MATH 793C – TOPOLOGY

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24. Complete Metric Spaces

Let M be a pseudometric space with pseudometric d and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in M. We say that $(x_n)_{n\in\mathbb{N}}$ is <u>Cauchy</u> iff $\forall_{\varepsilon>0}\ \exists_{m\in\mathbb{N}}\ \forall_{n,k\geq m}$
$$d(x_n, x_k) < \varepsilon .$$

<u>Remark</u>

Every convergent sequence is Cauchy.

<u>Proof</u>

Let $(x_n)_{n\in\mathbb{N}}$ converge to $x \in M$.

Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} x_n = x$, there is $m \in \mathbb{N}$ s.t.

$$d(x_n, x) < \varepsilon/2 \quad \text{for each } n \geq m.$$

For each $n,k > m$ we have then
$$d(x_n, x_k) \leq d(x_n, x) + d(x_k, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

<u>Example</u> Let $X = (0,1)$ with standard metric. $x_n = \frac{1}{n}$, $n \in \mathbb{N}$

$(x_n)_{n\in\mathbb{N}}$ is Cauchy, but it does not converge in X.

A pseudometric space M is <u>complete</u> iff every Cauchy sequence in M converges to some $x \in M$. We also say that the pseudometric is <u>complete</u>.

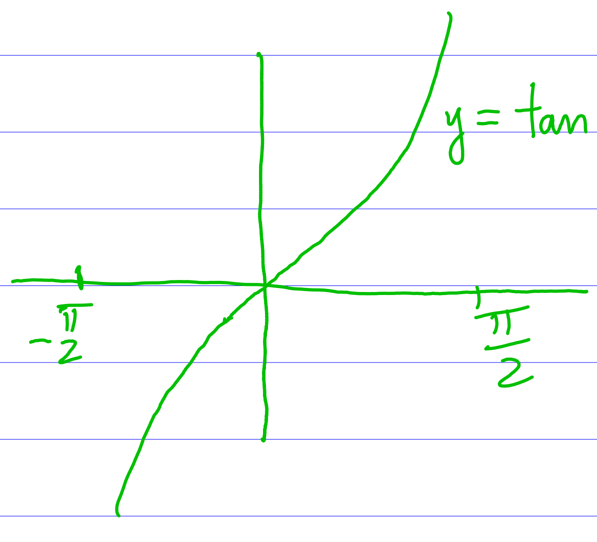
A top. space X is <u>completely metrizable</u> iff there exists a compatible (with respect to the top. of X) complete metric on X.

A <u>completely pseudometrizable</u> space is defined analogously.

<u>Example.</u>
$X = (0,1)$ - the open interval with standard metric and standard topology

Then X is not complete as a metric space.
However, X is completely metrizable as a top. space.

Let $d : X \times X \to [0,\infty)$ be def. by



$$d(x,y) = \left|\tan\left(\pi x - \frac{\pi}{2}\right) - \tan\left(\pi y - \frac{\pi}{2}\right)\right|$$

Homework: show d is a metric which is complete

$d\left(0, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} \infty$ so $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ is not Cauchy

d induces the standard topology on X.

__Example__
$X = \mathbb{R} \setminus \mathbb{Q}$ with standard metric and standard top.
X is not complete as a metric space, but we will show later that as a top. space X is completely metrizable.

__Remark__
$\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ with product topoloty where $\mathbb{N}$ has the discrete topology.

__Exercise__ Find a complete metric on $\mathbb{N}^{\mathbb{N}}$ compatible with the product topology.

__Example__
$X = \mathbb{Q}$ with standard topology is not completely metrizable.