MATH 793C – TOPOLOGY

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Let X be a topological space and $\mathcal{F}$ be a family of subsets of X.

We say that $\mathcal{F}$ is locally finite iff for every $x \in X$ there is a nbhd U of x in X such that

$$\{F \in \mathcal{F} : U \cap F \neq \emptyset\} \text{ is finite.}$$

We say that $\mathcal{F}$ is σ-locally finite iff there are $\mathcal{F}_1, \mathcal{F}_2, \ldots$ such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ and $\mathcal{F}_i$ is locally finite for each $i \in \mathbb{N}$

A top. space X is paracompact if it is Hausdorff and for every open cover $\mathcal{U}$ of X there exist a locally finite open cover $\mathcal{V}$ of X that is a refinement of $\mathcal{U}$.

Reminder

Assume X is T_3. Then X is paracompact if and only if

(*) for every open cover $\mathcal{U}$ of X there exist a σ-locally finite open cover $\mathcal{V}$ of X that is a refinement of $\mathcal{U}$.

Let X be a top. space.
A <u>development</u> for X is a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\{St(x, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a nbhd base at x for each $x \in X$.

<u>Lemma</u> (not in the textbook)
Let X be a topological space such that $\circledast$ holds and there exists a development for X. Then X has a σ-locally finite base.

<u>Proof</u> later

<u>Corollary</u>
Every metrizable space has a σ-locally finite base.

<u>Proof</u>
Let X be a metrizable space.
We have proved last semester that such X is paracompact.
Thus $\circledast$ holds for X.
Moreover X has a development.
Thus the Lemma implies that X has a σ-locally finite base.

Theorem 23.9 (Nagata, Smirnov)
A topological space is metrizable if and only if it is T_3 and has a σ-locally finite base.

Proof $\Rightarrow$ is clear
$\Leftarrow$ will be proved later

Let X be a topological space and $\mathcal{F}$ be a family of subsets of X.
We say that $\mathcal{F}$ is **discrete** iff for every $x \in X$ there is a nbhd U of x in X such that

$$|\{F \in \mathcal{F} : U \cap F \neq \emptyset\}| \leq 1.$$

We say that $\mathcal{F}$ is **σ-discrete** iff there are $\mathcal{F}_1, \mathcal{F}_2, \ldots$ such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots$ and $\mathcal{F}_i$ is discrete for each $i \in \mathbb{N}$

Remark (Bing Theorem)
A topological space is metrizable if and only if it is T_3 and has a σ-discrete base.

Proof $\Rightarrow$ (Exercise. Hint: Look at the proof of the theorem last semester that every metr. space is paracompact and modify the Lemma above as required)

$\Leftarrow$ follows from $\Leftarrow$ in the Nagata-Smirnov theorem.

A Moore space is a regular Hausdorff space with a development.

Corollary

A Moore space is metrizable if and only if it is paracompact.

Proof

Assume X is a paracompact Moore space.
Then X satisfies (*) and has a development.
Thus X has a σ-locally finite base.
Since X is also T_3, the Nagata-Smirnov Theorem implies that X is metrizable.

Proof of the Lemma

Let X be a top. satisfying (*) and having a development $\mathcal{U}_1, \mathcal{U}_2, \ldots$.

For each $n \in \mathbb{N}$, $\mathcal{U}_n$ is an open cover of X.
Thus there exists an open cover $\mathcal{V}_n$ of X is σ-locally finite and refines $\mathcal{U}_n$.
Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$. Then $\mathcal{B}$ is σ-locally finite.

It suffices to show that $\mathcal{B}$ is base.
Let U be open in X and $x \in U$.
Since $\mathcal{U}_1, \mathcal{U}_2, \dots$ is a development, there is $n \in \mathbb{N}$ s.t. $St(x, \mathcal{U}_n) \subseteq U$.

$\mathcal{V}_n$ is a cover of X so there is $V \in \mathcal{V}_n$ s.t. $x \in V$.
Since $\mathcal{V}_n$ refines $\mathcal{U}_n$, there is $U' \in \mathcal{U}_n$ s.t. $V \subseteq U'$.
Thus $x \in U'$, so $U' \subseteq St(x, \mathcal{U}_n)$. Hence $U' \subseteq U$.
Thus $\mathcal{B}$ is a base.

Proof of the Nagata-Smirnov Theorem

Let X be a T_3-space with a σ-locally finite base.

<u>Claim:</u> X is paracompact.
Let $\mathcal{U}$ be an open cover of X.
Let $\mathcal{B}$ be a σ-locally finite base for X.
For each $U \in \mathcal{U}$ and each $x \in U$, let $B_{U,x} \in \mathcal{B}$ be such that $x \in B_{U,x} \subseteq U$.

Let $\mathcal{V} = \{ B_{U,x} : U \in \mathcal{U} \text{ and } x \in U \}$.

It is clear that $\mathcal{V}$ is a refinement of $\mathcal{U}$.
It is also clear that $\mathcal{V}$ is an open cover of X.

Since $\mathcal{V} \subseteq \mathcal{B}$ and $\mathcal{B}$ is σ-locally finite, it follows that $\mathcal{V}$ is σ-locally finite.

Thus X is paracompact and the claim is proved.

It follows that X is normal.