MATH 793C — TOPOLOGY

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Let (M,d) and (N,ϱ) be pseudometric spaces. A function $f: M \to N$ is called an <u>isometry</u> iff $\varrho(f(x), f(y)) = d(x,y)$ for each $x, y \in M$.

<u>Theorem 24.4</u>

Let (M,d) be a metric space. There exists a complete metric space (N,ϱ) and an isometry $f: M \to N$ s.t. $f[M]$ is dense in N.

If (N', ϱ') is a complete metric space and $f': M \to N'$ is an isometry s.t. $f'[M]$ is dense in N', then there exists a bijective isometry $g: N \to N'$ s.t. $f' = g \circ f$.

$$\begin{tikzcd} & N \arrow[dd, bend left, "g"] \\ M \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & N' \end{tikzcd}$$

Lemma

Let M be a pseudometric space with a pseudometric d. Then there exists a complete pseudometric space (N, ϱ) and an isometry $g: M \to N$ s.t. $g[M]$ is dense in N.

Remark

No uniqueness statement for pseudometric spaces.

Proof

Let N be the set of all Cauchy sequences in M. Define $\varrho: N \times N \to [0, \infty)$ as follows:

$$\varrho\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\right) = \lim_{n\to\infty} d(x_n, y_n)$$

The limit exists since the sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and $\mathbb{R}$ is complete.

Proof Let $\varepsilon > 0$. There is $m \in \mathbb{N}$ s.t. $d(x_n, x_k) < \varepsilon/2$ and $d(y_n, y_k) < \varepsilon/2$ for each $n, k \geq m$.

Then $d(x_n, y_n) \leq d(x_n, x_k) + d(x_k, y_k) + d(y_n, y_k)$ for each $n, k \in \mathbb{N}$.

$$d(x_n, y_n) < d(x_k, y_k) + \varepsilon$$

for each $n, k \geq m$. Similarly, $d(x_k, y_k) < d(x_n, y_n) + \varepsilon$

So $\left| d(x_n, y_n) - d(x_k, y_k) \right| < \varepsilon$.

Thus $\left(d(x_n, y_n)\right)_{n \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$.

ρ is a pseudometric on N (Exercise).

We show that (N, ρ) is complete.

Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in N where

$$x_n = (x_{n,m})_{m \in \mathbb{N}} \in N \quad \text{for each } n \in \mathbb{N}$$

For each $k \in \mathbb{N}$, let $n_k, m_k \in \mathbb{N}$ be defined by induction on k as follows:

Let $n_1 = m_1 = 1$. Assume $k > 1$ and n_{k-1} and m_{k-1} are defined. Let $n_k > n_{k-1}$ be such that

$$\rho(x_r, x_s) < 1/k \quad \text{for every } r, s \geq n_k.$$

Let $m_k > m_{k-1}$ be such that

$$d(x_{n_k, r}, x_{n_k, s}) < 1/k \quad \text{for every } r, s \geq m_k.$$

$$\begin{array}{cccccc}
x_1 & — & x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\
x_2 & — & x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\
x_3 & — & x_{3,1} & x_{3,2} & x_{3,3} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \\
x_{n_k} & — & x_{n_k,1} & x_{n_k,2} & x_{n_k,3} & \cdots \quad x_{n_k,m_k} \quad x_{n_k,m_{k+1}} \quad \cdots \\
\vdots & & \vdots & \vdots & \vdots &
\end{array}$$

$d(x_{n_k,r}, x_{n_k,s}) < 1/k$ for $r,s \geq m_k$

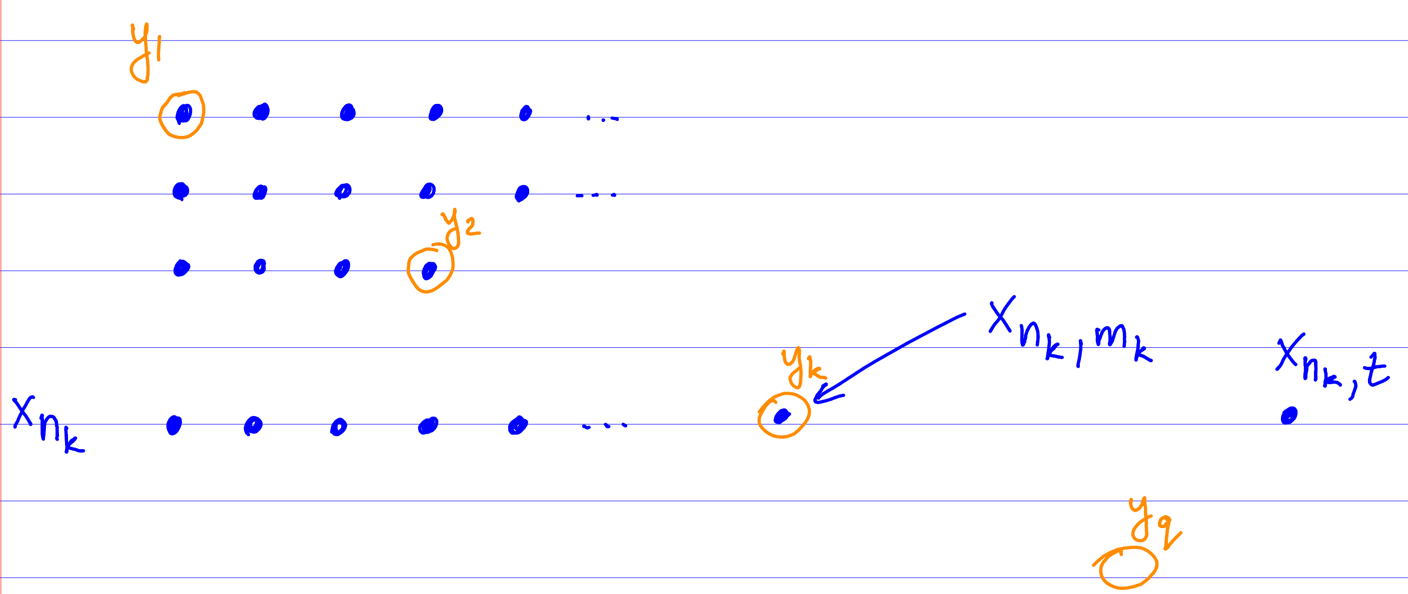
$x_{n_k,m_k} = y_k$

$\rho(x_r, x_s) < 1/k$ for $r,s \geq n_k$

Define $y_k = x_{n_k,m_k}$ for each $k \in \mathbb{N}$. Let

$$y = (y_k)_{k\in\mathbb{N}} \quad - \text{ sequence in } M.$$

We show that $y \in N$ and that $y = \lim_{n\to\infty} x_n$.



Claim

$$d(x_{n_k,t}, y_q) < 3/k$$

for each $k > 1$ and each $t \geq m_k$ and $q \geq k$.

The claim implies that $y \in N$.

Let $\varepsilon > 0$. Let $r \in \mathbb{N}$ be such that $3/r < \varepsilon$.
If $k, q \geq r$, then the claim implies that

$$d(y_k, y_q) = d(x_{n_k, m_k}, y_q) < 3/k \leq 3/r < \varepsilon .$$

Thus $(y_k)_{k \in \mathbb{N}}$ is Cauchy so $y \in N$.

The claim also implies that $\lim_{n \to \infty} x_n = y$.