# MATH 793C — TOPOLOGY

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Let $(M, d)$ be a pseudometric space. A <u>completion</u> of $(M, d)$ is a complete pseudometric space $(N, \varrho)$ s.t. $M$ is a dense subset of $N$ and $d$ is the restriction of $\varrho$ to $M \times M$.

<u>Corollary</u>

Each pseudometric space has a completion. Every metric space has a completion that is a metric space.

If $(M, d)$ is a metric space and both $(N, \varrho)$ and $(N', \varrho')$ are metric completions of $(M, d)$, then there exists a bijective isometry $g: N \to N'$ s.t. $g(x) = x$ for each $x \in M$.

<u>Examples</u>

ⓐ $[0,1]$ is a completion of $(0,1)$

ⓑ $\mathbb{R}$ is a completion of $\mathbb{Q}$

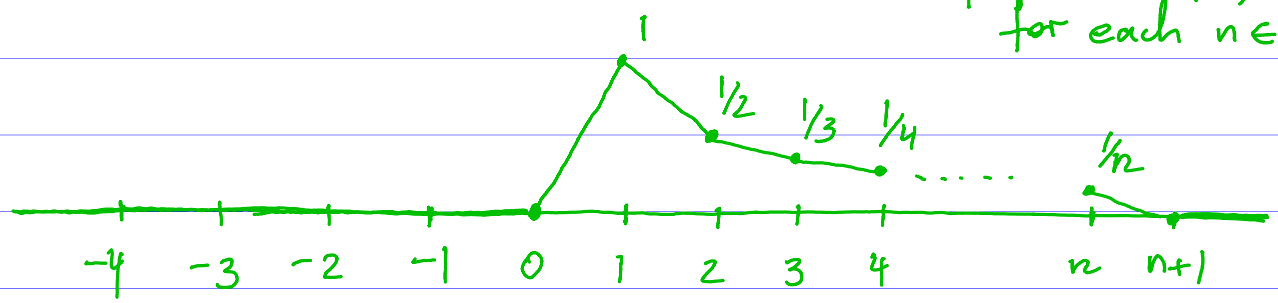
(c) Let $X$ be a top. space and $C_{00}(X)$ be the set of cont. functions $f: X \to \mathbb{R}$ s.t. there is a compact $K \subseteq X$ with $f(x) = 0$ for each $x \in X \setminus K$.

$C_{00}(X)$ is the set of cont. funct. $X \to \mathbb{R}$ with compact support.

Let $\rho(f,g) = \sup_{x \in X} |f(x) - g(x)|$ for each $f, g \in C_{00}(X)$.

Then $(C_{00}(X), \rho)$ is a metric space that is not complete is general.

Take $X = \mathbb{R}$ and $f_n : X \to \mathbb{R}$ $\quad f_n \in C_{00}(X)$ for each $n \in \mathbb{N}$




$(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $C_{00}(X)$.

Proof

Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ be such that $1/m < \varepsilon$

If $n, k \geq m$, then $\rho(f_n, f_k) < 1/m < \varepsilon$.

There are no $f \in C_{00}(X)$ s.t. $f = \lim_{n \to \infty} f_n$.

The completion of $C_{00}(X)$ is $C_0(X)$ which is the set of cont. functions $f: X \to \mathbb{R}$ s.t. for each $\varepsilon > 0$ there is a compact $K_\varepsilon \subseteq X$ s.t. $|f(x)| < \varepsilon$ for each $x \in X \setminus K_\varepsilon$.

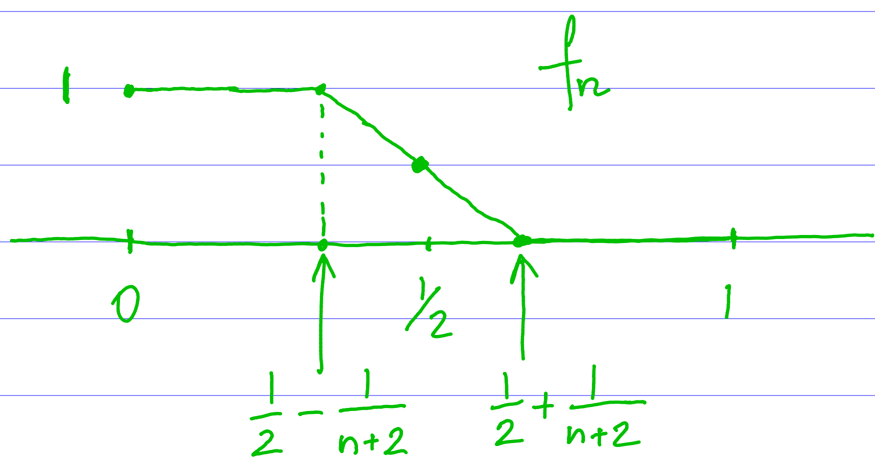
The metric on $C_0(X)$ is defined as on $C_{00}(X)$.

(d) Let $M$ be the set of all cont. $f: [0,1] \to \mathbb{R}$.

Let
$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx \quad \text{(Riemann integr.)}$$

Then $(M,d)$ is a metric space that is not complete.

Let $f_n: [0,1] \to \mathbb{R}$ be




$(f_n)_{n \in \mathbb{N}}$ is Cauchy

Proof Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ be such that $\frac{2}{m+2} < \varepsilon$

If $n, k \geq m$, then $d(f_n, f_k) \leq \frac{2}{m+2} \cdot 1 < \varepsilon$

$f_n(x) = f_k(x)$ for $x$ outside $\left(\frac{1}{2} - \frac{1}{m+2}, \frac{1}{2} + \frac{1}{m+2}\right)$

$|f_n(x) - f_k(x)| \leq 1$ for $x \in$

$(f_n)_{n \in \mathbb{N}}$ does not converge in $M$.

Let $N$ be the set of all Lebesgue integrable $f : [0,1] \longrightarrow \mathbb{R}$ and

$$\varrho(f, g) = \int_0^1 |f(x) - g(x)| \, dx \quad (\text{Leb. int.}) \quad (f, g \in N)$$

Then $(N, \varrho)$ is a pseudometric space that is a completion of $(M, d)$.

24.12. Theorem (Alexandroff)
Let $M$ be a complete metric space and $N$ be a subspace of $M$ that is a $G_\delta$-set in $M$. Then $N$ is completely metrizable.

Example
$\mathbb{R} \setminus \mathbb{Q}$ is a $G_\delta$-set in $\mathbb{R}$ so it is completely metrizable.

24.12. Theorem (Mazurkiewicz)

Let $M$ be a metric space and $N$ be a subspace of $M$ that is completely metrizable. Then $N$ is a $G_\delta$-set in $M$.

Example

$\mathbb{Q}$ is not a $G_\delta$-set in $\mathbb{R}$ so it is not completely metrizable.

Corollary

Let $N$ be a metrizable space. The following are equivalent:

ⓐ $N$ is completely metrizable.

ⓑ For every metric space $M$, if $N$ is a subspace of $M$, then $N$ is a $G_\delta$-set in $M$.

Proof Assume ⓐ. Then ⓑ follows by Mazurkiewicz Theorem.

Assume ⓑ. Let $M$ be a metric completion of $N$. By ⓑ $N$ is a $G_\delta$-set in $M$. Then ⓐ follows by Alexandroff Theorem.