MATH 793C — TOPOLOGY

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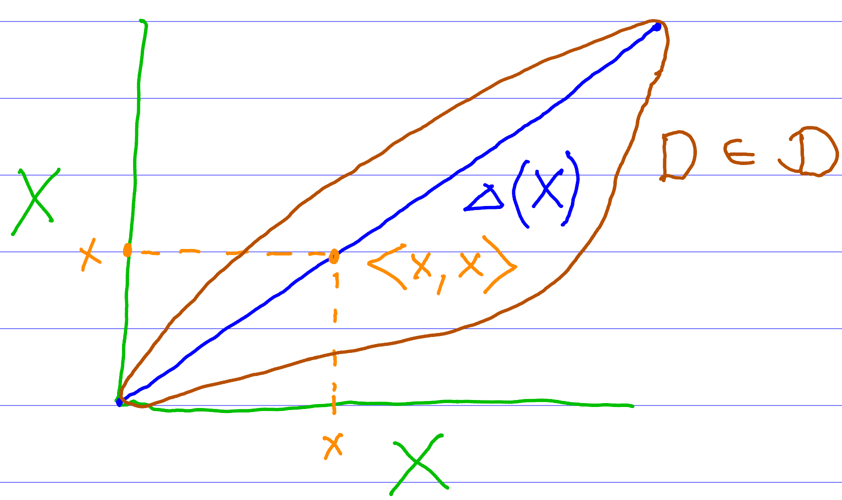
$\boxed{\text{Definition of a uniformity}}$ – review

Let X be a set. A <u>uniformity</u> on X is a filter $\mathcal{D}$ on $X \times X$ s.t. for each $D \in \mathcal{D}$ the following conditions hold:

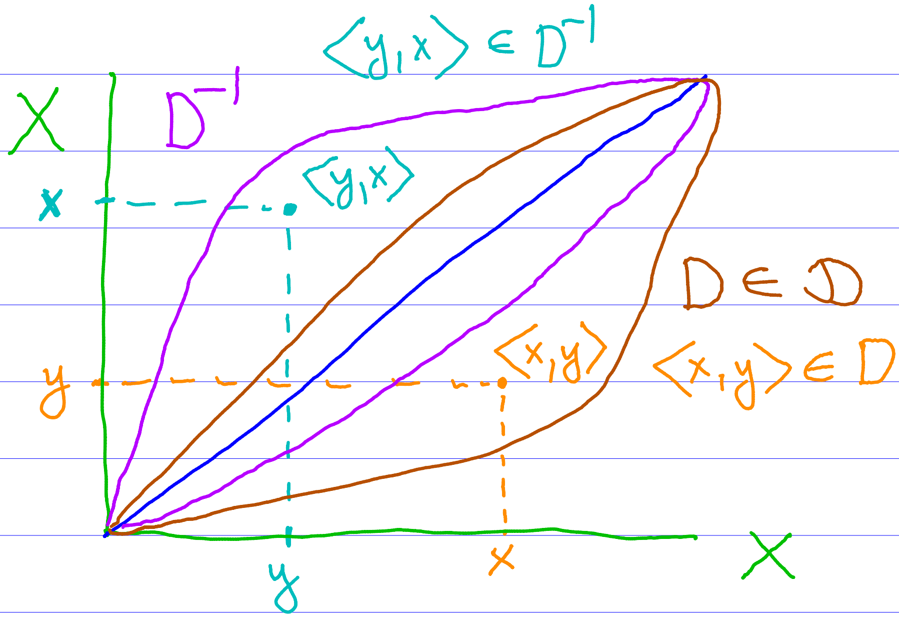
1. $\Delta(X) \subseteq D$ $\quad \Delta(X) = \{\langle x, x\rangle : x \in X\}$
2. $D^{-1} \in \mathcal{D}$
3. there is $E \in \mathcal{D}$ s.t. $E \circ E \subseteq D$.

The members of a uniformity are called <u>surroundings</u>.

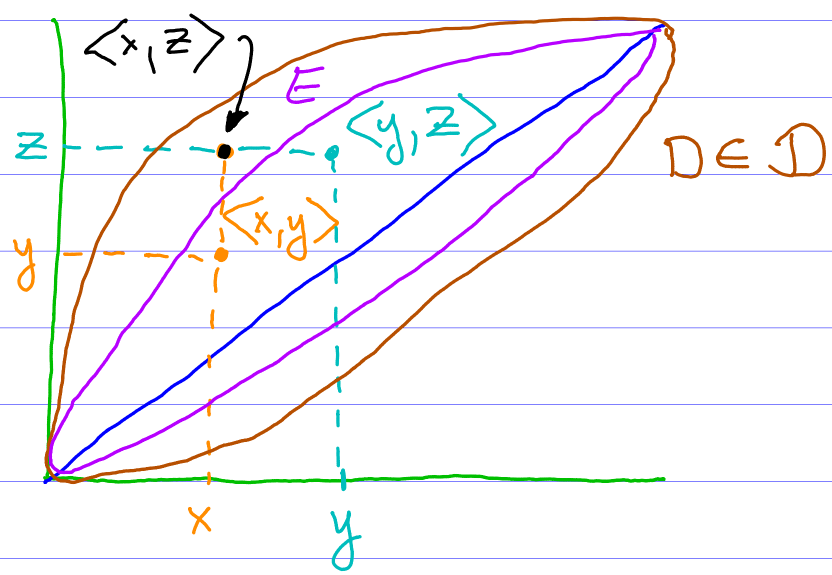
1. $\Delta(X) \subseteq D$ $\quad \Delta(X) = \{\langle x, x\rangle : x \in X\}$



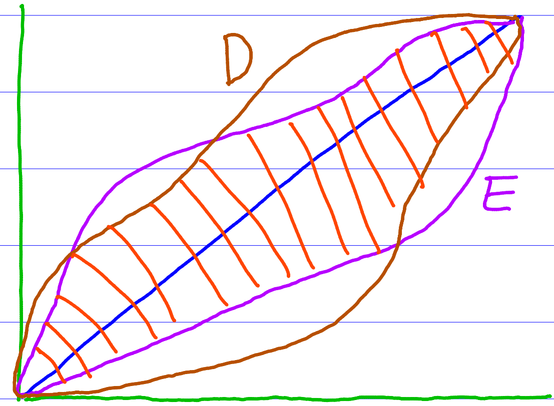
② $D^{-1} \in \mathcal{D}$



③ there is $E \in \mathcal{D}$ s.t. $E \circ E \subseteq D$.



$\mathcal{D}$ is a filter on $X \times X$

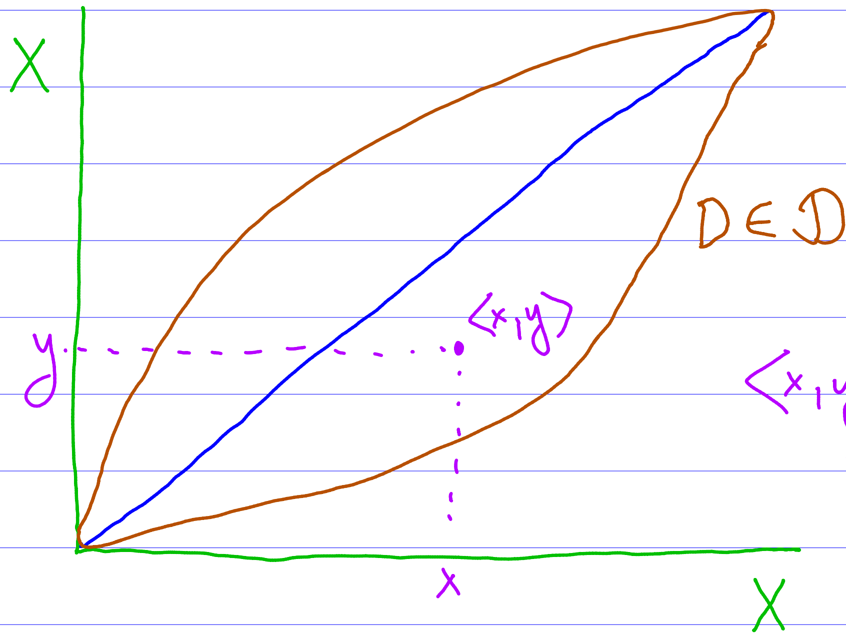


If $D, E \in \mathcal{D}$, then $D \cap E \in \mathcal{D}$

$\left. \begin{array}{l} E \subseteq X \times X \\ E \supseteq D \end{array} \right) \Longrightarrow E \in \mathcal{D}$

$\mathcal{D} \neq \emptyset$

Intuitions



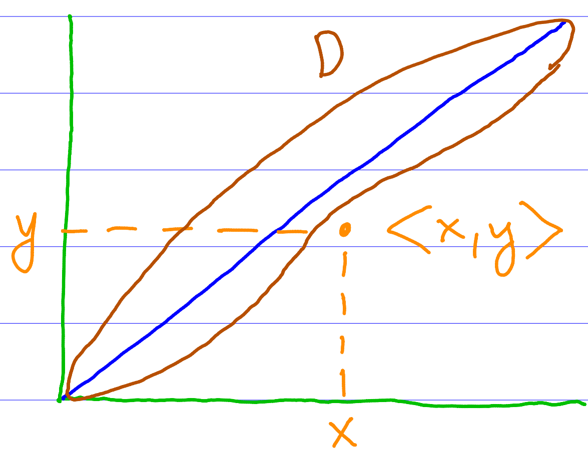
$\mathcal{D}$- uniformity on X

$\langle x, y\rangle \in D$

x, y are D-close

Separating uniformities – review

A uniformity $\mathcal{D}$ on X is <u>separating</u> iff for each $x, y \in X$ with $x \neq y$ there is $D \in \mathcal{D}$ s.t. $\langle x, y\rangle \notin D$.

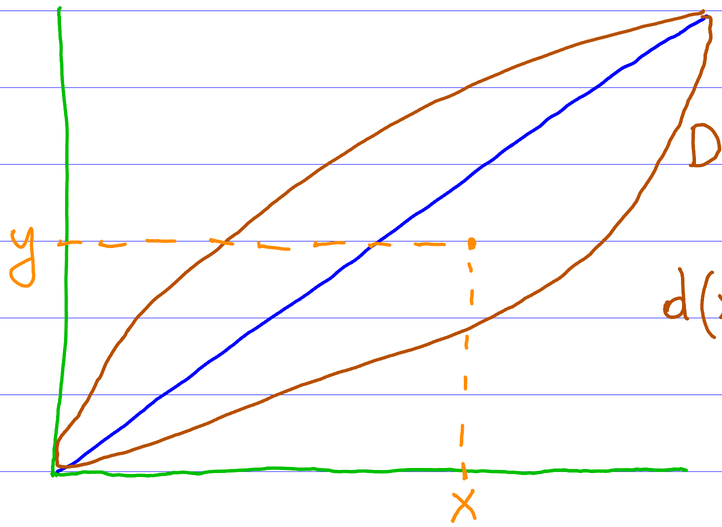


there is $D \in \mathcal{D}$

$x \neq y$

Uniformity from a pseudometric – review

If d is a pseudometric on X then $\mathcal{D}_d$ is the uniformity on X s.t. $D \in \mathcal{D}_d$ iff there exists $\varepsilon > 0$ with $\langle x, y \rangle \in D$ for every $x, y \in X$ with $d(x,y) < \varepsilon$.



$d(x,y) < \varepsilon \Rightarrow \langle x, y \rangle \in D$

$\mathcal{D}_d$ is separating if and only if d is a metric

Equivalent metrics inducing different uniformities

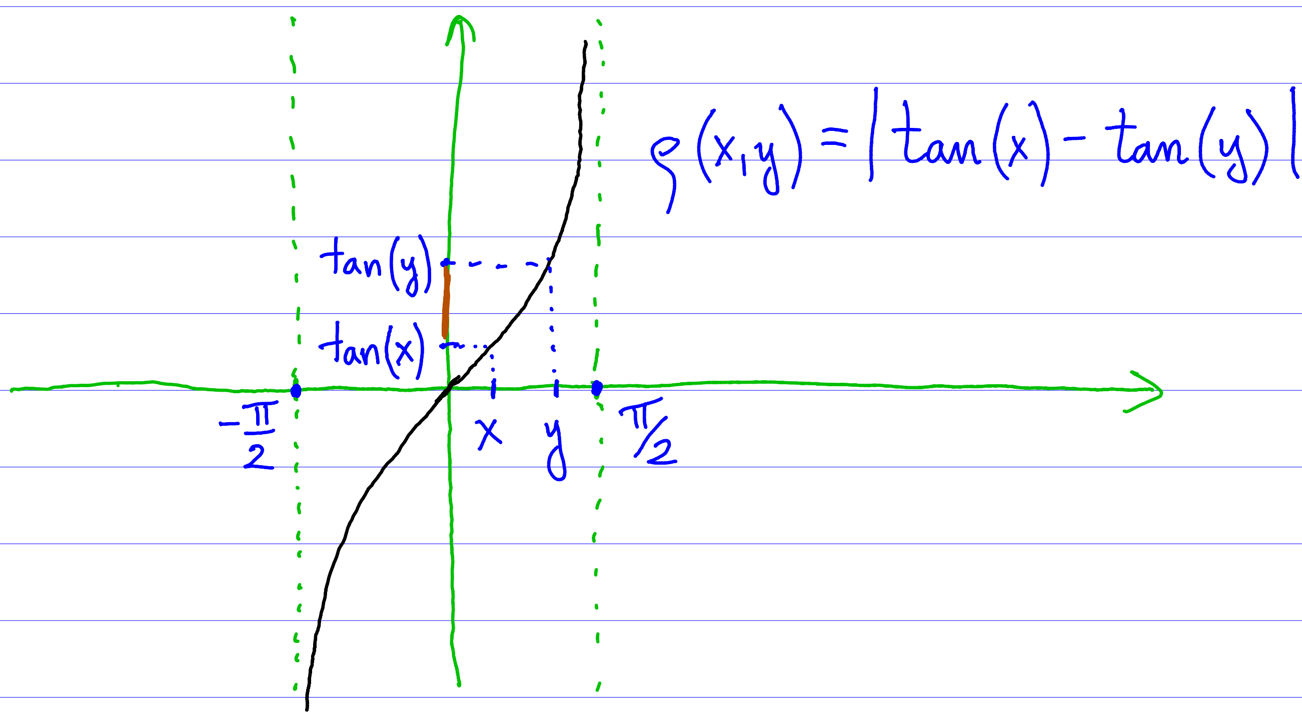
Example

Let $X = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Let d be the standard metric on X

$$d(x,y) = |x-y|$$

and ϱ be the metric defined by

$$\varrho(x,y) = |\tan(x) - \tan(y)|.$$



Let $\mathcal{D}_d$ and $\mathcal{D}_\varrho$ be the uniformities on X induced by d and ϱ, respectively.

Then $\mathcal{D}_d \subseteq \mathcal{D}_\varrho$, but $\mathcal{D}_d \neq \mathcal{D}_\varrho$.

Proof

① $\mathcal{D}_d \subseteq \mathcal{D}_\varrho$

Let $D \in \mathcal{D}_d$. There exists $\varepsilon > 0$ s.t. $\langle x, y \rangle \in D$ whenever $d(x,y) < \varepsilon$.

Let $x, y \in X$ with $\varrho(x,y) < \varepsilon$. Then

$$d(x,y) = |x-y| \leq |\tan(x) - \tan(y)| < \varepsilon$$

so $\langle x, y \rangle \in D$.

Thus $D \in \mathcal{D}_\varrho$.

② $\mathcal{D}_d \neq \mathcal{D}_\varrho$

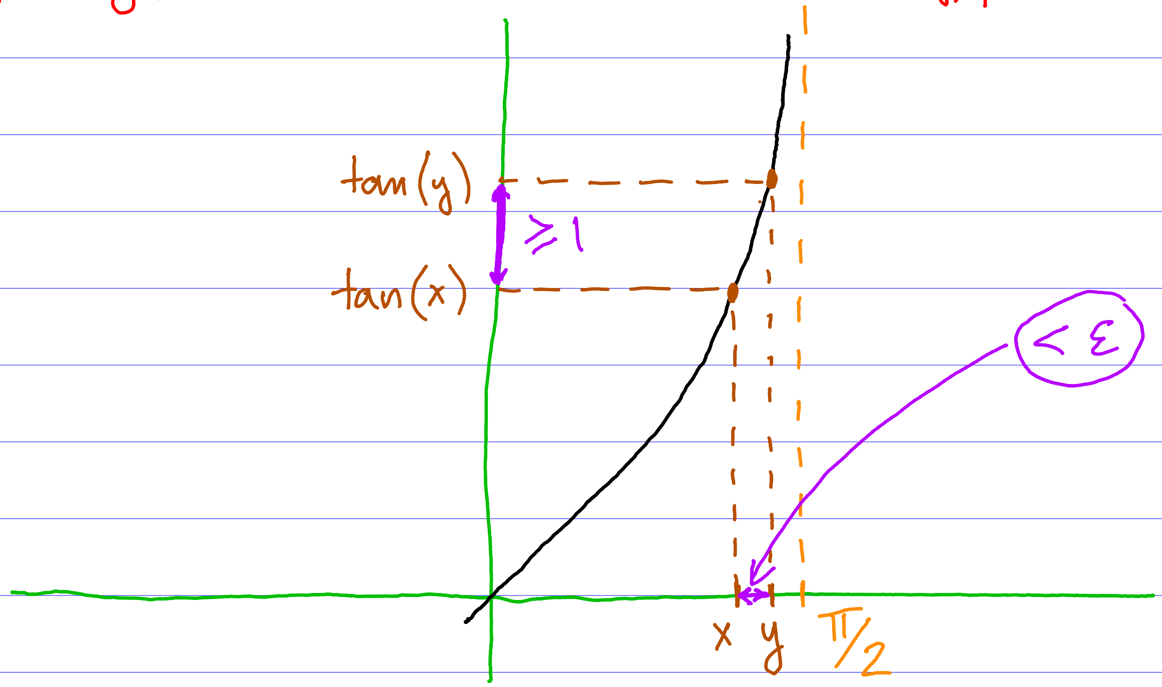
Let $D = \{ \langle x, y \rangle \in X \times X : |\tan(x) - \tan(y)| < 1 \}$

We will show that $D \in \mathcal{D}_\varrho$ but $D \notin \mathcal{D}_d$.

$D \in \mathcal{D}_\rho$ since $\langle x,y\rangle \in D$ whenever $\rho(x,y) < \varepsilon$ for $\varepsilon = 1$.

Now, we show that $D \notin \mathcal{D}_d$

If $\varepsilon > 0$, then there are $x, y \in X$ s.t.
$|x-y| < \varepsilon$ but $|\tan(x) - \tan(y)| \geq 1$.



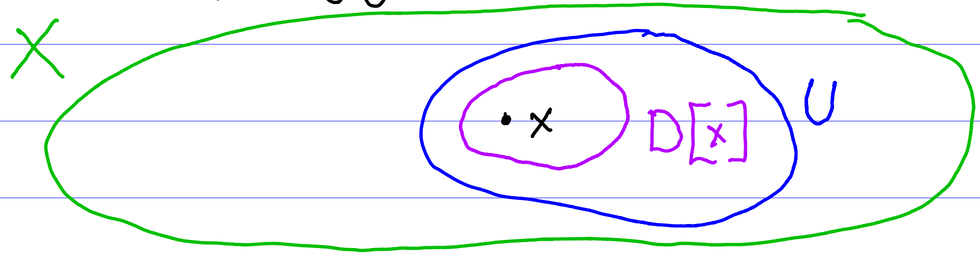
Thus for any $\varepsilon > 0$, there are $x, y \in X$ with
$d(x,y) < \varepsilon$ but $\langle x,y\rangle \notin D$.

Hence $D \notin \mathcal{D}_d$.

Topology induced by a uniformity

If $D \subseteq X \times X$ and $x \in X$, then let
$$D[x] = \{y \in X : \langle x, y \rangle \in D\}.$$

<u>Theorem</u>
Let $\mathcal{D}$ be a uniformity on X and $\tau \subseteq \mathcal{P}(X)$ consist of all $U \subseteq X$ s.t. for each $x \in U$ there exists $D \in \mathcal{D}$ with $D[x] \subseteq U$.
Then τ is a topology on X.



We call τ the <u>topology of $\mathcal{D}$.</u>

<u>Proof</u>
$\emptyset \in \tau$ is obvious.
$X \in \tau$ since $\mathcal{D} \neq \emptyset$. If $x \in X$ and $D \in \mathcal{D}$ is arbitrary, then $D[x] \subseteq X$.

Let $\tau' \subseteq \tau$. Let $U = \bigcup \tau'$. We want to show that $U \in \tau$.

Let $x \in U$. Then $x \in V$ for some $V \in \tau'$ so there is $D \in \mathcal{D}$ with $D[x] \subseteq V$. Since $V \subseteq U$, it follows that $D[x] \subseteq U$. Thus $U \in \tau$.

Let $\tau' \subseteq \tau$ be finite and nonempty. Let $U = \bigcap \tau'$.

Let $x \in U$. Then $x \in V$ for each $V \in \tau'$.
Given $V \in \tau'$, let $D_V \in \mathcal{D}$ be such that $D_V[x] \subseteq V$.
Since τ' is finite

$$D := \bigcap \{D_V : V \in \tau'\} \in \mathcal{D}.$$

$$D[x] \subseteq D_V[x] \subseteq V \quad \text{for each } V \in \tau'$$

Thus $D[x] \subseteq U$. Hence $U \in \tau$.

Exercises

Exercise

Let $\mathcal{D}$ be a uniformity on a set X and let $A \subseteq X$. Define $U = \{x \in A : \exists D \in \mathcal{D}\, (D[x] \subseteq A)\}$. Prove that U is open in the topology of $\mathcal{D}$.

Exercise

Let $\mathcal{D}$ be a uniformity on a set X and τ be the topology of $\mathcal{D}$. Prove that for each $x \in X$ the family $\mathcal{U}_x = \{D[x] : D \in \mathcal{D}\}$ is the nbhd system at x for the topology τ ($\mathcal{U}_x$ is the family of all nbhds of x).

Exercise

Let $\mathcal{D}$ be a uniformity on a set X and τ be the topology of $\mathcal{D}$. Prove that τ is Hausdorff if and only if $\mathcal{D}$ is separating.