MATH 793C – TOPOLOGY

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Let X be a top. space.
A <u>development</u> for X is a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X such that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\{St(x, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a nbhd base at x for each $x \in X$.

<u>Example</u>
If $\mathcal{U}_1, \mathcal{U}_2, \dots$ is a compatible normal sequence of open covers of X, then it is a development for X.

<u>Remark</u>
Let $\mathcal{V}_1, \mathcal{V}_2, \dots$ be a sequence of open covers of X such that $\{St(x, \mathcal{V}_n) : n \in \mathbb{N}\}$ is a nbhd base at x for each $x \in X$. Let for each $n \in \mathbb{N}$

$$\mathcal{U}_n = \{V_1 \cap V_2 \cap \dots \cap V_n \neq \phi : V_i \in \mathcal{V}_i, i = 1, 2, \dots, n\}.$$

Then $\mathcal{U}_1, \mathcal{U}_2, \dots$ is a development for X.

① $\mathcal{U}_n$ is an open cover of X.

Let $x \in X$. For each $i = 1, 2, \dots, n$, there is $V_i \in \mathcal{V}_i$ s.t. $x \in V_i$.

Then $x \in V_1 \cap V_2 \cap \dots \cap V_n \in \mathcal{U}_n$

② For each $x \in X$, $\mathcal{B}_x = \{St(x, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a nbhd base at x.

Let $x \in U \subseteq X$ with U open. There is $n \in \mathbb{N}$ s.t. $St(x, \mathcal{V}_n) \subseteq U$.

Then $St(x, \mathcal{U}_n) \subseteq U$ since $\mathcal{U}_n$ refines $\mathcal{V}_n$. Thus $\mathcal{B}_x$ is a nbhd base at x.

③ For each $n \in \mathbb{N}$, $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$.

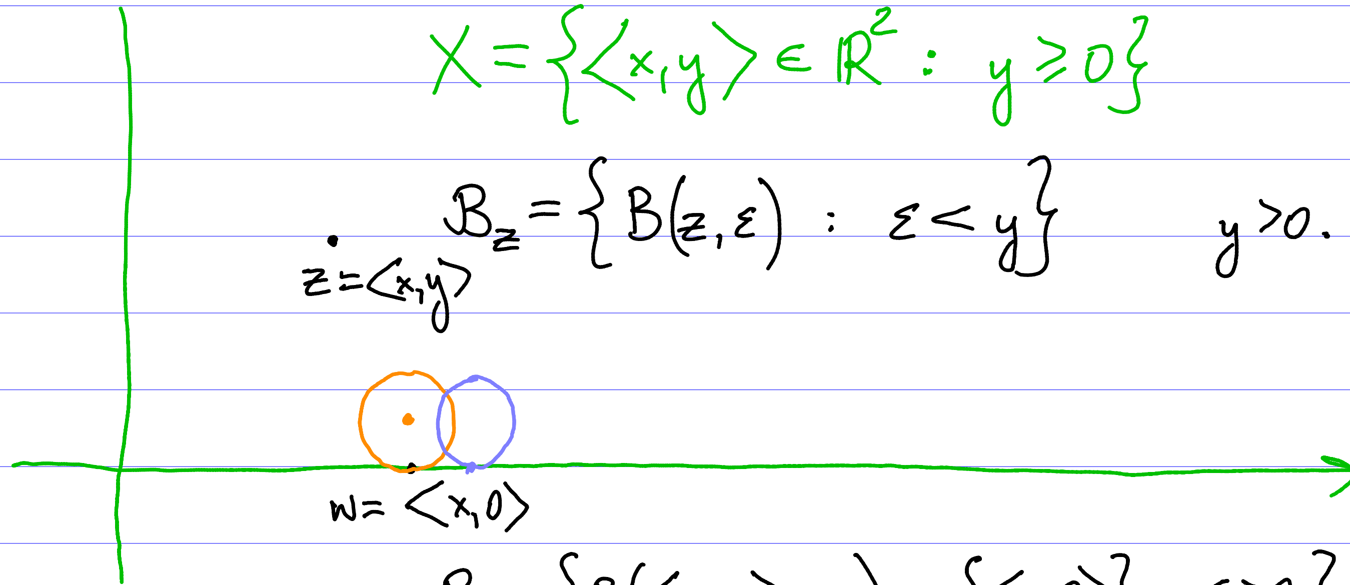
A Moore space is a regular Hausdorff space with a development.

Example

The Moore plane is a Moore space.

$$X = \{ \langle x, y \rangle \in \mathbb{R}^2 : y \geq 0 \}$$

$$\mathcal{B}_z = \{ B(z, \varepsilon) : \varepsilon < y \} \qquad y > 0.$$



$$\mathcal{B}_w = \{ B(\langle x, \varepsilon \rangle, \varepsilon) \cup \{ \langle x, 0 \rangle \} : \varepsilon > 0 \}$$

Take $\mathcal{U}_n = \{ B(z, \frac{1}{n}) : z = \langle x, y \rangle, \, y > \frac{1}{n} \}$

$$\cup \{ B(z, \tfrac{1}{n}) \cup \{ \langle x, 0 \rangle \} : z = \langle x, \tfrac{1}{n} \rangle \}$$

Then $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ is a development for Γ (the Moore plane).

Γ is not metrizable since it is not normal.

Remark

We will show later that a Moore space is metrizable if and only if it is paracompact.

Normal Moore space conjecture.

Every normal Moore space is metrizable.

Remark
This conjecture can't be proved nor disproved using the standard axioms ZFC of set theory.

Theorem 23.7 (Alexandroff and Urysohn 1923)
A top. space X is pseudometrizable if and only if X has a development $\mathcal{U}_1, \mathcal{U}_2, \ldots$ such that for every $n \in \mathbb{N}$ and for every $U, V \in \mathcal{U}_{n+1}$ with $U \cap V \neq \emptyset$, there exists $W \in \mathcal{U}_n$ with $U \cup V \subseteq W$.

Example
Let Γ be the Moore plain. The development for Γ that was defined before does not satisfy the requirements in the statement of the theorem above.

Proof of the Theorem.
Assume X is pseudometrizable. Let d a comp. pseudom.
Let $\mathcal{U}_n = \{B_d(x, \frac{1}{2^n}) : x \in X\}$. Then $\mathcal{U}_1, \mathcal{U}_2, \ldots$

is a required development for X.

Assume $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a development for X as in the statement of the theorem.

We will apply the following.

Theorem 23.5 (Nagata)

A top. space X is pseudometrizable iff for each $x \in X$ there exists a countable nblhd base $\mathcal{B}_x = \{U_{x,n} : n \in \mathbb{N}\}$ at x, s.t. the following conditions hold for each $n \in \mathbb{N}$ and each $x, y \in X$:

(a) $y \in U_{x,n+1}$ implies that $U_{y,n+1} \subseteq U_{x,n}$

(b) $U_{y,n+1} \cap U_{x,n+1} \neq \emptyset$ implies that $y \in U_{x,n}$

Define $U_{x,n} = St(x, \mathcal{U}_n)$ for each $x \in X$ and $n \in \mathbb{N}$.

$$= \bigcup \{U \in \mathcal{U}_n : x \in U\}$$

We verify (a) and (b).

Let $x, y \in X$ and $n \in \mathbb{N}$ be s.t. $y \in U_{x,n+1}$.
Let $z \in U_{y,n+1}$. $y \in U_{x,n+1} \cap U_{y,n+1}$

Thus there are $U, U' \in \mathcal{U}_{n+1}$ s.t.

$x, y \in U$ and $y, z \in U'$

$U \cap U' \neq \emptyset$ since $y \in U \cap U'$.

Thus there is $W \in \mathcal{U}_n$ s.t. $U \cup U' \subseteq W$.
Then $x, z \in W$, so
$$z \in St(x, \mathcal{U}_n) = U_{x,n}$$

Exercise. Verify (b).