MATH 793C – TOPOLOGY

Jerzy Wojciechowski — Spring 2020

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Uniform Spaces.

Let X be a set. A <u>uniformity</u> on X (diagonal uniformity) is a set $\mathcal{D}$ of reflexive binary relations on X s.t. $\mathcal{D}$ is a filter on $X \times X$ and for each $D \in \mathcal{D}$ there is $D_1 \in \mathcal{D}$ s.t. $D_1^{-1} \subseteq D$ and $D_2 \in \mathcal{D}$ s.t. $D_2 \circ D_2 \subseteq D$.

binary relation on X is a subset of $X \times X$

$\Delta(X) = \{\langle x, x\rangle : x \in X\}$

$D \subseteq X \times X$ $\quad$ D is reflexive iff $\Delta(X) \subseteq D$

$\mathcal{D}$ is a filter iff $D_1 \cap D_2 \in \mathcal{D}$ for $D_1, D_2 \in \mathcal{D}$ and if $D \in \mathcal{D}$ and $D \subseteq D' \subseteq X \times X$, then $D' \in \mathcal{D}$.

$D_1, D_2 \in \mathcal{D}$,

$$D_1 \circ D_2 = \{\langle x, z\rangle \in X \times X : \exists y \in X \ \langle x, y\rangle \in D_2 \text{ and } \langle y, z\rangle \in D_1\}$$

$D \in \mathcal{D}$, $D^{-1} = \{\langle x,y\rangle \in X \times X : \langle y,x\rangle \in D\}$

<u>Examples</u>

<u>Discrete uniformity</u>

X – any nonempty set

$\mathcal{D} = \{D \subseteq X \times X : \Delta(X) \subseteq D\}$

– the family of all reflexive relations on X.

The discrete uniformity is separating.

<u>Trivial uniformity</u>

X – any nonempty set

$\mathcal{D} = \{X \times X\}$.

If X has at lest two elements, $\mathcal{D}$ is not separating.

A uniformity $\mathcal{D}$ on X is <u>separating</u> iff

$$\bigcap \mathcal{D} = \{\langle x,x\rangle : x \in X\} = \Delta(X) \quad \text{iff}$$

for each $x, y \in X$ with $x \neq y$ there is $D \in \mathcal{D}$ s.t. $\langle x,y\rangle \notin D$.

The members of a uniformity are called <u>surroundings</u>.

A surrounding D is <u>symmetric</u> iff $D^{-1} = D$.

Example

Let d be a pseudometric on X. Let $\mathcal{D}_d \subseteq \mathcal{P}(X \times X)$ be defined by $D \in \mathcal{D}_d$ iff there exists $\varepsilon > 0$ s.t. $\langle x, y \rangle \in D$ for every $x, y \in X$ with $d(x,y) < \varepsilon$. Then $\mathcal{D}_d$ is a uniformity on X.

Let $D \in \mathcal{D}_d$. Then there is $\varepsilon > 0$ s.t. $\langle x, y \rangle \in D$ for any $x, y \in X$ with $d(x,y) < \varepsilon$.
In particular, $\langle x, x \rangle \in D$ for any $x \in X$.
Thus $\Delta(X) \subseteq D_d$.

Let $D_1, D_2 \in \mathcal{D}_d$. Let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\langle x, y \rangle \in D_1$ whenever $d(x,y) < \varepsilon_1$ and $\langle x, y \rangle \in D_2$ whenever $d(x,y) < \varepsilon_2$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.
If $d(x,y) < \varepsilon$, then $\langle x, y \rangle \in D_1 \cap D_2$. Thus $D_1 \cap D_2 \in \mathcal{D}_d$.

Let $D \in \mathcal{D}_d$ and $D \subseteq D' \subseteq X \times X$. Let $\varepsilon > 0$ be such that $\langle x, y \rangle \in D$ whenever $d(x,y) < \varepsilon$. Then $\langle x, y \rangle \in D'$ whenever $d(x,y) < \varepsilon$ so $D' \in \mathcal{D}_d$.

Let $D \in \mathcal{D}_d$. Let $\varepsilon > 0$ be such that $\langle x, y \rangle \in D$ whenever $d(x,y) < \varepsilon$. Let

$$D_1 = \{\langle x, y \rangle \in X \times X : d(x,y) < \varepsilon/2\}$$

We want $D_1 \circ D_1 \subseteq D$. Let $\langle x, z \rangle \in D_1 \circ D_1$.
There is $y \in X$ s.t. $\langle x, y \rangle, \langle y, z \rangle \in D_1$.
Then

$$d(x,y) < \varepsilon/2 \quad \text{and} \quad d(y,z) < \varepsilon/2.$$

Then $d(x,z) < \varepsilon$ so $\langle x, z \rangle \in D$.

Let $D \in \mathcal{D}_d$. Let $\varepsilon > 0$ be such that
$\langle x, y \rangle \in D$ whenever $d(x,y) < \varepsilon$.
Then $\langle x, y \rangle \in D^{-1}$ whenever $d(x,y) < \varepsilon$.

Ex. Let $X = \mathbb{R}$ with standard metric d.
Let $D = \{ \langle x, y \rangle \in X \times X : |x-y| < 1 \text{ or } x < y \}$
Then $D \in \mathcal{D}_d$, but $D^{-1} \neq D$.
$\langle 5, 1 \rangle \in D^{-1}$ $\quad$ $\langle 5, 1 \rangle \notin D$

Moreover, $\mathcal{D}_d$ is separating if and only if d is a metric.

If d is a metric then for any $x, y \in X$ with $x \neq y$ let $0 < \varepsilon < d(x,y)$ and

$$D = \{ \langle w, z \rangle \in X \times X : d(w,z) < \varepsilon \}$$

Then $D \in \mathcal{D}_d$ but $\langle x, y \rangle \notin D$. Thus $\mathcal{D}_d$ is separ.

If d is not a metric, there are $x, y \in D$ $x \neq y$ with $d(x,y) = 0$.

Then $\langle x, y \rangle \in D$ for each $D \in \mathcal{D}_d$ so $\mathcal{D}_d$ is not separating.

Skype Id: jerzy512