MATH 793C — TOPOLOGY

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Theorem 24.10.
Let M be a complete pseudometric space. If A is a closed subset of M, then A is complete (with the same pseudometric).

Remark
If M is a metric space and A is a subspace of M that is complete (in the same metric), then A is closed in M.

Proof
Let $x \in \overline{A}$. Let $(x_n)_{n\in\mathbb{N}}$ be a seq. in A that conv. to x. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy so it converges to some $y \in A$. Since M is Hausdorff, we have $x = y$ so $x \in A$. Thus A is closed.

Proof of the theorem.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy seq. in A.

Since M is complete there is $x \in M$ s.t.

$$x = \lim_{n \to \infty} x_n .$$

Since A is closed, we have $x \in A$.
Thus A is complete.

Theorem 24.11

Let X_n be a nonempty metric space for each $n \in \mathbb{N}$ and $X = \prod_{n \in \mathbb{N}} X_n$. Then X is completely metrizable if and only if each X_n is completely metrizable.

Proof

($\Leftarrow$)

Let ρ be a complete metric on X and a_n be a fixed element of X_n for each $n \in \mathbb{N}$.

For each $m \in \mathbb{N}$, let $\varphi_m : X_m \to X$ be defined by

$$\varphi_m(x) = (x_n)_{n \in \mathbb{N}}$$

where $x_m = x$ and $x_n = a_n$ for $n \neq m$.

Then φ_m is an embedding and $\varphi_m[X_m]$ is closed in X.
Thus X_m is completely metrizable for each $m \in \mathbb{N}$.

$(\Longrightarrow)$

Assume X_m is completely metrizable for each $m \in \mathbb{N}$.
Let d_m be a complete metric on X_m, $m \in \mathbb{N}$.

WLOG

We can assume that d_n is bounded by 1 for each $n \in \mathbb{N}$.

Let $d : X \times X \longrightarrow [0, \infty)$ be defined by

$$d\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}$$

We know (see class 2) that d is a metric on X that is compatible with the product topology on X.

We show that d is complete.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X,

where $x_n = (x_{n,m})_{m\in\mathbb{N}}$, $x_{n,m} \in X_m$, $n, m \in \mathbb{N}$

Claim

For each $m \in \mathbb{N}$, $(x_{n,m})_{n\in\mathbb{N}}$ is a Cachy seq. in X_m

Let $m \in \mathbb{N}$ be fixed.

Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ s.t. $\forall n, l \geq k$ we have

$$d(x_n, x_l) < \frac{\varepsilon}{2^m}$$

Then $d_m(x_{n,m}, x_{l,m}) < \varepsilon$ for each $n, l \geq k$.

Thus the claim is proved.

Let $y_m = \lim_{n \to \infty} x_{n,m}$ for each $m \in \mathbb{N}$.

It follows that $(y_m)_{m \in \mathbb{N}} = \lim_{n \to \infty} x_n$.

Recall

If Y_α is a top. space for each $\alpha \in A$ and $Y = \prod_{\alpha \in A} Y_\alpha$ has the product top, then a net $(y_\beta)_{\beta \in I}$ in Y converges to $y \in Y$ if and only if for each $\alpha \in A$ the net $\left(\pi_\alpha(y_\beta)\right)_{\beta \in I}$ converges to $\pi_\alpha(y)$.

Lemma

Let M be a complete metric space and G be a subspace of M that is open in M. Then G is completely metrizable.

Proof (Later)

Lemma

Let X be a topological space and X_α be a nonempty subspace of X for each $\alpha \in A$ ($A \neq \phi$). If $Y = \prod_{\alpha \in A} X_\alpha$, $Z = \bigcap_{\alpha \in A} X_\alpha$ and $f : Z \to Y$ is given by $f(x) = (x_\alpha)_{\alpha \in A}$, where $x_\alpha = x$ for each $\alpha \in A$, then f is an embedding.

Proof It is clear that f is injective. For each $\alpha \in A$, the composition $\pi_\alpha \circ f : Z \to X_\alpha$ is an embedding (hence it is cont.). Thus f is cont.

identity on Z

Let U be open in Z. Let $\beta \in A$. There is open U_β in X_β s.t. $U = U_\beta \cap Z$.

Let $U' = \prod_{\alpha \in A} U_\alpha$, where $U_\alpha = X_\alpha$ for $\alpha \neq \beta$.

Then U' is open in Y and $f[U] = U' \cap f[Z]$.

Thus $f[U]$ is open in $f[Z]$.

24.12. Theorem (Alexandroff)

Let M be a complete metric space and N be a subspace of M that is a G_δ-set in M. Then N is completely metrizable.

Proof

Let $N = \bigcap_{n \in \mathbb{N}} G_n$ with G_n open in M for each $n \in \mathbb{N}$.

Then $G = \prod_{n \in \mathbb{N}} G_n$ is completely metrizable

Let $f : N \longrightarrow G$ be defined by $f(x) = (x_i)_{i \in \mathbb{N}}$ where $x_i = x$ for each $i \in \mathbb{N}$.

$f[N]$ is closed in G hence completely metrizable.

Since $f : N \to f[N]$ is a homeomorphism, it follows that N is completely metrizable.

Office hours 1:30 – 2:00 pm

MWF