MATH 793C — TOPOLOGY

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Let (M,d) and (N,ϱ) be pseudometric spaces. A function $f: M \longrightarrow N$ is called an <u>isometry</u> iff $\varrho(f(x), f(y)) = d(x,y)$ for each $x, y \in M$.

<u>Remark</u>

If M is a metric space and $f: M \longrightarrow N$ is an isometry, then f is injective.

<u>Theorem 24.4</u>

Let (M,d) be a metric space. There exists a complete metric space (N,ϱ) and an isometry $f: M \longrightarrow N$ s.t. $f[M]$ is dense in N.

If (N', ϱ') is a complete metric space and $f': M \to N'$ is an isometry s.t. $f'[M]$ is dense in N', then there exists a bijective isometry $g: N \longrightarrow N'$ s.t. $f' = g \circ f$.

Proof

Let (N', ϱ') be a complete pseudometric space and $f: M \to N'$ be an isometry s.t. $f[M]$ is dense in N'.

Let $\sim$ be the equivalence relation on N' defined by $x \sim y$ iff $\varrho'(x,y) = 0$. Let N be a subspace of N' obtained by choosing exactly one element from each equivalence class of $\sim$. If an equivalence class contains $f(x)$ for some $x \in X$, then such x is unique. Choose $f(x)$ in from such an equivalence class.
This implies that $f[M] \subseteq N$.

Let ϱ be the restriction of ϱ' to $N \times N$.
Then (N, ϱ) is a subspace of (N', ϱ')

If $y, z \in N$ are distinct, then $\varrho(y,z) > 0$ so ϱ is a metric and (N, ϱ) is a metric space.
Moreover $f: M \to N$ is an isometry.

Now we show that N is complete.
Let $(y_n)_{n \in \mathbb{N}}$ be a Cauchy seq. in N.
Since N' is complete, there is $z \in N'$ s.t. $z = \lim_{n \to \infty} y_n$.
Let $y \in N$ be s.t. $y \sim z$. Then $y = \lim_{n \to \infty} y_n$.

Since $f[M]$ is dense in N', it is dense in N.

Assume that (N, ϱ) and (N', ϱ') are complete metric spaces with isometries $f: M \to N$ and $f': M \to N'$ s.t. $f[M]$ is dense in N and $f'[M]$ are dense in N'.

We want a bijective isometry $g: N \to N'$ s.t. $f' = g \circ f$.

Define g as follows. Let $x \in N$. Since $f[M]$ is dense in N, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in M s.t. $x = \lim_{n\to\infty} f(y_n)$. Then $(f(y_n))_{n\in\mathbb{N}}$ is Cauchy in N implying that $(y_n)_{n\in\mathbb{N}}$ is Cauchy in M (since f is an isometry). Consequently, $(f'(y_n))_{n\in\mathbb{N}}$ is Cauchy in N' (since f' is an isometry). Since N' is complete there is $x' \in N'$ s.t. $x' = \lim_{n\to\infty} f'(y_n)$. Such x' is unique since N' is Hausdorff.
Define $g(x) = x'$.

Note that the def. of $g(x)$ does not depend on the choice of the sequence $(y_n)_{n\in\mathbb{N}}$.
If $(z_n)_{n\in\mathbb{N}}$ is a seq. in M s.t. $x = \lim_{n\to\infty} f(z_n)$, then

$$0 = \lim_{n\to\infty} \varrho\big(f(z_n), f(y_n)\big) = \lim_{n\to\infty} d(z_n, y_n)$$
$$= \lim_{n\to\infty} \varrho'\big(f'(z_n), f'(y_n)\big)$$

Thus $\rho'\left(\lim_{n\to\infty} f'(z_n), \lim_{n\to\infty} f'(y_n)\right)$.

Since ρ' is a metric, it follows that

$$\lim_{n\to\infty} f'(z_n) = \lim_{n\to\infty} f'(y_n).$$

Now we show that g is an isometry.
Let $x, t \in N$, $x = \lim_{n\to\infty} f(y_n)$, $t = \lim_{n\to\infty} f(z_n)$

where $y_n, z_n \in M$ for each $n \in \mathbb{N}$.

$$g(x) = \lim_{n\to\infty} f'(y_n) \quad \text{and} \quad g(t) = \lim_{n\to\infty} f'(z_n).$$

$$\rho'(g(x), g(t)) = \lim_{n\to\infty} \rho'(f'(y_n), f'(z_n))$$
$$= \lim_{n\to\infty} d(y_n, z_n) = \lim_{n\to\infty} \rho(f(y_n), f(z_n))$$
$$= \rho(x, t) \qquad \text{so } g \text{ is an isometry}$$

Now, we show that g is a bijection.
Let $z \in N'$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in M s.t. $z = \lim_{n\to\infty} f'(x_n)$.
Arguing as above, there is $y \in N$ s.t.
$y = \lim_{n\to\infty} f(x_n)$.
Then $z = g(y)$.

We proved that g is surjective. Since N is a metric space and g is an isometry, it is injective.

If $x \in M$, then $g \circ f(x) = f'(x)$ since the sequence $(y_n)_{n \in \mathbb{N}}$ in M s.t.

$$\lim_{n \to \infty} f(y_n) = f(x)$$

can be chosen to be constant with value x.

<u>Remark</u>

In particular, g is a homeomorphism.