Every connected subspace of $\mathbb{R}$ is an interval so it is path-connected.

3.3.5. Example (topologist's sine curve).

Let

$$X := \left\{\langle x, y\rangle \in \mathbb{R}^2 : x \in (0,1]\,,\, y = \sin\frac{1}{x}\right\} \cup (\{0\} \times [-1,1])$$

with the subspace topology inherited from $\mathbb{R}^2$. Then X is connected, but it is not path-connected.

Proof. Since

$$X' := \left\{\langle x, y\rangle \in \mathbb{R}^2 : x \in (0,1]\,,\, y = \sin\frac{1}{x}\right\}$$

is the image of continuous function on a connected space $(0,1]$, it follows that X' is connected. Since X is the closure of X' in $\mathbb{R}$, it is connected.

Suppose, for a contradiction, that X is path-connected. Let f be a path in X from $\langle 0,0\rangle$ to $\langle 1/\pi, 0\rangle$. Let

$$t := \sup\{s \in [0,1] : (f_1)(s) = 0\}\,,$$

where f_1 is the composition of f with the projection on the first coordinate. We get a contradiction by showing that f_2 is not continuous at t, wheref_2 is the composition of f with the projection on the second coordinate.

Since f_1 is continuous, it follows that $f_1(t) = 0$ so $t < 1$. By the intermediate value property of the function f_1 there are sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ in $(t, 1]$ both converging to t such that $f_2(c_n) = 1$ and $f_2(d_n) = -1$ for each $n \in \mathbb{N}$. This proves the discontinuity of f_2 and provides the required contradiction. □

3.3.6. Theorem (path-connectedness and continuity).

If $f : X \to Y$ is continuous and X is path-connected, then $f[X]$ is path-connected.

Proof. Let $x, y \in X$ and g be a path in X from x to y. Then $f \circ g$ is a path in Y from $f(x)$ to $f(y)$. □

3.3.7. Theorem (path-connectedness and products).

The product of a family of path-connected spaces is path-connected.

Proof. Let X_α be path-connected for each $\alpha \in A$ and $X := \prod_{\alpha\in A} X_\alpha$. Let $x := (x_\alpha)_{\alpha\in A}$ and $y := (y_\alpha)_{\alpha\in A}$ be point in X. For each $\alpha \in A$ there is a path f_α in X_α from x_α to y_α. Then

$$f := \prod_{\alpha\in A} f_\alpha,$$

defined by

$$f(t) := (f_\alpha(t))_{\alpha\in A},$$

for each $t \in [0, 1]$, is a path in X from x to y. □

3.3.8. Path components.

Let X be a topological space and $x \in X$. The *path component* of x in X is the union of all path-connected subsets of X that contain x.

Example.

Let X be be the topologist's sine curve. Then X has two path components.

3.3.9. Theorem (Space-Filling curve).

There exists a continuous surjection $f : [0,1] \to [0,1]^2$.

Sketch of proof. Let $X_n := \{0,1\}$ with the discrete topology and $X := \prod_{n\in\mathbb{N}} X_n$. There is a continuous surjection $g : X \to [0,1]$ defined by

$$g\big((x_n)_{n\in\mathbb{N}}\big) := \sum_{n\in\mathbb{N}} \frac{x_n}{2^n}.$$

Then $g \times g : X^2 \to [0,1]^2$ is a continuous surjection. There is a homeomorphism $h : X \to X^2$ defined by

$$h\big((x_n)_{n\in\mathbb{N}}\big) := \big\langle (x_{2n})_{n\in\mathbb{N}}, (x_{2n+1})_{n\in\mathbb{N}} \big\rangle .$$

and there is a homeomorphism $\varphi : X \to C$, where $C \subseteq [0,1]$ is the Cantor set, defined by

$$\varphi\big((x_n)_{n\in\mathbb{N}}\big) = \sum_{n\in\mathbb{N}} \frac{2x_n}{3^n}.$$

Then $\varphi^{-1} : C \to X$ can be extended to a continuous function $\psi : [0,1] \to X$ by defining ψ to be linear on each open interval removed during the construction of C. Then

$$f := (g \times g) \circ h \circ \psi$$

is a continuous surjection $[0,1] \to [0,1]^2$ as required. □

Remark.

Every connected subspace of $[0,1]$ is path-connected, but there exists a connected subspace of $[0,1]^2$ that is not path-connected. If follows that $[0,1]$ and $[0,1]^2$ are not homeomorphic. We will show later that any continuous surjection $[0,1] \to [0,1]^2$ would be a homeomorphism, which implies that there are no such continuous surjection.

3.3.10. Exercises.

3.4. Local Connectivity

3.4.1. Locally connected spaces.

A topological space X is *locally connected* at $x \in X$ if for every nbhd U of x there is a connected nbhd V of x with $V \subseteq U$. We say that X is locally connected if it is locally connected at each $x \in X$.

Examples.

A trivial space is locally connected. A discrete space is locally connected. An Euclidean space $\mathbb{R}^n$ is locally connected. The topologist's sine curve is not locally connected.

3.4.2. Theorem (criterion for local connectedness).

A topological space X is locally connected if and only if the components of every open subset of X are open in X.

Proof. Assume that X is locally connected and U is open in X. If C is a component of U and $x \in C$, then U is a nbhd of x in X so there is a connected nbhd V of x in X with $x \in V \subseteq U$. Then $V \subseteq C$ so C is a nbhd of x in X. Since C is a nbhd of each $x \in C$, it follows that C is open in X.

Now assume that the components of every open subset of X are open in X. Let $x \in X$ and U be a nbhd of x. Let U' be an open nbhd of x with $U' \subseteq U$ and V be the component of U' containing x. Then V is a connected nbhd of x with $V \subseteq U$. Thus X is locally connected. □

Remark.

Every open subspace of a locally connected topological space is locally connected.

3.4.3. Theorem (continuity and local connectedness).

Let $f : X \to Y$ be a continuous closed surjection. If X is locally connected, then Y is locally connected.

Proof. Let U be open in Y and C be a component of U. We will show that $f^{-1}[C]$ is open in X. Since f is a closed surjection, we have

$$f\big[X \setminus f^{-1}[C]\big] = Y \setminus C,$$

so it will follow then that $Y \setminus C$ is closed in Y and consequently that is C is open. Let $x \in f^{-1}[C]$. Then $x \in f^{-1}[U]$, which is open in X. Let V be the component of $f^{-1}[U]$ that contains x. Then $f(x) \in f[V]$ and $f[V]$ is connected so $f[V] \subseteq C$ and hence $V \subseteq f^{-1}[C]$. Since V is open, $f^{-1}[C]$ is a nbhd of x. Since $f^{-1}[C]$ is a nbhd of each $x \in f^{-1}[C]$, it follows that $f^{-1}[C]$ is open, as required. $\square$

Remark.

The above result also holds when f is a continuous open surjection.

Example.

Let $X := \{0\} \cup \mathbb{N}$ with discrete topology and

$$Y := \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

with the topology inherited from $\mathbb{R}$. If $f : X \to Y$ is defined by $f(0) := 0$ and $f(n) := 1/n$ for each $n \in \mathbb{N}$, then f is a continuous surjection. The space X is locally connected, but Y is not.

3.4.4. Theorem (local connectedness and products).

Let X_α be a topological space for each $\alpha \in A$ and $X := \prod_{\alpha \in A} X_\alpha$. Then X is locally connected if and only if X_α is locally connected for each $\alpha \in A$ and all but finitely many X_α are connected.

Proof. Assume that X is locally connected. Let $\beta \in A$ and $x \in X_\beta$. We show that X_β is locally connected at x. Let U_β be an open nbhd of x in X_β. Then $p_\beta^{-1}[U_\beta]$ is open in X. Let $\xi = (\xi_\alpha)_\alpha \in p_\beta^{-1}[U_\beta]$ with $\xi_\alpha = x$. There is a connected nbhd V of ξ in X with $V \subseteq p_\beta^{-1}[U_\beta]$. Then

$$x \in p_\beta[V] \subseteq U_\beta$$

and $p_\beta[V]$ is a connected nbhd of x since p_β is continuous and open. Thus X_β is locally connected at x.

Now we show that all but finitely many of X_α are connected. Let C be a component of X. Then C is open so there is finite $A' \subseteq A$ and open $U_\alpha \in X_\alpha$ for every $\alpha \in A$ such that

$$\varnothing \neq B := \bigcap_{\alpha \in A'} p_\alpha^{-1}[U_\alpha] \subseteq C.$$

If $\alpha \in A \setminus A'$, then $p_\alpha[B] = X_\alpha$ so $p_\alpha[C] = X_\alpha$, which implies that X_α is connected.

Now assume that X_α is locally connected for each $\alpha \in A$ and $A' \subseteq A$ is finite and such that X_α is connected for every $\alpha \in A \setminus A'$. Let $\xi = (\xi_\alpha)_{\alpha \in A} \in X$. To show that X is locally connected at ξ, it suffices to show that for every finite $B \subseteq A$ and an open nbhd U_α of ξ_α for every $\alpha \in B$, the set

$$V := \bigcap_{\alpha \in B} p_\alpha^{-1}[U_\alpha]$$

contains a connected nbhd of ξ. Given a set V as described above, let V_α be a connected nbhd of ξ with $V_\alpha \subseteq U_\alpha$ for every $\alpha \in B$ and let V_α be any connected nbhd of ξ_α for each $\alpha \in A' \setminus B$. Let

$$C := \bigcap_{\alpha \in A' \cup B} p_\alpha^{-1}[V_\alpha] = \prod_{\alpha \in A} V_\alpha,$$

where $V_\alpha = X_\alpha$ for $\alpha \in A \setminus (A' \cup B)$. Then C is a nbhd of ξ contained in V and C is connected by Theorem 3.1.12, since V_α is connected for each $\alpha \in A$. □

3.4.5. Local path-connectedness.

A topological space X is *locally path-connected at* $x \in X$ if for each nbhd U of x there exists a path-connected nbhd V of x with $V \subseteq U$. We say that X is *locally* *path-connected* if it is locally path-connected at each $x \in X$.

3.4.6. Proposition (criterion for local path-connectedness).

A topological space X is locally path-connected if and only if the path-components of every open subspace of X are open.

Proof. Assume that the path-components of every open subspace of X are open. If $x \in X$ and U is an open nbhd of x in X, then the component of U that contains x is the required path-connected nbhd of x that is contained in U. Thus X is locally path-connected.

Assume that X is locally path-connected. Let U be open in X and P be a path-component of U. If $x \in P$, there is a path-connected nbhd V of x with $V \subseteq U$. Then $P \cup V$ is path-connected so $P \cup V = P$ and so P is a nbhd of x. Since P is a nbhd of each $x \in P$, it is open. □

Remark.

A topological space X is locally path-connected if and only if it has a basis consisting of path-connected sets.

3.4.7. Proposition (components of locally path-connected space)

Let X be a locally-path connected space. Then each path-component of X is clopen and is a component of X.

Proof. Let P be a path component of X. Then P is open. Since each other path component is also open, it follows that P is closed. Since P is clopen, no proper superset of P can be connected. Since P is connected, it is a component. □

Example.

The topologist's sine curve is connected, but it is not locally path-connected. It's path components are neither closed nor open.

Remark.

A connected locally-path connected space is path-connected.

3.4.8. Theorem (continuity and local path-connectedness).

Let $f : X \to Y$ be a continuous surjection that is open or closed. If X is locally path-connected, then Y is also locally path-connected.

3.4.9. Theorem (local path-connectedness and products).

Let X_α be a topological space for each $\alpha \in A$ and $X := \prod_{\alpha \in A} X_\alpha$. Then X is locally path-connected if and only if X_α is locally path-connected for each $\alpha \in A$ and all but finitely many X_α are path-connected.

3.4.10. Exercises.

4. Convergence

4.1. Sequences

4.1.1. Convergence of sequences.

Let X be a topological and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ *converges* to $x \in X$ provided for each nbhd U of x there is $k \in \mathbb{N}$ such that $x_n \in U$ for each $n \geq k$.

4.1.2. Cluster points of sequences.

Let X be a topological and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $x \in X$ is a *cluster point* of $(x_n)_{n\in\mathbb{N}}$ provided for each nbhd U of x and each $k \in \mathbb{N}$ there is $n \geq k$ such that $x_n \in U$.

Remark.

The point x is a cluster point of $(x_n)_{n\in\mathbb{N}}$ if and only if for each nbhd U of x the set

$$\{n \in \mathbb{N} : x_n \in U\}$$

is infinite.

4.1.3. Proposition (sequences and closure in metric spaces).

Let X be a metric space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to x.

Proof. Assume that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to x and U be a nbhd of x. Then there is $k \in \mathbb{N}$ with $x_n \in U$ for every $n \geq k$. In particular, $x_k \in U$ so $A \cap U \neq \varnothing$. If follows that $x \in \overline{A}$.

Now assume that $x \in \overline{A}$. Let $U_n := B(x, 1/n)$ be an open ball for each $n \in \mathbb{N}$. Then U_n is a nbhd of x for each $n \in \mathbb{N}$ so there is $x_n \in A \cap U_n$. If U is any nbhd of x, then there is $k \in \mathbb{N}$ with $U_k \subseteq U$. Then $x_k \in U$ for every $k \geq n$ so $(x_n)_{n\in\mathbb{N}}$ converges to x. □

4.1.4. Proposition (continuity and sequences in metric spaces).

Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to $x \in X$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

Proof. Assume that f is continuous and $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$. Let U be a nbhd of $f(x)$ in Y. Then $f^{-1}[U]$ is a nbhd of x in X so there is $k \in \mathbb{N}$ such that $x_n \in f^{-1}[U]$ for every $n \geq k$. Then $f(x_n) \in U$ for every $n \geq k$, implying that $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

Assume that for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to $x \in X$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$. Let $A \subseteq X$ and $x \in \overline{A}$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to x in X. Since $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$, it follows that $f(x) \in \overline{f[A]}$. Since $f\left[\overline{A}\right] \subseteq \overline{f[A]}$, it follows that f is continuous. □

4.1.5. Subsequences.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a set X and $(k_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ with $k_1 < k_2 < \ldots$, then $(y_n)_{n\in\mathbb{N}}$ is a *subsequence* of $(x_n)_{n\in\mathbb{N}}$, where $y_n := x_{k_n}$ for each $n \in \mathbb{N}$.

Remark.

For any topological space and $A \subseteq X$. If a sequence $(x_n)_{n\in\mathbb{N}}$ in A converges to x, then $x \in \overline{A}$.

4.1.6. Proposition (subsequences and cluster points).

Let X be metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $x \in X$ is a cluster point of $(x_n)_{n\in\mathbb{N}}$ if and only if there is a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges to x.

Proof. Assume that x is a cluster point of $(x_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let $U_n := B(x, 1/n)$. Let $k_1 \in \mathbb{N}$ be such that $x_{k_1} \in U_1$ and for each $n \in \mathbb{N}$, let $k_{n+1} > k_n$ be such that $x_{k_{n+1}} \in U_{n+1}$. If $y_n := x_{k_n}$for each $n \in \mathbb{N}$, then $(y_n)_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to x.

Assume that there is a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges to x. If U is a nbhd of x, then there is $k \in \mathbb{N}$ such that $y_n \in U$ for every $n \geq k$. Thus the set $\{n \in \mathbb{N} : x_n \in U\}$ is infinite, so x is a cluster point of $(x_n)_{n\in\mathbb{N}}$. □

Example.

Let $X := \mathbb{R}$ be the set of real numbers with the cocountable topology and $A := \mathbb{R} \setminus \mathbb{Q}$. Then no sequence $(x_n)_{n\in\mathbb{N}}$ in A converges to 0 since

$$U := \mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}$$

is open and $0 \in U$. However $0 \in \overline{A}$ as any nbhd U of 0 is cocountable so $U \cap A \neq \varnothing$.

Example.

Let

$$X := \{\langle 0, 0\rangle\} \cup \mathbb{N} \times \mathbb{N}$$

with a topology such that $\{\langle m,n\rangle\}$ is open for every $m,n\in\mathbb{N}$ and U containing $\langle 0,0\rangle$ is open when there is $m_0\in\mathbb{N}$ such that $\{\langle m,n\rangle\notin U : n\in\mathbb{N}\}$ is finite for every $m\geq m_0$. If $(x_n)_{n\in\mathbb{N}}$ is a bijective sequence in $\mathbb{N}\times\mathbb{N}$, then $\langle 0,0\rangle$ is a cluster point of $(x_n)_{n\in\mathbb{N}}$, but no sequence in $\mathbb{N}\times\mathbb{N}$ converges to $\langle 0,0\rangle$ so, in particular, no subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to $\langle 0,0\rangle$.

4.1.7. Exercises.

4.2. Nets

4.2.1. Directed set.

A *directed set* is a set A with a binary relation $\leq$ that is reflexive and transitive and for every $\alpha,\beta\in A$ there is $\gamma\in A$ with $\alpha\leq\gamma$ and $\beta\leq\gamma$.

Examples.

The set $\mathbb{N}$ with standard linear order is a directed set. The family of finite subsets of a set X is directed by inclusion. The family of nbhds of $x\in X$ for a topological space X is directed by inverted inclusion.

4.2.2. Definition of a net.

A *net* in a set X is a function $(x_\alpha)_{\alpha\in A}$ from a directed set A to X.

4.2.3. Convergence of nets in topological spaces.

Let $(x_\alpha)_{\alpha\in A}$ be a net in a topological space X. If $Y\subseteq X$, then $(x_\alpha)_{\alpha\in A}$ is *eventually in* Y provided there is $\beta\in A$ such that $x_\alpha\in Y$ for every $\alpha\geq\beta$.

A net $(x_\alpha)_{\alpha\in A}$ in X *converges to* $x\in X$ provided for every nbhd U of x the net $(x_\alpha)_{\alpha\in A}$ is eventually in U.

Example.

Let X be a trivial space. Then any net in X converges to any point of X.

4.2.4. Hausdorff spaces.

A topological space X is *Hausdorff* if for every distinct $x, y \in X$ there are disjoint open sets U, V with $x \in U$ and $y \in V$.

Example.

The Sierpiński space is not Hausdorff. Any discrete space is Hausdorff. Any metric space is Hausdorff. Any ordered space is Hausdorff.

4.2.5. Theorem (uniqueness of limits in Hausdorff spaces).

A topological space X is Hausdorff if and only if every net in X converges to at most one point in X.

Proof. Assume that X is Hausdorff. Suppose, for a contradiction, that $x, y \in X$ are distinct and $(x_\alpha)_{\alpha \in A}$ is a net in X that converges to both x and y. Let U and V be disjoint and open in X with $x \in U$ and $y \in V$. There are $\beta, \gamma \in A$ with $x_\alpha \in U$ for $\alpha \geq \beta$ and $x_\alpha \in V$ for $\alpha \geq \gamma$. Let $\delta \in A$ be such that $\alpha \leq \delta$ and $\beta \leq \delta$. Then $x_\delta \in U \cap V$, which is a contradiction.

Assume that every net in X converges to at most one point in X. Let $x, y \in X$ be distinct and suppose, for a contradiction, that $U \cap V \neq \varnothing$ for any open U, V in X with $x \in U$ and $y \in V$. Let

$$A := \{\langle U, V \rangle : U, V \text{ are open nbhds of } x \text{ and } y, \text{ respectively}\}.$$

Define $\leq$ on A by

$$\langle U, V \rangle \leq \langle U', V' \rangle$$

if $U' \subseteq U$ and $V' \subseteq V$. Then A becomes a directed set. Let $x_\alpha \in U \cap V$ for every $\alpha := \langle U, V \rangle \in A$. Then $(x_\alpha)_{\alpha \in A}$ converges to x since for any open nbhd U of x we have $x_\alpha \in U$ for $\alpha \geq \langle U, X \rangle$. Similarly, $(x_\alpha)_{\alpha \in A}$ converges to y, which is a contradiction. $\square$

4.2.6. Theorem (nets and closure).

Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if there is a net $(x_\alpha)_{\alpha \in A}$ in A that converges to x.

Proof. Assume that there is a net $(x_\alpha)_{\alpha \in A}$ in A that converges to x. Let U be a nbhd of x. Then there is $\beta \in A$ such that $x_\alpha \in U$ for every $\alpha \geq \beta$. In particular, $x_\beta \in U$ so $U \cap A \neq \varnothing$. Thus $x \in \overline{A}$.

Now assume that $x \in \overline{A}$. Let D be the set of all nbhds of x directed by inverted inclusion. For each $\alpha \in D$ there is $x_\alpha \in \alpha \cap A$. Then $(x_\alpha)_{\alpha \in A}$ converges to x. $\square$

4.2.7. Theorem (nets and continuity).

Let X and Y be topological spaces and $f : X \to Y$. Then f is continuous if and only if for every net $(x_\alpha)_{\alpha \in A}$ in X that converges to $x \in X$, the net $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$.

Proof. Assume that f is continuous and $(x_\alpha)_{\alpha \in A}$ is a net in X that converges to $x \in X$. Let U be a nbhd of $f(x)$ in Y. Then $f^{-1}[U]$ is a nbhd of x in X so there is $\beta \in A$ such that $x_\alpha \in f^{-1}[U]$ for every $\alpha \geq \beta$. Thus $f(x_\alpha) \in U$ for every $\alpha \geq \beta$ so $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$.

Now assume that for every net $(x_\alpha)_{\alpha \in A}$ in X that converges to $x \in X$, the net $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$. If $B \subseteq X$, and $x \in \overline{B}$, then there is a net $(x_\alpha)_{\alpha \in A}$ in X that converges to x. Since $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$, it follows that $f(x) \in \overline{f[A]}$. Since $f\left[\overline{A}\right] \subseteq \overline{f[A]}$ for any $A \subseteq X$, it follows that f is continuous. $\square$

4.2.8. Theorem (convergence of nets in product spaces).

Let X_α be a topological space for each $\alpha \in A$ and $X := \prod_{\alpha\in A} X_\alpha$. A net $(x_\beta)_{\beta\in B}$ in X converges to $x \in X$ if and only of the net $(p_\alpha(x_\beta))_{\beta\in B}$ converges to $p_\alpha(x)$ for every $\alpha \in A$.

Proof. Assume that $(x_\beta)_{\beta\in B}$ converges to x. Since p_α is continuous, it follows that $(p_\alpha(x_\beta))_{\beta\in B}$ converges to $p_\alpha(x)$ for every $\alpha \in A$.

Assume that $(p_\alpha(x_\beta))_{\beta\in B}$ converges to $p_\alpha(x)$ for every $\alpha \in A$. Let U be a nbhd of x. There is a finite $A' \subseteq A$ and open U_α in X_α for every $\alpha \in A'$ such that

$$\prod_{\alpha\in A} U_\alpha \subseteq U,$$

where $U_\alpha := X_\alpha$ for every $\alpha \in A \setminus A'$. Let

$$A' := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

For each $\alpha \in A'$ there is $\beta_\alpha \in B$ such that $p_\alpha(x_\beta) \in U_\alpha$ for every $\beta \geq \beta_\alpha$. Since A' is finite, there is $\gamma \in B$ such that $\beta_\alpha \leq \gamma$ for every $\alpha \in A'$. If $\beta \geq \gamma$, then $p_\alpha(x_\beta) \in U_\alpha$ for every $\alpha \in A$ so $(x_\beta)_{\beta\in B}$ converges to x. □

4.2.9. Cluster points of nets.

Let $(x_\alpha)_{\alpha\in A}$ be a net in a set X and $Y \subseteq X$. We say that $(x_\alpha)_{\alpha\in A}$ is *frequently* in Y if for every $\beta \in A$ there is $\alpha \geq \beta$ with $x_\alpha \in Y$. If X is a topological space, $(x_\alpha)_{\alpha\in A}$ is a net in X and $Y \subseteq X$, then $x \in X$ is a *cluster point* of $(x_\alpha)_{\alpha\in A}$ if for every nbhd U of x the net $(x_\alpha)_{\alpha\in A}$ is frequently in U.

4.2.10. Subnet.

Let $(x_\alpha)_{\alpha\in A}$ be a net in a set X. A subnet of $(x_\alpha)_{\alpha\in A}$ is a net $(y_\beta)_{\beta\in B}$ such that there is a function $\varphi : B \to A$ which satisfies:

1. for every $\alpha \in A$ there is $\beta \in B$ with $\alpha \leq \varphi(\beta')$ for $\beta' \geq \beta$, and

2. $y_\beta = x_{\varphi(\beta)}$ for every $\beta \in B$.

We will say that φ *defines* the subnet.

4.2.11. Theorem (cluster points and subnets).

Let $(x_\alpha)_{\alpha\in A}$ be a net in a topological space X. Then $x \in X$ is a cluster point of $(x_\alpha)_{\alpha\in A}$ if and only if there exists a subnet $(y_\beta)_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$ that converges to x.

Proof. Assume that there exists a subnet $(y_\beta)_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$ that converges to x. Let U be a nbhd of x in X. There is $\beta_0 \in B$ such that $y_\beta \in U$ for every $\beta \geq \beta_0$. Let $\varphi : B \to A$ define the subnet. Let $\alpha \in A$. There is $\beta_1 \in B$ such that $\varphi(\beta) \geq \alpha$ for every $\beta \geq \beta_1$. Let $\beta \in B$ be such that $\beta \geq \beta_0$ and $\beta \geq \beta_1$. then

$$x_{\varphi(\beta)} = y_\beta \in U$$

and $\varphi(\beta) \geq \alpha$. Thus $(x_\alpha)_{\alpha\in A}$ is frequently in U, which implies that x is a cluster point of $(x_\alpha)_{\alpha\in A}$.

Now assume that x is a cluster point of $(x_\alpha)_{\alpha\in A}$. Let B be the family of all pairs $\langle\alpha, U\rangle$ with $\alpha \in A$ and U being a nbhd of x with $x_\alpha \in U$. Define a direction $\leq$ on B by

$$\langle\alpha, U\rangle \leq \langle\alpha', U'\rangle$$

if $\alpha \leq \alpha'$ and $U' \subseteq U$. Then $\varphi : B \to A$ given by $\varphi(\alpha, U) := \alpha$ defines a subnet $(y_\beta)_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$. We show that $(y_\beta)_{\beta\in B}$ converges to x. Let U be a nbhd of x. There is $\alpha \in A$ with $x_\alpha \in U$. Let $\beta_0 := \langle\alpha, U\rangle$. If $\beta := \langle\alpha', U'\rangle \geq \beta_0$, then

$$y_\beta = x_{\varphi(\beta)} = x_{\alpha'} \in U' \subseteq U,$$

as required. □

4.2.12. Universal net (ultranet).

A net $(x_\alpha)_{\alpha\in A}$ in a set X is a *universal net* (also called *ultranet*) if for every $Y \subseteq X$ the net $(x_\alpha)_{\alpha\in A}$ is eventually in Y or in $X \setminus Y$.

Remark.

Let $(x_\alpha)_{\alpha\in A}$ be a net in a set X such that there is $\beta \in A$ and $y \in X$ with $x_\alpha = y$ for every $\alpha \geq \beta$. Then $(x_\alpha)_{\alpha\in A}$ is an ultranet in X.

A net $(x_\alpha)_{\alpha\in A}$ in a set X is frequently in some $Y \subseteq X$ if and only if $(x_\alpha)_{\alpha\in A}$ is not eventually in $X \setminus Y$, hence a universal net in a topological space converges to any of it's cluster points.

4.2.13. Exercises.

4.3. Filters

4.3.1. Definition of a filter.

Let X be a set. A *filter* on X is a family $\mathscr{F}$ of subsets of X such that:

1. $\mathscr{F} \neq \varnothing$ and $\varnothing \notin \mathscr{F}$;
2. if $F \in \mathscr{F}$ and $F \subseteq F' \subseteq X$, then $F' \in \mathscr{F}$;
3. if $F_1, F_2 \in \mathscr{F}$, then $F_1 \cap F_2 \in \mathscr{F}$.

Examples.

If X is any nonempty set and $\mathscr{F} := \{X\}$, then $\mathscr{F}$ is a filter on X.

Let X be an infinite set and $\mathscr{F}$ consist of all cofinite sets. Then $\mathscr{F}$ is a filter on X.

4.3.2. Filter generated by a net.

Let $(x_\alpha)_{\alpha\in A}$ be a net in a set X and $\mathscr{F}$ consist of those $F \subseteq \mathscr{F}$ for which there exist $\beta \in A$ such that $x_\alpha \in F$ for every $\alpha \geq \beta$, that is $F \in \mathscr{F}$ provided the net is eventually in F. Then $\mathscr{F}$ is a filter on X.

4.3.3. Nbhd filter.

Let X be a topological space, $x \in X$ and $\mathscr{F}$ consist of all nbhds of x. Then $\mathscr{F}$ is a filter. It is called the *nbhd filter* at x.

4.3.4. Comparing filters.

Let $\mathscr{F}$ and $\mathscr{F}'$ be filters on a set X. Then $\mathscr{F}$ is *finer* than $\mathscr{F}'$ when $\mathscr{F}' \subseteq \mathscr{F}$.

4.3.5. Convergence of filters.

Let $\mathscr{F}$ be a filter in a topological space X. Then $\mathscr{F}$ *converges* to $x \in X$ provided $\mathscr{F}$ is finer than the nbhd filter at x.

4.3.6. Theorem.

Let X be a topological space, $(x_\alpha)_{\alpha \in A}$ be a net in X and $\mathscr{F}$ be the filter generated by $(x_\alpha)_{\alpha \in A}$. Then, for every $x \in X$, the net $(x_\alpha)_{\alpha \in A}$ converges to x if and only if $\mathscr{F}$ converges to x .

Proof. (to be written) □

4.3.7. Ultrafilters.

A filter $\mathscr{F}$ on a set X is an *ultrafilter* provided every filter on X that is finer than $\mathscr{F}$ must be equal $\mathscr{F}$.

4.3.8. Theorem.

A filter $\mathscr{F}$ on a set X is an ultrafilter if and only if for every $Y \subseteq X$ either $Y \in \mathscr{F}$ or $X \smallsetminus Y \in \mathscr{F}$.

Proof. (to be written) □

4.3.9. Theorem.

Let $(x_\alpha)_{\alpha\in A}$ be a net in a set X and $\mathscr{F}$ be the filter generated by $(x_\alpha)_{\alpha\in A}$. Then $\mathscr{F}$ is an ultrafilter if and only if $(x_\alpha)_{\alpha\in A}$ is an ultranet.

Proof. (to be written) □

4.3.10. Zorn's Lemma.

If X is a partially ordered set and each chain in X has an upper bound, then there exists a maximal element in X.

4.3.11. Theorem.

Let X be a set and $\mathscr{F}$ be a filter on X. Then there exists an ultrafilter on X that is finer than $\mathscr{F}$.

Proof. (to be written) □

4.3.12. Theorem.

Let X be a set and $(x_\alpha)_{\alpha\in A}$ be a net in X. Then there exists a subnet $(y_\beta)_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$ that is an ultranet.

Proof. Let $\mathscr{F}$ be the filter on X that is generated by $(x_\alpha)_{\alpha\in A}$ and $\mathscr{U}$ be an ultrafilter on X that is finer than $\mathscr{F}$. Define

$$B := \{\langle \alpha, U\rangle : \alpha \in A,\, U \in \mathscr{F},\, x_\alpha \in U\}$$

and let B be directed by

$$\langle \alpha, U\rangle \leq \langle \alpha', U'\rangle$$

if $\alpha \leq \alpha'$ and $U' \subseteq U$. If $\varphi : B \to A$ is given by $\varphi(\alpha, U) := \alpha$, then φ defines a subnet $(y_\beta)_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$. We show that $(y_\beta)_{\beta\in B}$ is an ultranet.

Let $Y \subseteq X$. Then either Y or $X \setminus Y$ belongs to $\mathscr{U}$. If $Y \in \mathscr{U}$, then $X \setminus Y \notin \mathscr{U}$, which implies that $X \setminus Y \notin \mathscr{F}$. Thus the net $(x_\alpha)_{\alpha \in A}$ is not eventually in $X \setminus Y$ and hence $(x_\alpha)_{\alpha \in A}$ is frequently in Y. Let $\alpha_0 \in A$ be such that $x_{\alpha_0} \in Y$. Then

$$\beta_0 := \langle \alpha_0, Y \rangle \in B.$$

If $\beta := \langle \alpha, U \rangle \geq \beta_0$, then $x_\alpha \in U \subseteq Y$ so $y_\beta = x_\alpha \in Y$. Thus $(y_\beta)_{\beta \in B}$ is eventually in Y. If $X \setminus Y \in \mathscr{U}$, then a similar argument shows that $(y_\beta)_{\beta \in B}$ is eventually in $X \setminus Y$. □

4.4. Hausdorff Spaces

4.4.1. Proposition.

Let X be a topological space. The following conditions are equivalent:

1. X is Hausdorff.
2. For each $x \in X$ the intersection of all closed nbhds of x is $\{x\}$.
3. The diagonal $\Delta := \{\langle x, x \rangle \in X\}$ is closed in $X \times X$.

Proof. Assume 1. We show that 2. holds. Let $x \in X$ and let C be the intersection of all closed nbhds of x. Suppose, for a contradiction, that $C \neq \{x\}$. Then there is $y \in C \setminus \{x\}$. Let U, V be disjoint open sets with $x \in U$ and $y \in V$. Then $X \setminus V$ is a closed nbhd of x so $C \subseteq X \setminus V$. Since $y \in C \cap V$, we have a contradiction.

Assume 2. We show that 3. holds. Let

$$\langle x, y \rangle \in (X \times X) \setminus \Delta.$$

Then $x \neq y$ so there is a closed nbhd N of x with $y \notin N$. Let U be open with $x \in U \subseteq N$ and $V := X \setminus N$. Then $U \times V$ is open in $X \times X$ with $\langle x, y \rangle \in U \times V$ and

$$(U \times V) \cap \Delta = \varnothing.$$

Thus $(X \times X) \setminus \Delta$ is open in $X \times X$, which implies that Δ is closed in $X \times X$.

Assume 3. We show that 1. holds. Let $x, y \in X$ be distinct. Then $\langle x, y\rangle \in (X \times X) \smallsetminus \Delta$ so there are open U and V in X with

$$\langle x, y\rangle \in U \times V \subseteq (X \times X) \smallsetminus \Delta.$$

Thus U and V are disjoint with $x \in U$ and $y \in V$. □

4.4.2. Corollary.

Assume that Y is a Hausdorff space.

1. If $f : X \to Y$ is continuous for some topological space X, then the graph of f is closed in $X \times Y$.
2. If $f, g : X \to Y$ are continuous for some topological space X, then

   $$\{x \in X : f(x) = g(x)\}$$

   is closed in X.

Proof. Assume that $f : X \to Y$ is continuous for some topological space Y. The graph of f is the set

$$G := \{\langle x, f(x)\rangle : x \in X\}.$$

Let $1_Y : Y \to Y$ be the identity function and $\Delta := \{\langle y, y\rangle \in Y\}$. Since the function

$$f \times 1_Y : X \times Y \to Y \times Y$$

is continuous, Δ is closed and $G = (f \times 1_Y)^{-1}[\Delta]$, it follows that G is closed.

Now assume that $f, g : X \to Y$ are continuous for some topological space X and

$$G := \{x \in X : f(x) = g(x)\}.$$

Let $h : X \to X \times Y$ be defined by $h(x) := \langle f(x), g(x)\rangle$. Since h is continuous and $G = h^{-1}[\Delta]$, it follows that G is closed. □

Remark.

Any subspace of a Hausdorff space is Hausdorff.

4.4.3. Theorem (product of Hausdorff spaces).

If X_α is a Hausdorff space for every $\alpha \in A$ and $X := \prod_{\alpha \in A} X_\alpha$, then X is Hausdorff.

Proof. Let $(x_\alpha)_{\alpha \in A}$ and $(y_\alpha)_{\alpha \in A}$ be distinct elements of X. There is $\beta \in A$ with $x_\beta \neq y_\beta$. Since X_β is Hausdorff, there are disjoint open U_β and V_β in X_β with $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$. Let $U_\alpha := V_\alpha := X_\alpha$ for every $\alpha \in A \setminus \{\beta\}$ and

$$U := \prod_{\alpha \in A} U_\alpha \qquad \text{and} \qquad V := \prod_{\alpha \in A} V_\alpha.$$

Then U and V are disjoint open sets in X with $x \in U$ and $y \in V$. □

4.4.4. T_0 spaces and T_1 spaces.

A T_0 space is a topological space such that for any distinct $x, y \in X$ there is an open set U with $x \in U$, but $y \notin U$. A T_0 space is a topological space such that for any distinct $x, y \in X$ there is an open set U with $\{x, y\} \cap U$ having exactly one element.

Remark.

Any Hausdorff space is T_1 and any T_1 space is T_0.

Examples.

The Sierpiński space is T_0 but not T_1. The space of natural numbers with the cofinite topology is T_1 but is not Hausdorff.

4.4.5. Proposition (characterization of T_1 spaces).

Let X be a topological space. The following conditions are equivalent:

1. X is T_1;
2. $\{x\}$ is closed for every $x \in X$;
3. the intersection of all nbhds of x is equal $\{x\}$ for every $x \in X$.

Proof. Assume X is T_1. Let $x \in X$. For every $y \in X \setminus \{x\}$ there is open U_y with $y \in U_y$ and $x \notin U_y$. Then

$$U := \bigcup_{y \in X \setminus \{x\}} U_y = X \setminus \{x\}$$

is open, so $\{x\}$ is closed.

Assume that $\{x\}$ is closed for every $x \in X$. To prove 3. suppose for a contradiction that there is $y \in X \setminus \{x\}$ such that y belongs to every nbhd of x. Since $\{y\}$ is closed $X \setminus \{y\}$ is open and contains x, which is a contradiction.

Now assume that 3. holds. If $x, y \in X$ are distinct, then there is a nbhd U of x with so X is T_1. $\square$

5. Compactness

5.1. Compact Spaces

5.2. Countable Compact Spaces

5.3. Compact Metric Spaces

5.4. Locally Compact Spaces

5.5. Proper Maps