MATH 793C – TOPOLOGY

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24.8. Lemma. Let X be a metric space, Y be a complete metric space and $A \subseteq X$. If $f : A \to Y$ is continuous, then there exists a G_δ-set A^* in X and a continuous $f^* : A^* \to Y$ such that $A \subseteq A^* \subseteq \bar{A}$ and $f^*(x) = f(x)$ for every $x \in A$.

Proof. Let

$$\begin{aligned} A^* &= \left\{x \in \bar{A} : \operatorname{osc}(f,x) = 0\right\} \\ &= \bigcap_{n\in\mathbb{N}} \left\{x \in \bar{A} : \operatorname{osc}(f,x) < \frac{1}{n}\right\}. \end{aligned}$$

Then A^* is a G_δ-set in X and $A \subseteq A^* \subseteq \bar{A}$.

If $x \in A^*$, then $x \in \bar{A}$ so there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to x. Since $\operatorname{osc}(f,x) = 0$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy. Define

$$f^*(x) = \lim_{n\to\infty} f(x_n)$$

exists since Y is complete

Let $(y_n)_{n\in\mathbb{N}}$ a sequence in A that converges to x.

Let $(z_n)_{n\in\mathbb{N}}$ be defined by $z_{2n-1} = x_n$ and $z_{2n} = y_n$ for each $n \in \mathbb{N}$.
Then $(z_n)_{n\in\mathbb{N}}$ converges to x so $(f(z_n))_{n\in\mathbb{N}}$ is Cauchy.
It follows that $(f(z_n))_{n\in\mathbb{N}}$ converges to some $z \in Y$.
Then both $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ converge to z.
Since Y is Hausdorff $z = f^*(x)$.

Thus $\lim_{n\to\infty} f(y_n) = f^*(x)$.

Thus the definition of $f^*(x)$ does not depend on the choice of $(x_n)_{n\in\mathbb{N}}$.

If $x \in A$ and $x_n = x$ for each $n \in \mathbb{N}$, then

$$f^*(x) = \lim_{n\to\infty} f(x_n) = f(x)$$

so f^* is an extension of x.

It remains to show that f^* is continuous. It suffices to show that $\operatorname{osc}(f^*, x) = 0$ for each $x \in A^*$.

Let $x \in A^*$ and $\varepsilon > 0$. We want to find a nbhd U of x in X such that $\boxed{\operatorname{diam}(f^*[A^* \cap U]) < \varepsilon}$.
Since $\operatorname{osc}(f, x) = 0$, there is a nbhd U of x in X such that $\boxed{\operatorname{diam}(f[A \cap U]) < \varepsilon/3}$.

Let $x, y \in A^* \cap U$ and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be sequences in A with

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

Let ϱ be the metric of Y.

$$f^*(x) = \lim_{n\to\infty} f(x_n) \qquad f^*(y) = \lim_{n\to\infty} f(y_n)$$

Let $n \in \mathbb{N}$ be such that $\varrho(f(x_n), f^*(x)) < \varepsilon/3$, $\qquad \varrho(f(y_n), f^*(y)) < \varepsilon/3$

and $x_n, y_n \in U$.

Then $\varrho(f(x_n), f(y_n)) < \varepsilon/3$ since $x_n, y_n \in U \cap A$

so $\quad \varrho(f^*(x), f^*(y)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

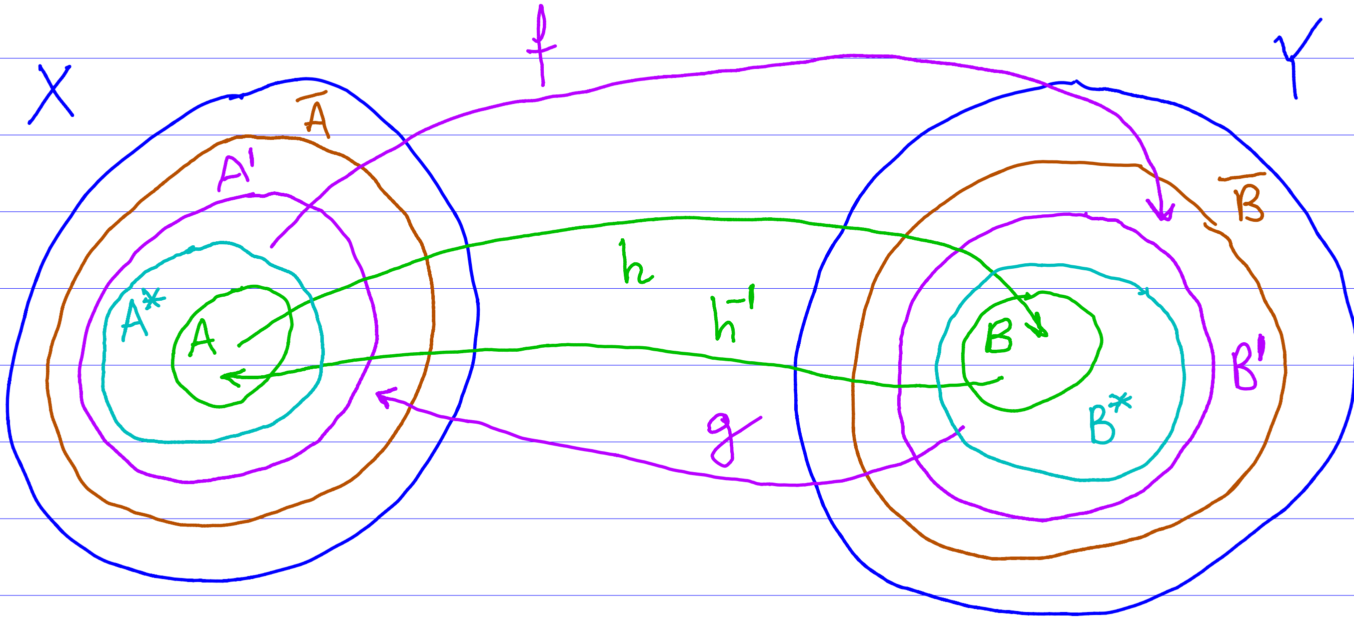
We proved that $\operatorname{diam}(f^*[A^* \cap U]) < \varepsilon$, so

$\operatorname{osc}(f^*, x) = 0$

Thus f^* is continuous at x.

24.9. Theorem (Lavrentieff). Let X and Y be complete metric spaces and $h : A \to B$ be a homeomorphism for some $A \subseteq X$ and $B \subseteq Y$. Then there are G_δ-sets A^* and B^* in X and Y, respectively, such that $A \subseteq A^* \subseteq \overline{A}$ and $B \subseteq B^* \subseteq \overline{B}$ and h can be extended to a homeomorphism $h^* : A^* \to B^*$.

Proof Let A', B' be G_δ-sets in X, Y, respectively and $f : A' \to Y$, $g : B' \to Y$ be continuous extensions of $h : A \to B$ and $h^{-1} : B \to X$, respectively, with $A \subseteq A' \subseteq \overline{A}$ and $B \subseteq B' \subseteq \overline{B}$.



Let $A^* = \{x \in A' : f(x) \in B'\}$ and $B^* = \{y \in B' : g(y) \in A'\}$

Since f is continuous and B' is a G_δ-set in Y, it follows that $A^* = f^{-1}[B']$ is a G_δ-set in A'.
Since A' is a G_δ-set in X, it follows that A^* is a G_δ-set in X.

Similarly, B^* is a G_δ-set in Y.

It is clear that $A \subseteq A^* \subseteq \overline{A}$ and $B \subseteq B^* \subseteq \overline{B}$.

$g \circ f : A^* \longrightarrow X$ and $f \circ g : B^* \longrightarrow Y$ are continuous extensions of the identity maps $A \to A$ and $B \to B$, respectively.

$(g \circ f)(x) = x$ for $x \in A$ and $(f \circ g)(y) = y$ for $y \in B$

$\varphi : A^* \to X$ be the identity function $\varphi(x) = x$ for $x \in A^*$

continuous

φ and $g \circ f$ agree on A, A is dense in A^* and A^* is Hausdorff, thus $\varphi = g \circ f$.

Since X is Housdorff and A is dense in A^*, it follows that $g \circ f \restriction A^*$ is the identity map on A^*.

Similarly, $f \circ g \restriction B^*$ is the identity map on B^*.

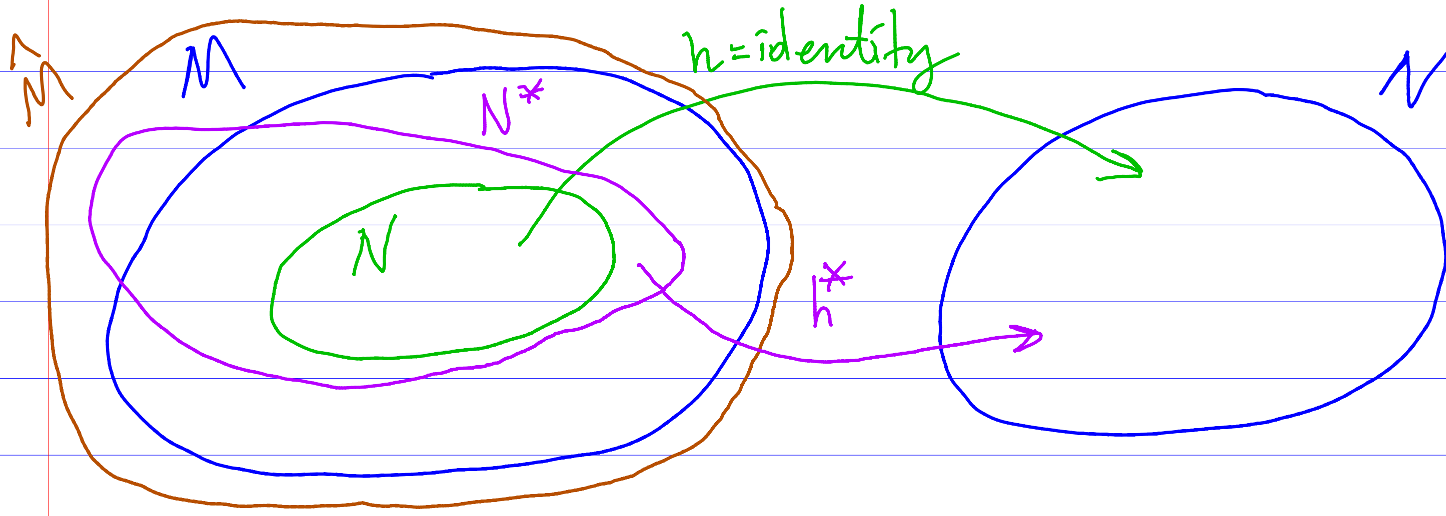
Let h^* be the restriction of f to A^*.

The restriction of g to B^* is the inverse of h^*

Thus h^* is a homeomorphism $A^* \to B^*$.

<u>24.12. Theorem</u> (Mazurkiewicz)
Let M be a metric space and N be a subspace of M that is completely metrizable. Then N is a G_δ-set in M.

<u>Proof</u>
Let ρ be a compatible complete metric on N.



$\hat{M}$ – the completion of M

Let $h : N \longrightarrow N$ be the identity function.
Then there is a G_δ-set N^* in $\hat{M}$ with
$N \subseteq N^* \subseteq cl_{\hat{M}}(N)$ and a homeomorphism $h^* : N^* \longrightarrow N$
that extends h. (by Lavrentieff Theorem)

We must have $N^* = N$ and $h^* = h$.
Thus N is a G_δ-set in $\hat{M}$. It follows that
N is a G_δ-set in M.