MATH 793C — TOPOLOGY

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Let X be a topological space.

We say that X is a Baire space iff the intersection of any countable family of dense open sets in X is dense in X.

A subset $A \subseteq X$ is nowhere dense in X iff $\overline{A}$ has empty interior (iff $X \setminus \overline{A}$ is dense iff there exist a dense open U in X with $A \cap U = \emptyset$).

A subset $A \subseteq X$ is meager in X iff A is a countable union of nowhere dense subsets of X.

Example $\mathbb{Q}$ is meager in $\mathbb{Q}$ (with the standard top.)

Remark
X is Baire iff $X \setminus A$ is dense in X for each $A \subseteq X$ that is meager in X iff no nonempty open subset of X is meager in X.

Theorem (Baire)

Let X be a G_δ-set in a compact Hausdorff space Y. Then X is a Baire space (with the subspace top.)

Corollary

Every locally compact space is Baire.
Every completely metrizable space is Baire.

Exercise

Prove that $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$ with the standard topology.

Proof of Baire Thm.

Assume first that $X = Y$.
Let $G_1, G_2, \dots$ be open dense in X and $U \neq \emptyset$ be open in X.

We want to show that $\bigcap_{n=1}^{\infty} G_n \cap U \neq \emptyset$.

Since $U \cap G_1 \neq \emptyset$ and since X is regular, there is an open $V_1 \neq \emptyset$ in X with $\overline{V_1} \subseteq U \cap G_1$.
If V_n is defined (open and non-empty), then let $V_{n+1} \neq \emptyset$ be open in X with $\overline{V_{n+1}} \subseteq V_n \cap G_{n+1}$.

Since X is compact and since $\bigcap_{n=1}^{m} \overline{V_n} \neq \emptyset$ for each $m \in \mathbb{N}$,

$\boxed{V_{m+1} \subseteq}$

it follows that $\bigcap_{n=1}^{\infty} \overline{V_n} \neq \emptyset$.

It follows that $U \cap \bigcap_{n=1}^{\infty} G_n \neq \emptyset$ since

$$\bigcap_{n=1}^{\infty} \overline{V_n} \subseteq U \cap \bigcap_{n=1}^{\infty} G_n$$

Thus $\bigcap_{n=1}^{\infty} G_n$ is dense in X.

Now assume that $X = \bigcap_{n=1}^{\infty} H_n$, where H_n is open in Y for each $n \in \mathbb{N}$.
We can assume WLOG that X is dense in Y, since Y can be replaced by $\overline{X}$ otherwise.
It follows that H_n is dense in Y for each $n \in \mathbb{N}$.
Let $G_1, G_2, \ldots$ be open dense in X.
There are open $G_1', G_2', \ldots$ in Y with $G_n = X \cap G_n'$ for each $n \in \mathbb{N}$.
For each $n \in \mathbb{N}$, the set G_n' is dense in Y.
Then

$$\bigcap_{n=1}^{\infty} G_n = X \cap \bigcap_{n=1}^{\infty} G_n' = \bigcap_{n=1}^{\infty} H_n \cap \bigcap_{n=1}^{\infty} G_n'$$

is dense in Y so it is dense in X.

Exercise

Let $f:[0,\infty) \to [0,\infty)$ be continuous and such that for each $y \in (0,\infty)$ we have

$$\lim_{n\to\infty} f(ny) = 0.$$

Prove that $\lim_{x\to\infty} f(x) = 0$.

Hint: Suppose not. Then there is $\varepsilon > 0$ s.t. for each $m \in \mathbb{N}$ there is $x \geq m$ with $f(x) > \varepsilon$.
For each $m \in \mathbb{N}$, let

$$G_m = \{ y \in (0,\infty) : \text{there is } n \geq m \text{ with } f(ny) > \varepsilon \}$$

Prove that G_m is dense open in $[0,\infty)$ for each $m \in \mathbb{N}$ and use the Baire Thm. to get a contradiction.

Theorem 25.5

There exists a continuous $f:[0,1] \to \mathbb{R}$ such that for every $x \in [0,1]$ the derivative $f'(x)$ does not exist.

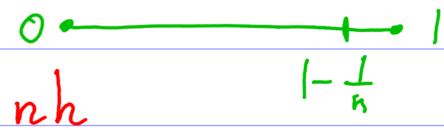
Proof

Let X be the set of all cont. $f : [0,1] \to \mathbb{R}$ with the sup-metric d, where

$$d(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}.$$

It can be proved that d is complete.

For each $n \in \mathbb{N}$, let $A_n \subseteq X$ consist of all $f \in X$ such that there is $x \in [0, 1-\frac{1}{n}]$ with the property that



$$|f(x+h) - f(x)| \leq nh$$

for each $h \in (0, \frac{1}{n}]$.

If $f \in X$ has the right derivative at some $x \in [0,1)$ then $f \in A_n$ for some $n \in \mathbb{N}$.

We show that A_n is closed and has empty interior for each $n \in \mathbb{N}$ (see the book for details).

It follows that the set of functions in X that have the right derivative somewhere in $[0,1)$ is meager in X. Similarly for the set of functions in X that have the left derivative somewhere in $(0,1]$.

The space X is complete metric so Baire Theorem implies that the set of functions in X that does not have the left derivative at any $x \in (0,1]$ and does not have the right derivative at any $x \in [0,1)$ is dense in X. In particular, this set is non-empty.