MATH 793C – TOPOLOGY

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<u>Theorem 23.9</u> (Nagata, Smirnov)
A topological space is metrizable if and only if it is T_3 and has a σ-locally finite base.

<u>Proof</u> $\Leftarrow$

Let X be a T_3-space with a σ-locally finite base.

We have proved that X is normal.

Let $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ be a base for the topology of X, where $\mathcal{B}_n$ is locally finite for each $n \in \mathbb{N}$.

<u>Claim:</u> Every closed set in X is a $\underbrace{G_\delta\text{-set}}_{\text{countable intersection of open sets}}$.

Let F be a closed set in X.
For each $n \in \mathbb{N}$, let

$$C_n = \bigcup \{\overline{B} : B \in \mathcal{B}_n, \ \overline{B} \cap F = \phi\}.$$

Since $\mathcal{B}_n$ is locally finite, it follows that C_n is closed. Let $U_n = X \setminus C_n$. U_n is open.

We show that $F = \bigcap_{n \in \mathbb{N}} U_n$.

$\subseteq$ holds since $F \subseteq U_n$ for each $n \in \mathbb{N}$.

To show $\supseteq$, let $x \notin F$. We show there is $n \in \mathbb{N}$ s.t. $x \notin U_n$.

Since $\mathcal{B}$ is a base for the top. of X, and since X is regular, there is
$B \in \mathcal{B}$ s.t. $x \in B$ and $\overline{B} \cap F = \emptyset$
There is $n \in \mathbb{N}$ s.t. $B \in \mathcal{B}_n$.
Then $\overline{B} \subseteq C_n$ so $x \notin U_n$.

<u>Claim:</u> For every closed set $F \subseteq X$, there is continuous $f: X \to [0,1]$ with $F = f^{-1}[\{0\}]$.

Let F be a closed set in X.
Let

$$F = \bigcup_{n \in \mathbb{N}} U_n$$ with U_n open in X for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $f_n : X \to [0,1]$ be continuous with
$$f_n[F] \subseteq \{0\} \text{ and } f_n[X \setminus U_n] \subseteq \{1\}.$$

Such f_n exists since X is normal (Urysohn's Lemma)

Let $f : X \to [0,1]$ be defined by
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x).$$

If $x \in F$, then $f_n(x) = 0$ for each $n \in \mathbb{N}$ so $f(x) = 0$.
If $x \notin F$, then $f_n(x) = 1$ for some $n \in \mathbb{N}$ so $f(x) \neq 0$.

It also can be proved that f is continuous.
(See a homework problem last semester)

Thus the claim is proved.

Without loss of generality, we can assume that $\mathcal{B}_i \cap \mathcal{B}_j = \phi$ for $i \neq j$.

For each $B \in \mathcal{B}$, let n_B be the unique element of $\mathbb{N}$ with $B \in \mathcal{B}_{n_B}$.

Let $M = \prod_{B \in \mathcal{B}} [0, 1/n_B]$.

We will consider M with two topologies: the product topology and the sup topology.

The sup topology is induced by the metric d on M defined by:

$$d\left((x_B)_{B\in\mathcal{B}}, (y_B)_{B\in\mathcal{B}}\right) = \sup\left\{|x_B - y_B| : B\in\mathcal{B}\right\}.$$

Note that the sup topology is finer than the product topology.

For each $B\in\mathcal{B}$, let $g_B : X \longrightarrow [0, 1/n_B]$ be continuous with

$$X \setminus B = g_B^{-1}[\{0\}].$$

Let $g : X \longrightarrow M$ be defined by

$$g(x) = (g_B(x))_{B\in\mathcal{B}}.$$

<u>Claim:</u> g is an embedding when M has the product topology.

g is continuous since $\pi_B \circ g = g_B : X \to [0, 1/n_B]$ is continuous for each $B\in\mathcal{B}$, where $\pi_B : M \to [0, 1/n_B]$ is the projection map.

g is injective:

Let $x, y \in X$ be distinct.
There is $B \in \mathcal{B}$ s.t. $x \in B$, $y \notin B$.
Then $g_B(x) \neq 0$ and $g_B(y) = 0$ so $g(x) \neq g(y)$.

g is closed:

Let C be closed in X and
$$y = (y_B)_{B \in \mathcal{B}} \in g[X] \setminus g[C].$$

We show that $y \notin \overline{g[C]}$.

Let $x \in X$ be such that $y = g(x)$.
Then $x \notin C$.
There is $B' \in \mathcal{B}$ s.t. $x \in B'$ and $B' \cap C = \emptyset$.
Then $g_{B'}(x) = \delta > 0$ and $g_{B'}[C] \subseteq \{0\}$.

Let $V = \prod_{B \in \mathcal{B}} V_B$, with $V_{B'} = (\delta/2, 1/n_{B'}]$ and $V_B = [0, 1/n_B]$ for $B \neq B'$.

Then V is open in M and $g(x) \in V$ but $g[C] \cap V = \emptyset$. Thus
$$x \notin \overline{g[C]}.$$

Claim: g is an embedding when M has the sup topology.

g is injective.
g is closed since the sup topology is stronger than the product topology.

It remains to prove that g is continuous.