MATH 793C – TOPOLOGY

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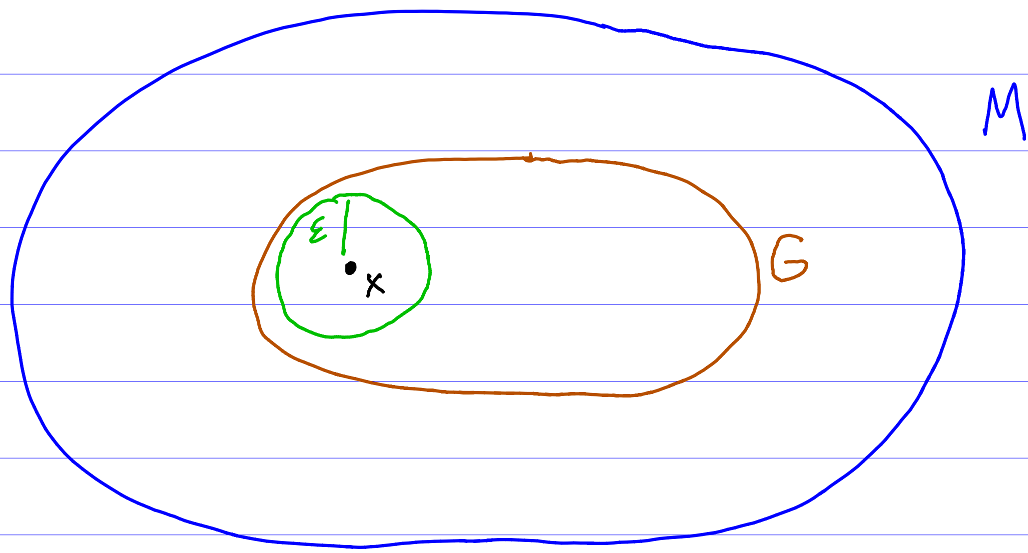
Lemma

Let M be a complete metric space and G be a subspace of M that is open in M. Then G is completely metrizable.

Proof

Let d be the metric of M. We can assume that $G \neq M$.

Define $f : G \to (0, \infty)$ by $f(x) = \dfrac{1}{\inf\{\underbrace{d(x,y) : y \in M \setminus G}_{\geq \varepsilon}\}}$



Then f is continuous (Exercise).

Define $\varrho : G \times G \longrightarrow [0,\infty]$ by

$$\varrho(x,y) = d(x,y) + |f(x) - f(y)|$$

Then ϱ is a metric on G (Exercise).

We show that ϱ induces on G the same topology as d and that G with ϱ is complete.

Let $(x_n)_{n\in\mathbb{N}}$ be a ϱ-Cauchy sequence in G.

Then $(x_n)_{n\in\mathbb{N}}$ is also d-Cauchy.
Let $y \in M$ be the d-limit of $(x_n)_{n\in\mathbb{N}}$.
We show that $y \in G$. (when we prove that ϱ and d induce the same topology on G, it will follow that y is the ϱ-limit of $(x_n)_{n\in\mathbb{N}}$).

Suppose BWOC that $y \in M \setminus G$.
Since $(x_n)_{n\in\mathbb{N}}$ is ϱ-Cauchy, there is $m \in \mathbb{N}$ s.t.
$\varrho(x_n, x_k) < 1$ so $|f(x_n) - f(x_k)| < 1$ for $n, k \geq m$.

Since $(x_n)_{n\in\mathbb{N}}$ d-converges to y there is $n \geq m$ s.t.

$$d(x_n, y) < \frac{1}{f(x_m)+1} \qquad \frac{1}{d(x_n,y)} > f(x_m) + 1$$

Then $f(x_n) > f(x_m) + 1$ so

$$|f(x_n) - f(x_m)| > 1$$ which is a contr.

Let τ' be the topology induced by ϱ on G and τ be the original top. on G (induced by d).

Since $\varrho(x,y) \geq d(x,y)$ for each $x, y \in G$, it follows that $U_\varrho(x,\varepsilon) \subseteq U_d(x,\varepsilon)$ for each $\varepsilon > 0$. Thus $\tau \subseteq \tau'$.

Let $x \in G$, $r = f(x)$ and $\varepsilon > 0$ (assume $\varepsilon < 2r$).

Let $\delta = \min\left\{ \varepsilon/2, \frac{1}{r} - \frac{1}{r + \varepsilon/2}, \frac{1}{r - \varepsilon/2} - \frac{1}{r - \varepsilon/3} \right\}$

open ball centered at x of radius δ

We show that $U_d(x,\delta) \subseteq U_\varrho(x,\varepsilon)$. (so $\tau' \subseteq \tau$)

If $z \in U_d(x,\delta)$ then $d(x,z) < \frac{1}{r} - \frac{1}{r+\varepsilon/2}$

$d(x,z) < \frac{\varepsilon}{2}$ and $d(x,z) < \frac{1}{r - \varepsilon/2} - \frac{1}{r - \varepsilon/3}$

$$r \geq \frac{1}{d(x,y)} \text{ for each } y \in M \setminus G$$

$$r - \varepsilon/3 < \frac{1}{d(x,y)} \quad \text{for some } y \in M \setminus G$$

true for any $y \in M$

For all $y \in M \setminus G$

For some $y \in M \setminus G$

$$\frac{1}{r - \varepsilon/3} > d(x,y) \geq \frac{1}{r}$$

$$d(x,y) - d(x,z) \leq d(z,y) \leq d(x,y) + d(x,z)$$

$$\frac{1}{r + \varepsilon/2} < d(z,y) < \frac{1}{r - \varepsilon/2}$$

$$\frac{1}{r + \varepsilon/2} \leq \inf\{d(z,y) : y \in M \setminus G\} < \frac{1}{r - \varepsilon/2}$$

$$r + \varepsilon/2 \geq f(z) > r - \varepsilon/2$$

Thus $|f(z) - f(x)| \leq \frac{\varepsilon}{2}$.

$$d(x,z) < \frac{\varepsilon}{2}$$

$$\rho(x,z) < \varepsilon \quad \text{so} \quad z \in U_\rho(x,\varepsilon)$$

Theorem 24.13.

Let X be a metric space. The following are equivalent.

a) X is completely metrizable.

b) X is a G_δ-set in the metric completion of X.

c) X is a G_δ-set in every metric space of which it is a subspace.

d) X is a G_δ-set in βX (the Stone-Čech compactification of X).

e) X is a G_δ-set in every Tychonoff space of which it is a dense subspace.

completely reg. Hausdorff

Proof We know that ⓐ $\Leftrightarrow$ ⓒ. Moreover ⓒ $\Rightarrow$ ⓑ $\Rightarrow$ ⓐ. ⓔ $\Rightarrow$ ⓑ is clear. ⓒ $\Rightarrow$ ⓓ will be proved later.

ⓓ $\Rightarrow$ ⓔ

Let Y be a Tychonoff space of which X is a dense subspace. Then X is a dense subspace of βY. (Since Y is dense in βY and X is dense in Y)

Let $f: \beta X \longrightarrow \beta Y$ be the continuous extension of the identity embedding $X \longrightarrow \beta Y$.

Such f exists since βX is the Stone-Čech compactification of X

For any compact Z and cont. $g: X \to Z$ there is a cont. ext. $g': \beta X \longrightarrow Z$.