MATH 793C — TOPOLOGY

Jerzy Wojciechowski

Spring 2020

Class 23 March 6

Theorem 24.13.

Let X be a metric space. The following are equivalent.

a) X is completely metrizable.

b) X is a G_δ-set in the metric completion of X.

c) X is a G_δ-set in every metric space of which it is a subspace.

d) X is a G_δ-set in βX (the Stone-Čech compactification of X).

e) X is a G_δ-set in every Tychonoff space of which it is a dense subspace.

Proof (a) $\Rightarrow$ (d)

Let d be a metric on X that is complete, compatible with the topology and bounded by 1.

For each $x \in X$, let $\varphi'_x : X \to [0,1]$ be defined by

$$\varphi'_x(y) = d(x,y).$$

Note that φ'_x is continuous. Let $\varphi_x : \beta X \to [0,1]$ be a continuous extension of φ'_x.

Define $\rho : \beta X \times \beta X \to \mathbb{R}$ by

$$\rho(a,b) = \inf\{\, \varphi_x(a) + \varphi_x(b) \; : \; x \in X \,\}$$

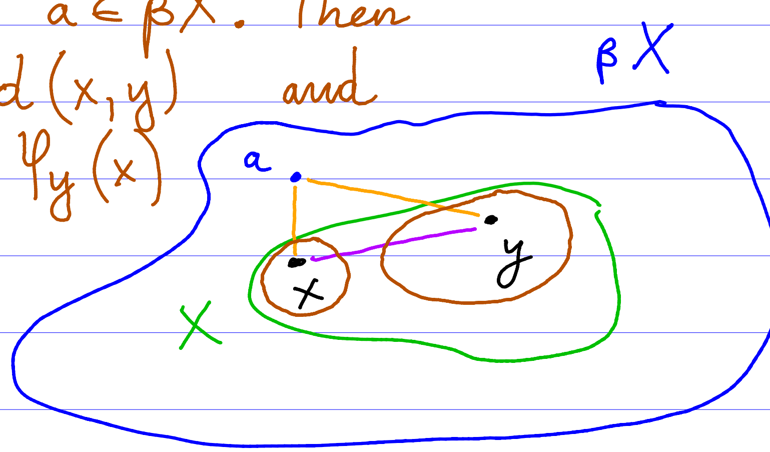
To complete the proof it remains to show that

ρ is a metric on βX such that
$\rho(a,x) = \varphi_x(a)$ for each $x \in X$ and $a \in \beta X$.

Claim 0 Let $x \in X$ and $a \in \beta X$.
If $\varphi_x(a) < \varepsilon$, then a is in the closure (in βX) of the ball $B_d(x,\varepsilon)$ (which is a subset of X)

Claim 1 Let $x, y \in X$ and $a \in \beta X$. Then
$\varphi_x(a) + \varphi_y(a) \geq d(x,y)$ and
$\varphi_x(a) + d(x,y) \geq \varphi_y(x)$

Exercise



Proof of Claim 1
Suppose, BWOC, that $\varphi_x(a) + \varphi_y(a) < d(x,y)$.
Let ε, δ be such that
$\varphi_x(a) < \varepsilon$, $\varphi_y(a) < \delta$ and $\varepsilon + \delta < d(x,y)$

Claim 0 implies that $a \in cl_{\beta X}(B_d(x, \varepsilon))$

Thus $a \in cl_{\beta X}(X \setminus B_d(y, \delta))$ since

$$B_d(x, \varepsilon) \cap B_d(y, \delta) = \phi.$$

Since φ_y is cont. and since $\varphi_y(z) \geq \delta$ for $z \in X \setminus B_d(y, \delta)$, it follows that $\varphi_y(a) \geq \delta$, which is a contradiction.

The proof of $\varphi_x(a) + d(x,y) \geq \varphi_y(x)$ is similar (Exercise)

<u>Claim 2</u>

If $x \in X$ and $a \in \beta X$, then $\varrho(a, x) = \varphi_x(a)$.

<u>Proof</u>

Let $y \in X$. Then the definition of ϱ implies that

$$\begin{aligned} \varrho(a, x) &\geq \varphi_y(a) + \varphi_y(x) \\ &= \varphi_y(a) + d(x, y) \end{aligned}$$

Claim 1 implies that $\varphi_y(a) + d(x,y) \geq \varphi_x(a)$

Thus $\varrho(a,x) \geq \varphi_x(a)$.

Since $\varrho(a,x) \leq \varphi_x(a) + \varphi_x(x) = \varphi_x(a)$

it follows that $\varrho(a,x) = \varphi_x(a)$.

Claim 3 If $a,b,c \in \beta X$, then $\varrho(a,b) + \varrho(b,c) \geq \varrho(a,c)$

Proof Suppose, BWOC, that

$$\varrho(a,b) + \varrho(b,c) < \varrho(a,c).$$

Let $x,y \in X$ be such that

$$\varphi_x(a) + \boxed{\varphi_x(b) + \varphi_y(b)} + \varphi_y(c) < \varrho(a,c)$$

Claim 1 implies that $\varphi_x(b) + \varphi_y(b) \geq \boxed{d(x,y)}$

and

$$d(x,y) + \varphi_y(c) \geq \varphi_x(c)$$

Thus $\varphi_x(a) + \varphi_x(c) < \varrho(a,c)$ which is a contr.

Claim 4. If $x \in X$ and $a \in \beta X \setminus X$, then $\varrho(x,a) > 0$.

Proof Since βX is Hausdorff there is $\varepsilon > 0$ such that $a \notin cl_{\beta X}(B_d(x,\varepsilon))$. Then $\varphi_x(a) \geq \varepsilon$, by Claim 0.

Claim 5

ϱ is a metric on βX.

Proof

It is clear that $\varrho(a,b) \geq 0$ and $\varrho(a,b) = \varrho(b,a)$ for each $a, b \in \beta X$.
Claim 2 implies that ϱ is a pseudometric on βX.

Let $a, b \in \beta X$ be distinct. If $a \in X$ or $b \in X$, then claim 4 implies that $\varrho(a,b) > 0$.

Assume $a, b \in \beta X \setminus X$. Suppose BWOC that $\varrho(a,b) = 0$. Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ in X s.t.

$$\varphi_{x_n}(a) + \varphi_{x_n}(b) < \frac{1}{n}.$$

The sequence $(x_n)_{n \in \mathbb{N}}$ is d-Cauchy so it d-converges to some $x \in X$

It follows that $\rho(x,a) = \rho(x,b) = 0$, which is a contradiction.

<u>Exercise</u>

Prove that for each $a \in \beta X \setminus X$ the singleton $\{a\}$ is open in the topology τ induced by ρ on βX.

Conclude that τ is finer than the original topology of βX.

<u>Exercise</u>

Assume that there exists a pseudo-metric ρ' on βX that extends the metric d on X and such that the original topology of βX is finer than the topology induced by ρ'. Prove that $\beta X = X$ so X is compact and $\rho' = d$.

Let X be a set and $f: X \to X$.
$x \in X$ is a fixed point iff $f(x) = x$.

Let (X, ϱ) be a metric space and $f: X \to X$.
We say that f is ϱ-contractive iff there exists $\alpha \in (0,1)$ s.t. $\varrho(f(x), f(y)) \leq \alpha \cdot \varrho(x,y)$ for all $x, y \in X$.

Theorem 24.16 (Banach)
If X is complete in a metric ϱ on X and $f: X \to X$ is ϱ-contractive, then f is continuous and has exactly one fixed point.

Proof
The uniqueness is clear. Cont. of f is clear.
Let $\alpha \in (0,1)$ be s.t. $\varrho(f(x), f(y)) \leq \alpha \cdot \varrho(x,y)$ $\forall x, y \in X$.
Let $x \in X$. Let $x_1 = x$, $x_2 = f(x_1), \dots, x_{n+1} = f(x_n)$ for each $n \in \mathbb{N}$.
$(x_n)_{n \in \mathbb{N}}$ is Cauchy since $\sum_{n=1}^{\infty} \alpha^n$ converges.

Let y be the limit of $(x_n)_{n \in \mathbb{N}}$.

Since f is cont. $f(y) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = y$

25. The Baire theorem.

Let X be a topological space.
We say that X is a Baire space iff the intersection of any countable family of dense open sets in X is dense in X.

<u>Example</u>
$\mathbb{Q}$ with standard top. is not a Baire space.
Let $q_1, q_2, \dots$ be an enumeration of $\mathbb{Q}$.
$U_n = \mathbb{Q} \setminus \{q_n\}$ is open in $\mathbb{Q}$ and dense in $\mathbb{Q}$.

$\bigcap_{n \in \mathbb{N}} U_n = \emptyset$ which is not dense in $\mathbb{Q}$.