MATH 793C — TOPOLOGY

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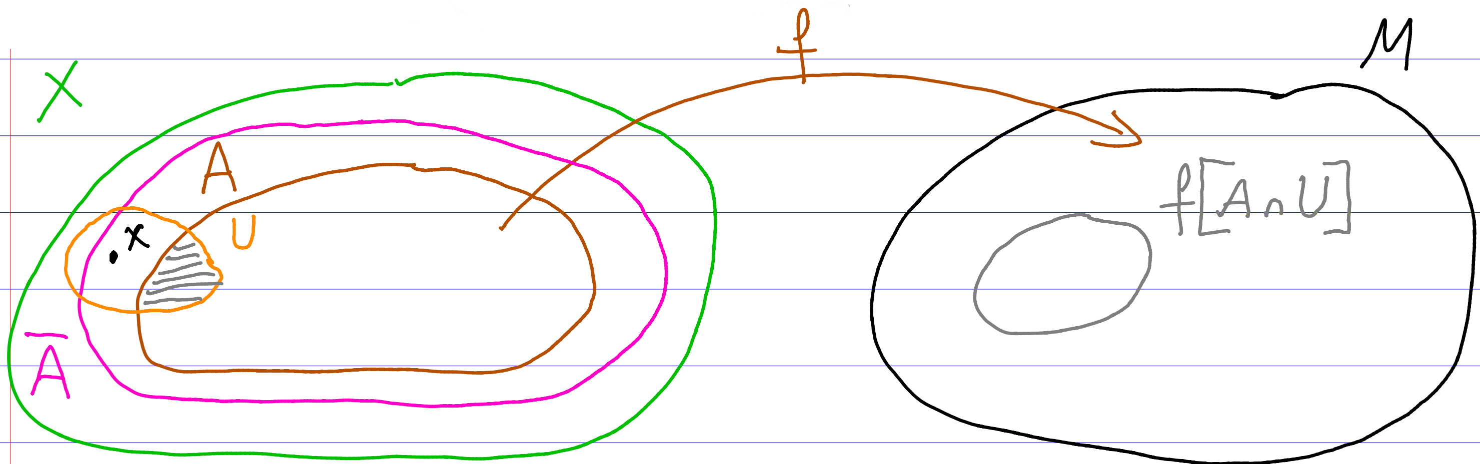
Oscilation. Let M be a metric space with metric d, let $A \subseteq X$ (X – topological space) and $f : A \to M$. If $x \in \overline{A}$, then let the oscilation of f at x, denoted $\operatorname{osc}(f,x)$, be defined by

$$\begin{aligned}\operatorname{osc}(f,x) &= \inf\{\operatorname{diam}(f[A\cap U]) : U \text{ is a nbhd of } x \text{ in } X\}\\ &= \inf\{\sup\{d(f(y),f(z)) : y,z \in A\cap U\} : U \text{ is a nbhd of } x \text{ in } X\}\end{aligned}$$



$$\operatorname{diam}(f[A\cap U]) = \sup\{d(f(x),f(y)) : x,y \in A\cap U\}$$

Remark. Note that if $x \in A$, then $\operatorname{osc}(f,x) = 0$ if and only if f is continuous at x.

Proof. Assume that f is continuous at x.
Let $\varepsilon > 0$. We will show that there is a nbhd U of x in X s.t. $\operatorname{diam}(f[A\cap U]) < \varepsilon$.

Since f is continuous at x, there is an open nbhd V

of x in A s.t. $f[V] \subseteq \boxed{U_d(f(x), \varepsilon/2)}$

open ball cent. at $f(x)$ and radius $\varepsilon/2$

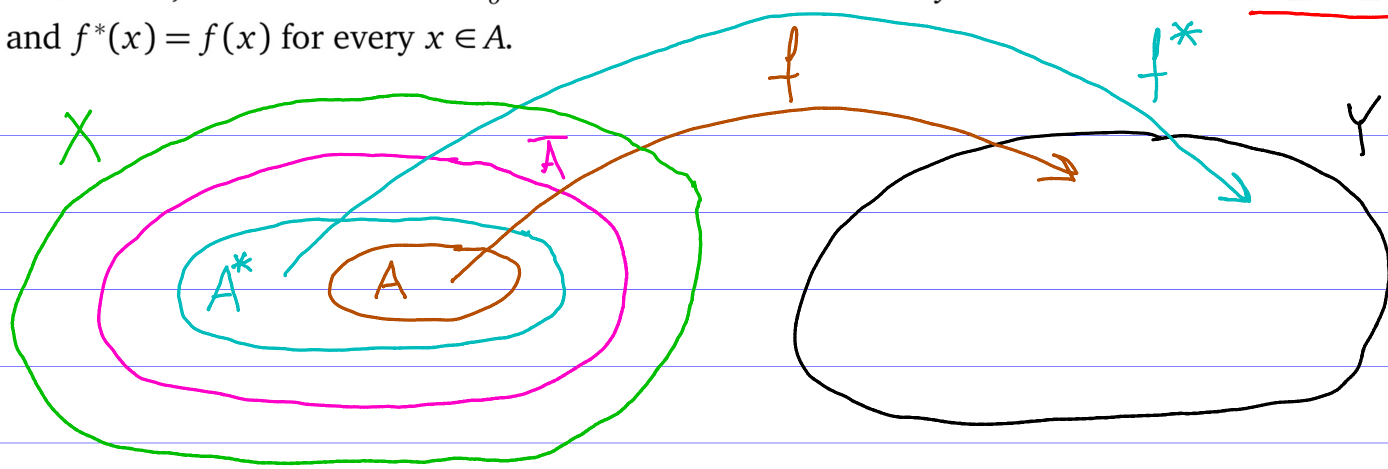
Let U be open in X s.t. $U \cap A = V$.
If $y, z \in V$, then

$$d(f(y), f(z)) \leq d(f(y), f(x)) + d(f(z), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Now assume that $osc(f, x) = 0$.
Let $\varepsilon > 0$. We want to find open nbhd V of x in A s.t. $f[V] \subseteq U_d(f(x), \varepsilon)$.

Since $osc(f, x) = 0$, there is a nbhd U of x in X s.t. $diam(f[U \cap A]) < \varepsilon$. WLOG U is open.
Let $V = U \cap A$. Then $f[V] \subseteq U_d(f(x), \varepsilon)$.

24.8. Lemma. Let X be a metric space, Y be a complete metric space and $A \subseteq X$. If $f : A \to Y$ is continuous, then there exists a G_δ-set A^* in X and a continuous $f^* : A^* \to Y$ such that $A \subseteq A^* \subseteq \bar{A}$ and $f^*(x) = f(x)$ for every $x \in A$.



Proof. Let

$$A^* = \{x \in \bar{A} : \operatorname{osc}(f,x) = 0\}$$
$$= \bigcap_{n\in\mathbb{N}} \underbrace{\boxed{\left\{x \in \bar{A} : \operatorname{osc}(f,x) < \frac{1}{n}\right\}}}_{A_n}.$$

open in $\bar{A}$

Then A^* is a G_δ-set in X and $A \subseteq A^* \subseteq \bar{A}$.

A_n is open in $\bar{A}$.

Let $x \in A_n$. Then $x \in \bar{A}$ and $\operatorname{osc}(f,x) < \frac{1}{n}$ so there is an open nbhd U of x in X s.t.

$$\operatorname{diam}(f[U \cap A]) < \frac{1}{n}.$$

nbhd of x in $\bar{A}$

If $y \in U \cap \bar{A}$, then $\operatorname{osc}(f,y) < \frac{1}{n}$. Thus $U \cap \bar{A} \subseteq A_n$.

Recall Every closed set in a metric space is a G_δ-set.

Thus $\bar{A} = \bigcap_{n\in\mathbb{N}} A'_n$, where A'_n is open in X.

$A_n = \bar{A} \cap A''_n$, where A''_n is open in X.

Thus $A^* = \left(\bigcap_{n\in\mathbb{N}} A'_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} A''_n\right)$ so A^* is a G_δ-set in X.

If $x \in A^*$, then $x \in \bar{A}$ so there is a sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to x. Since $\operatorname{osc}(f,x) = 0$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy. Define

$$f^*(x) = \lim_{n\to\infty} f(x_n).$$

$(f(x_n))_{n\in\mathbb{N}}$ is Cauchy.

Let $\varepsilon>0$. Since $\operatorname{osc}(f,x)=0$ there is a nbhd U of x in X s.t. $\operatorname{diam}(f[A\cap U])<\varepsilon$.

There is $m\in\mathbb{N}$ s.t. $x_n\in U$ for each $n\geq m$.

Let $n,k\geq m$. Then $d(f(x_n)-f(x_k))<\varepsilon$.