MATH 793C — TOPOLOGY

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Theorem 24.13.

Let X be a metric space. The following are equivalent.

a) X is completely metrizable.

b) X is a G_δ-set in the metric completion of X.

c) X is a G_δ-set in every metric space of which it is a subspace.

d) X is a G_δ-set in βX (the Stone-Čech compactification of X).

e) X is a G_δ-set in every Tychonoff space of which it is a dense subspace.

Proof

We know that (e) $\Rightarrow$ (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)

(d) $\Rightarrow$ (e)

Let Y be a Tychonoff space of which X is a dense subspace. Then X is a dense subspace of βY.

Let $f : \beta X \longrightarrow \beta Y$ be a continuous extension of the identity embedding $X \longrightarrow \beta Y$.

Let G_n be open in βX for each $n \in \mathbb{N}$ with

$$X = \bigcap_{n \in \mathbb{N}} G_n.$$

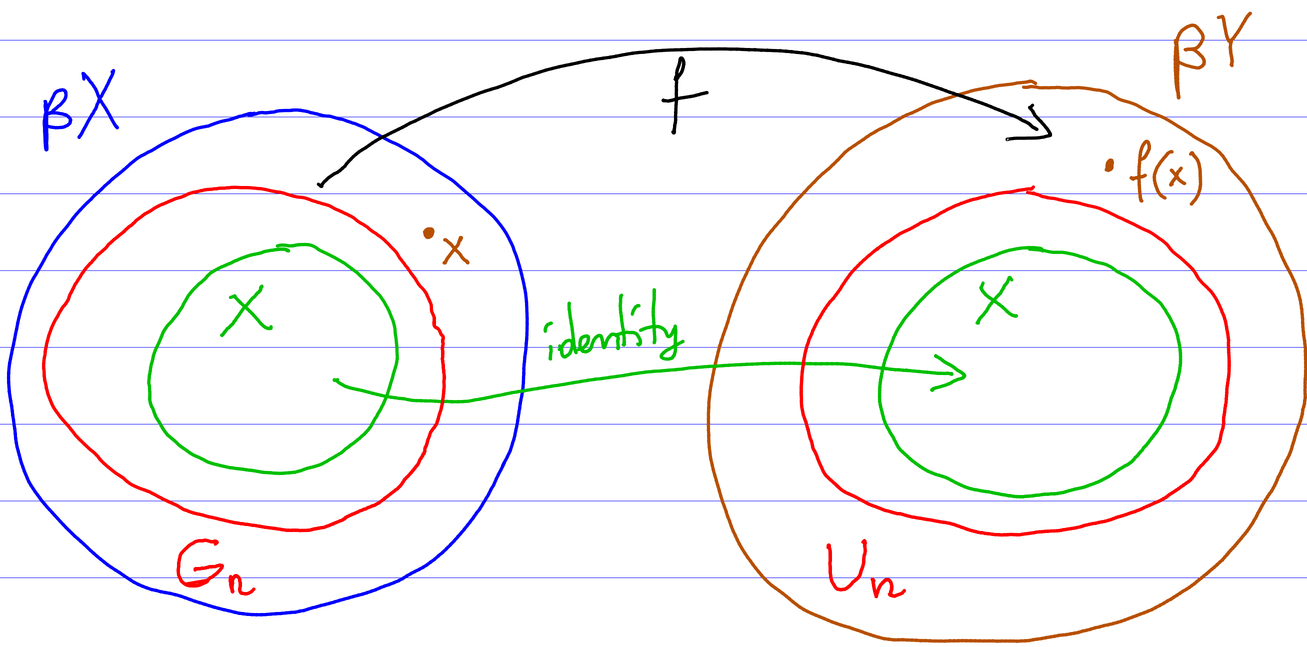
open in βY for each $n \in \mathbb{N}$

For each $n \in \mathbb{N}$, let $U_n = \beta Y \setminus f[\beta X \setminus G_n]$.

To complete the proof it suffices to show that

$$X = \bigcap_{n \in \mathbb{N}} U_n.$$

Hence $X = \bigcap_{n \in \mathbb{N}} (Y \cap U_n)$, where $Y \cap U_n$ is open in Y for each $n \in \mathbb{N}$.



By Theorem 19.8 in the textbook (covered last semester in class 19)

$$f[\beta X \setminus X] = \beta Y \setminus X$$

It follows that $X \subseteq \bigcap_{n\in\mathbb{N}} U_n$.

If $x \in \beta X \setminus X$, then there is $n \in \mathbb{N}$ s.t. $x \in \beta X \setminus G_n$, so $f(x) \notin U_n$.

Since $f[\beta X \setminus X] = [\beta Y \setminus X]$, if $y \in \beta Y \setminus X$, there is $x \in \beta X \setminus X$ with $y = f(x)$ so there is $n \in \mathbb{N}$ with $y \notin U_n$.

Thus $X = \bigcap_{n\in\mathbb{N}} U_n$.

(a) $\Rightarrow$ (d)

Let d be a metric on X that is complete, compatible with the topology and bounded by 1.

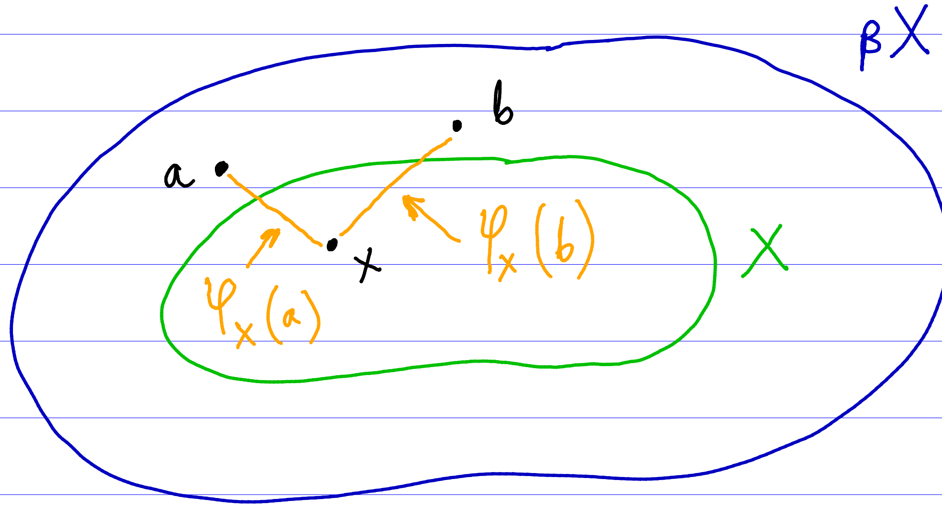
For each $x \in X$, let $\varphi'_x : X \to [0,1]$ be defined by

$$\varphi'_x(y) = d(x,y).$$

Note that φ'_x is continuous. Let $\varphi_x : \beta X \to [0,1]$ be a continuous extension of φ'_x.

Define $\rho : \beta X \times \beta X \to \mathbb{R}$ by

$$\rho(a,b) = \inf\{ \varphi_x(a) + \varphi_x(b) : x \in X\}$$



We will show that

ϱ is a metric on βX such that
$\varrho(a,x) = \varphi_x(a)$ for each $x \in X$ and $a \in \beta X$.

It follows that ϱ is an extension of d.

For each $n \in \mathbb{N}$, Let

$$G_n = \left\{ a \in \beta X : \varphi_x(a) < \frac{1}{n} \text{ for some } x \in X \right\}$$
$$= \bigcup_{x \in X} \left\{ a \in \beta X : \varphi_x(a) < \frac{1}{n} \right\}$$

Then G_n is open in βX since φ_x is cont. for each $x \in X$.

We show that $X = \bigcap_{n \in \mathbb{N}} G_n$.

It is clear that $X \subseteq \bigcap_{n \in \mathbb{N}} G_n$.

Let $a \in \bigcap_{n \in \mathbb{N}} G_n$. For each $n \in \mathbb{N}$, let $x_n \in X$ be such that $\varrho(x_n, a) < \frac{1}{n}$.

Then $(x_n)_{n \in \mathbb{N}}$ ϱ-converges to a.
Thus it is ϱ-Cauchy, hence d-Cauchy.
Since X is d-complete, it has a d-limit x in X. Since ϱ extends d, x is a ϱ-limit of $(x_n)_{n \in \mathbb{N}}$. Since ϱ is a metric on βX, it follows that $a = x$. Thus $a \in X$.

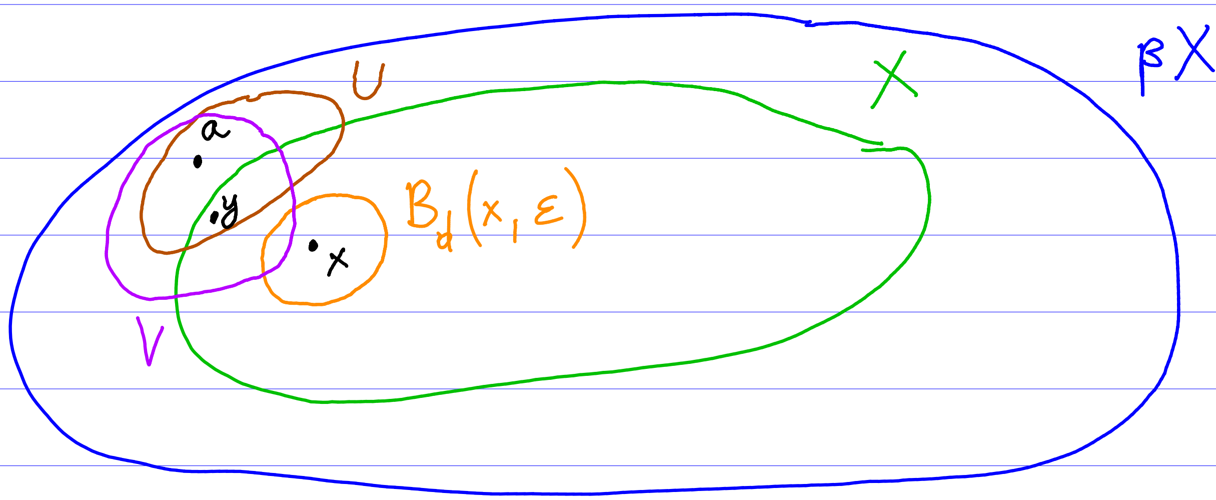
Hence $X = \bigcap_{n \in \mathbb{N}} G_n$.

<u>Exercise*</u>. Is it true that the top. induced by ϱ on βX is finer than the original top. of βX?

It remains to show that

ϱ is a metric on βX such that
$\varrho(a, x) = \varphi_x(a)$ for each $x \in X$ and $a \in \beta X$.

Claim 0 Let $x \in X$ and $a \in \beta X$.
If $\varphi_x(a) < \varepsilon$, then a is in the closure (in βX) of the ball $B_d(x, \varepsilon)$ (which is a subset of X)



Proof of Claim 0
Suppose that $a \notin cl_{\beta X}(B_d(x, \varepsilon))$. Let U be a nbhd of a in βX with

$$U \cap B_d(x, \varepsilon) = \phi.$$

There is a nbhd V of a in βX s.t. $\varphi_x[V] \subseteq [0, \varepsilon)$ (since φ_x is continuous).
Since X is dense in βX, there is $y \in U \cap V \cap X$.
Thus $\varphi_x(y) < \varepsilon$ so $y \in B_d(x, \varepsilon)$ $(\varphi_x(y) = d(x, y))$ which is a contradiction.