MATH 793C — TOPOLOGY

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Lemma
Let M be a pseudometric space with pseudometric d. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in M and $y \in M$. If a subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to y, then $(x_n)_{n\in\mathbb{N}}$ converges to y.

Proof Let $\varepsilon > 0$.
There is $m \in \mathbb{N}$ s.t. $d(x_n, x_k) < \varepsilon/2$ for $n, k \geq m$.
Let $(n_t)_{t\in\mathbb{N}}$ be an increasing seq. in $\mathbb{N}$ s.t.

$$y = \lim_{t\to\infty} x_{n_t}.$$

There is $r \in \mathbb{N}$ s.t. $d(x_{n_t}, y) < \varepsilon/2$ for $t \geq r$.

Let $l = \max(m, n_r)$. If $n \geq l$, then

$$d(x_n, y) \leq d(x_n, x_{n_s}) + d(x_{n_s}, y) < \varepsilon$$

where $s \in \mathbb{N}$ is such that $n_s \geq l$. $(s \geq r)$

Lemma

Let M be a pseudometric space with a pseudometric d. Then there exists a complete pseudometric space (N, ϱ) and an isometry $g: M \longrightarrow N$ s.t. $g[M]$ is dense in N.

Proof

Let N be the set of all Cauchy sequences in M. Define $\varrho: N \times N \longrightarrow [0, \infty)$ as follows:

$$\varrho\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \lim_{n\to\infty} d(x_n, y_n)$$

We show that (N, ϱ) is complete.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in N where

$$x_n = (x_{n,m})_{m\in\mathbb{N}} \in N \text{ for each } n \in \mathbb{N}$$

For each $k \in \mathbb{N}$, let $n_k, m_k \in \mathbb{N}$ be defined by induction on k as follows:

Let $n_1 = m_1 = 1$. Assume $k > 1$ and n_{k-1} and m_{k-1} are defined. Let $n_k > n_{k-1}$ be such that

$$\varrho(x_r, x_s) < 1/k \text{ for every } r, s \geq n_k.$$

Let $m_k > m_{k-1}$ be such that

$d(x_{n_k, r}, x_{n_k, s}) < 1/k$ for every $r, s \geq m_k$.

Define $y_k = x_{n_k, m_k}$ for each $k \in \mathbb{N}$. Let

$y = (y_k)_{k \in \mathbb{N}}$ - sequence in M.

We show that $y \in N$ and that $y = \lim_{n \to \infty} x_n$.

<u>Claim</u> $d(x_{n_k, t}, y_q) < 3/k$ for each $k > 1$
$t \geq m_k$ and $q \geq k$.

The claim implies that $y \in N$.

We show that the claim implies that $\lim_{n \to \infty} x_n = y$.

It suffices to show that $\lim_{k \to \infty} x_{n_k} = y$.

Let $\varepsilon > 0$. Let $l > 1$ be such that $3/l < \varepsilon$.
We will show that $\rho(x_{n_k}, y) < \varepsilon$ for each $k \geq l$.

Let $k \geq l$. We want $\lim_{m \to \infty} d(x_{n_k, m}, y_m) < \varepsilon$.

It suffices to show that $d(x_{n_k, m}, y_m) < \varepsilon$

for each $m \geq m_k$.

Let $m \geq m_k$. Then $m \geq k$ so the claim implies that

$$d(x_{n_k, m}, y_m) < 3/k \leq 3/l < \varepsilon.$$

Now we prove the claim.

$$\boxed{\underline{\text{Claim}} \quad d(x_{n_k, t}, y_q) < 3/k \quad \text{for each } k > 1,\ t \geq m_k \text{ and } q \geq k.}$$

Let $k > 1$, $t \geq m_k$, $q \geq k$.

We know that $\rho(x_{n_k}, x_{n_q}) < 1/k$ by def. of n_k.

$$\lim_{m \to \infty} d(x_{n_k, m}, x_{n_q, m}) < 1/k \quad \text{by the def. of } \rho.$$

There is $m' \geq \max(m_k, m_q)$ s.t.

$$d(x_{n_k, m'}, x_{n_q, m'}) < 1/k$$

Then

$$d(x_{n_k, t}, y_q) = d(x_{n_k, t}, x_{n_q, m_q})$$

$$\leq d(x_{n_k, t}, x_{n_k, m'}) + d(x_{n_k, m'}, x_{n_q, m'})$$

$$+ d(x_{n_q, m'}, x_{n_q, m_q}) < 1/k + 1/k + 1/q \leq \frac{3}{k}$$

We proved that (N, ϱ) is complete.

Let $g: M \to N$ be defined by $g(x) = (x_n)_{n \in \mathbb{N}}$ with $x_n = x$ for each $n \in \mathbb{N}$.

We show that g is an isometry.

Let $x, y \in M$. Then $\varrho(g(x), g(y)) = d(x, y)$.

We show that $g[M]$ is dense in N.

Let $y = (y_n)_{n \in \mathbb{N}} \in N$.

Let $\varepsilon > 0$. We want to show that

open ball of radius ε and center y

$$B_\varrho(y, \varepsilon) \cap g[M] \neq \emptyset.$$

$(y_n)_{n \in \mathbb{N}}$ is a Cauchy seq. in M

Let $m \in \mathbb{N}$ be such that $d(y_n, y_k) < \varepsilon$ for $n, k \geq m$.

Let $x = y_m$. Then $g(x)$ is the const. seq. with value y_m.

Then $\rho(y, g(x)) = \lim_{n \to \infty} d(y_n, y_m) < \varepsilon$.

Thus $g(x) \in B_\rho(y, \varepsilon) \cap g[M]$.

Thus $g[M]$ is dense in N.