MATH 793C — TOPOLOGY

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<u>Theorem 23.9</u> (Nagata, Smirnov)
A topological space is metrizable if and only if it is T_3 and has a σ-locally finite base.

<u>Proof</u> $\Leftarrow$

Let X be a T_3-space with a σ-locally finite base.

$\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a base for the topology of X, where $\mathcal{B}_n$ is locally finite for each $n \in \mathbb{N}$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ when $i \neq j$.

For each $B \in \mathcal{B}$, n_B is the unique element of $\mathbb{N}$ with $B \in \mathcal{B}_{n_B}$.

$$M = \prod_{B\in\mathcal{B}} [0, 1/n_B].$$

The sup topology on M is induced by the metric d defined by:

$$d\left((x_B)_{B\in\mathcal{B}}, (y_B)_{B\in\mathcal{B}}\right) = \sup\left\{|x_B - y_B| : B\in\mathcal{B}\right\}$$

The sup topology is finer than the product topology.

$M = \prod_{B\in\mathcal{B}} M_B$

For each $B\in\mathcal{B}$, $M_B = [0, 1/n_B]$ and

$g_B : X \longrightarrow M_B$ is continuous with

existence last class

$X \setminus B = g_B^{-1}[\{0\}]$.

$g_B(x) = 0$ if $x \notin B$
$g_B(x) > 0$ if $x \in B$

$g : X \longrightarrow M$ is defined by

$$g(x) = (g_B(x))_{B\in\mathcal{B}}.$$

g is an embedding when M has the product topology. proved last class

<u>Claim:</u> g is an embedding when M has the sup topology.

The claim implies that X is homeomorphic to a

subspace of the metric space M (with the sup topology).
Thus X is metrizable.

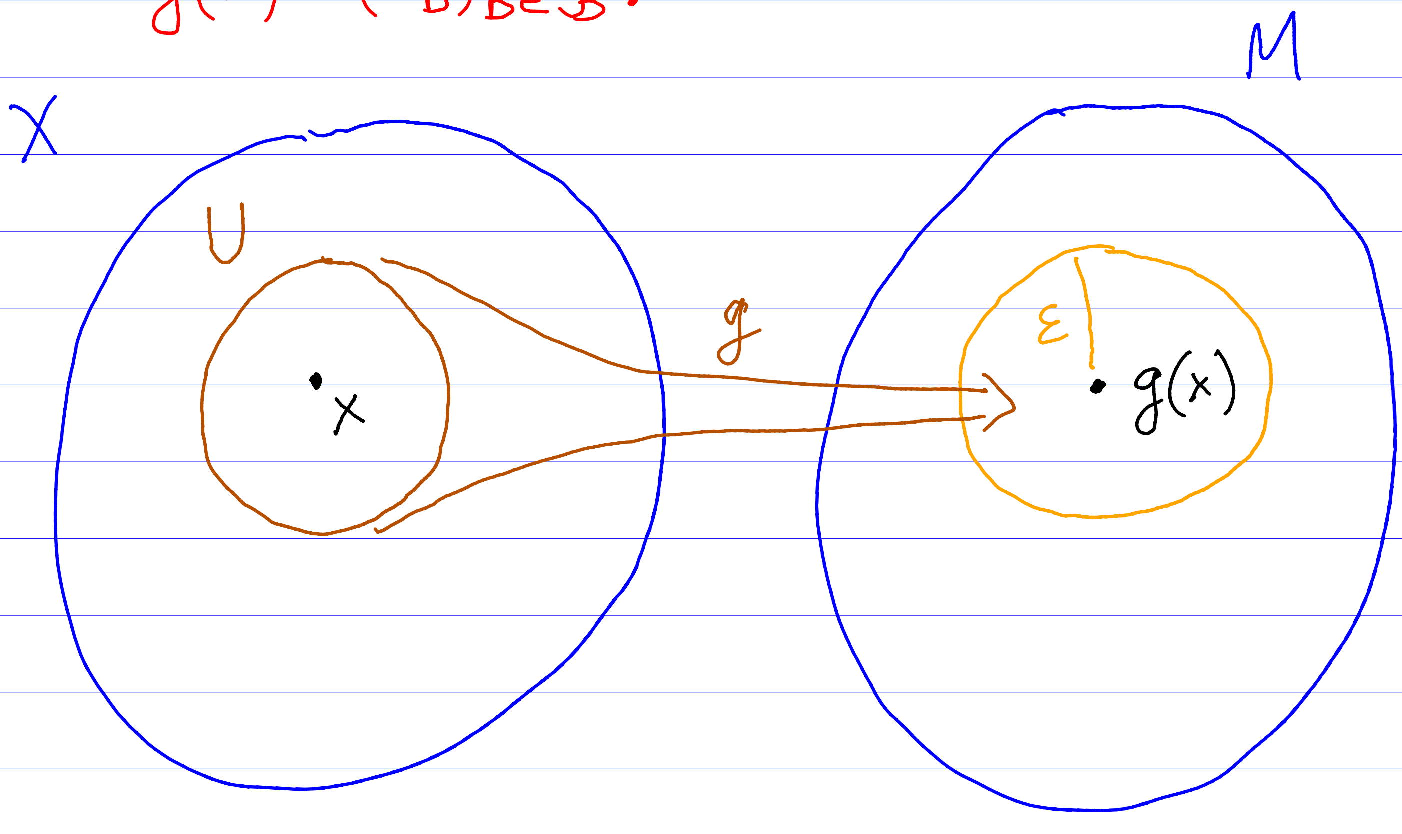
It remains to prove that g is continuous.

Let $x \in X$ and $g(x) = (y_B)_{B \in \mathcal{B}} \in M$.

Let $\varepsilon > 0$. We show that there exists open U in X such that for every $z \in U$ we have

$$|y_B - w_B| < \varepsilon \quad \text{for each } B \in \mathcal{B}$$

where $g(z) = (w_B)_{B \in \mathcal{B}}$.



Let $m \in \mathbb{N}$ be such that $1/m < \varepsilon$.

Let $\mathcal{B}' = \{B \in \mathcal{B} : n_B \geq m\} = \bigcup_{n \geq m} \mathcal{B}_n$

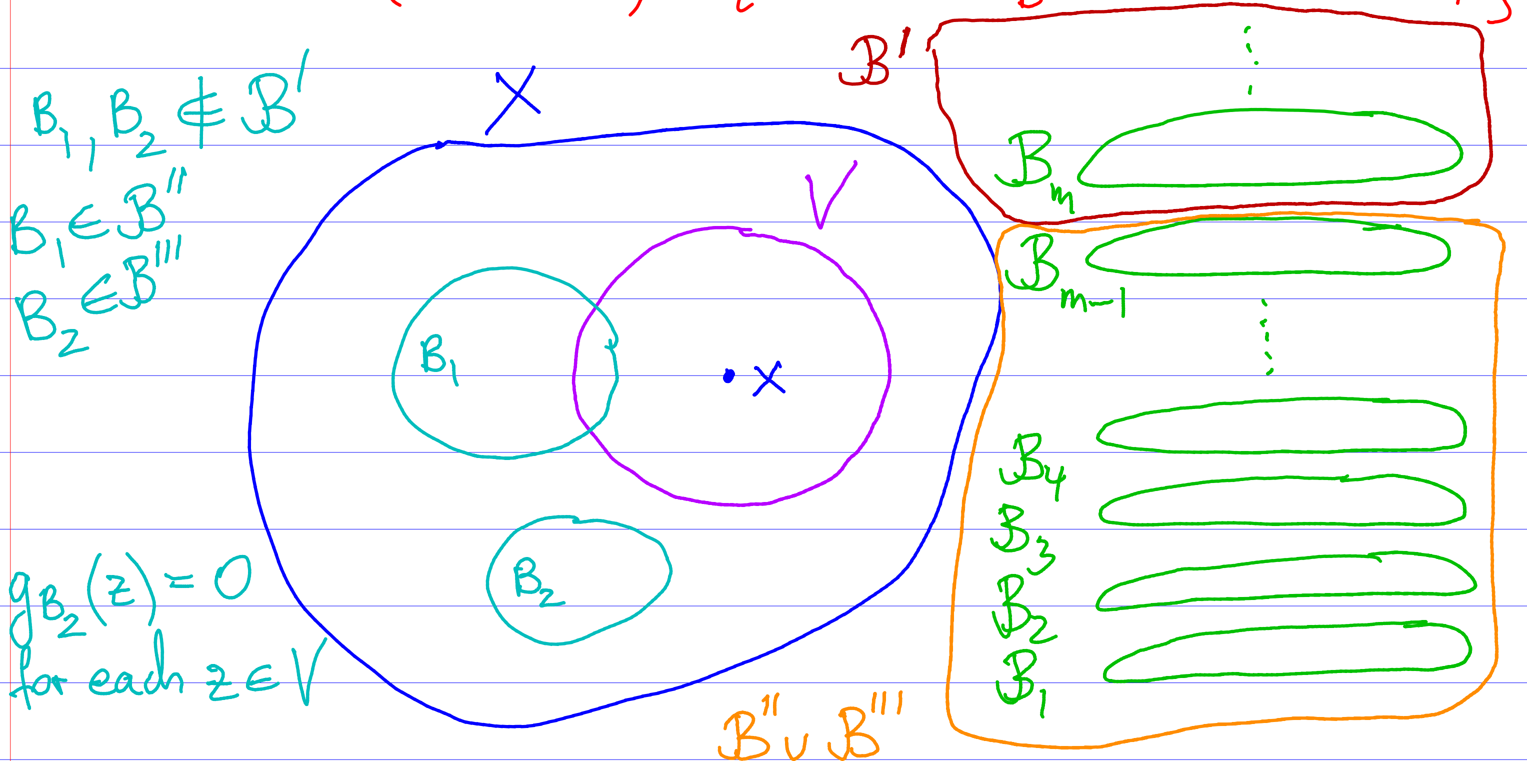
Note that $\mathcal{B} \setminus \mathcal{B}' = \bigcup_{n < m} \mathcal{B}_n$ is locally finite.

finite union of locally finite families is locally finite

because $\mathcal{B} \setminus \mathcal{B}'$ is locally finite

There is open V in X such that $x \in V$ and $\mathcal{B}'' = \{B \in \mathcal{B} \setminus \mathcal{B}' : B \cap V \neq \emptyset\}$ is finite. Let

$\mathcal{B}''' = \mathcal{B} \setminus (\mathcal{B}' \cup \mathcal{B}'') = \{B \in \mathcal{B} : n_B < m \text{ and } B \cap V = \emptyset\}$



Note that $g_B(z) = 0$ for each $z \in V$ and $B \in \mathcal{B}'''$.

We have $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}'' \cup \mathcal{B}'''$ ($\mathcal{B}', \mathcal{B}'', \mathcal{B}'''$ are pairwise disjoint)

For each $z \in V$ and $B \in \mathcal{B}' \cup \mathcal{B}'''$ we have

$$|g_B(z) - g_B(x)| < \varepsilon.$$

since $M_B = [0, 1/n_B]$
$1/n_B < \varepsilon$ for $B \in \mathcal{B}'$

and since

$\mathcal{B}''$ is finite.

Let $S = \prod_{B \in \mathcal{B}} S_B$ where

$S_B = (g_B(x) - \varepsilon, g_B(x) + \varepsilon) \cap M_B$ for $B \in \mathcal{B}''$ and

$S_B = M_B$ for $B \in \mathcal{B}' \cup \mathcal{B}'''$.

since $g : X \to M$ is cont. when M has the prod. top.

S is open in M with the product topology.
Thus $g^{-1}[S]$ is open in X and $x \in g^{-1}[S]$.

Let $U = V \cap g^{-1}[S]$. Then $x \in U$, U is open, and

$|g_B(x) - g_B(z)| < \varepsilon$ for each $B \in \mathcal{B}$ and each $z \in U$.

Thus U satisfies the requirements.