MATH 793C — TOPOLOGY

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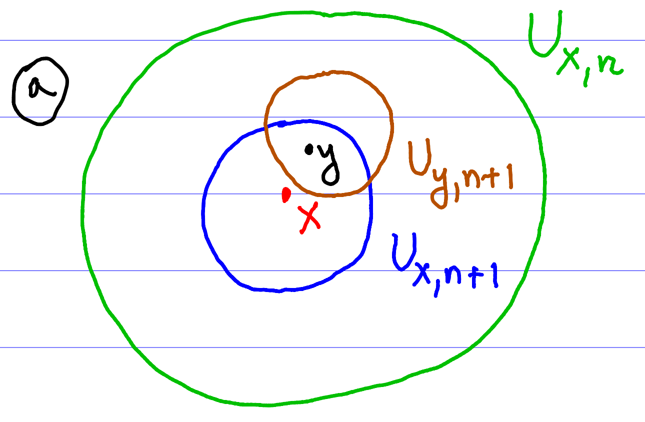
Class 8 — January 31

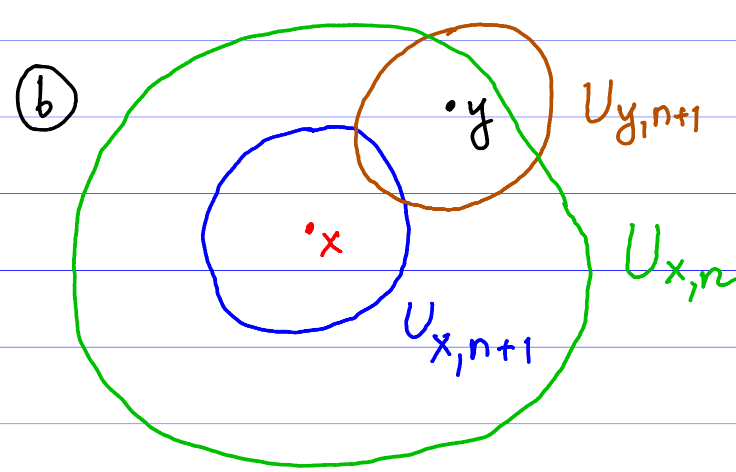
<u>Theorem 23.5</u> (Nagata)

A top. space X is pseudometrizable iff for each $x \in X$ there exists a countable nbhd base $\mathcal{B}_x = \{U_{x,n} : n \in \mathbb{N}\}$ at x, s.t. the following conditions hold for each $n \in \mathbb{N}$ and each $x, y \in X$:

(a) $y \in U_{x,n+1}$ implies that $U_{y,n+1} \subseteq U_{x,n}$

(b) $U_{y,n+1} \cap U_{x,n+1} \neq \emptyset$ implies that $y \in U_{x,n}$





<u>Proof</u>

Assume that X is pseudometrizable. Let d be a compatible pseudometric on X.

Let $U_{x,n} = B_d(x, 1/2^n)$. Then (a) and (b) hold.

Assume that ⓐ and ⓑ hold for each $n \in \mathbb{N}$ and each $x, y \in X$.

For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{ \mathrm{int}_X(U_{x,n}) : x \in X \}$.

We show that $St(U_{x,n+2}, \mathcal{U}_{n+2}) \subseteq U_{x,n}$ for each $n \in \mathbb{N}$ and each $x \in X$. $(*)$

Assume that $(*)$ is proved. Then let $\mathcal{V}_n = \mathcal{U}_{2n}$.
It follows that $\mathcal{V}_1, \mathcal{V}_2, \ldots$ is a compatible normal sequence of open covers of X. Thus X is pseudometrizable.

normal

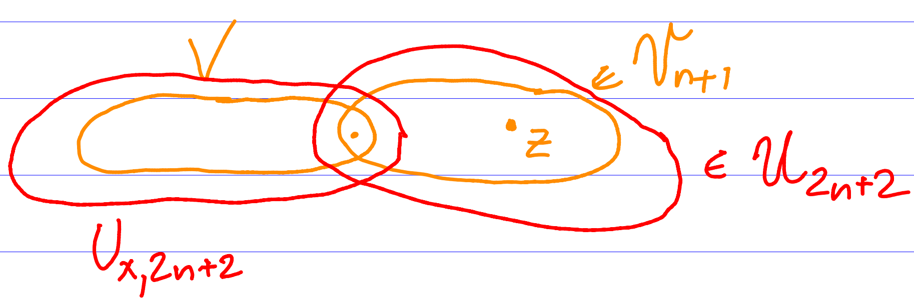
If $V \in \mathcal{V}_{n+1}$ then there is $V' \in \mathcal{V}_n$ s.t.

$$St(V, \mathcal{V}_{n+1}) \subseteq V'$$

Let $V \in \mathcal{V}_{n+1}$ $\quad V = \mathrm{int}_X(U_{x,2n+2})$ for some $x \in X$.

By $(*)$ $\underbrace{St(U_{x,2n+2}, \mathcal{U}_{2n+2})}_{\supseteq St(V, \mathcal{V}_{n+1})} \subseteq U_{x,2n}$

$\supseteq St(V, \mathcal{V}_{n+1})$ ← open



It follows that $St(V, \mathcal{V}_{n+1}) \subseteq \text{int}(U_{x,2n}) = V' \in \mathcal{V}_n$

compatible

Let U be open in X and $x \in U$.
There is $n \in \mathbb{N}$ s.t. $U_{x,2n} \subseteq U$.

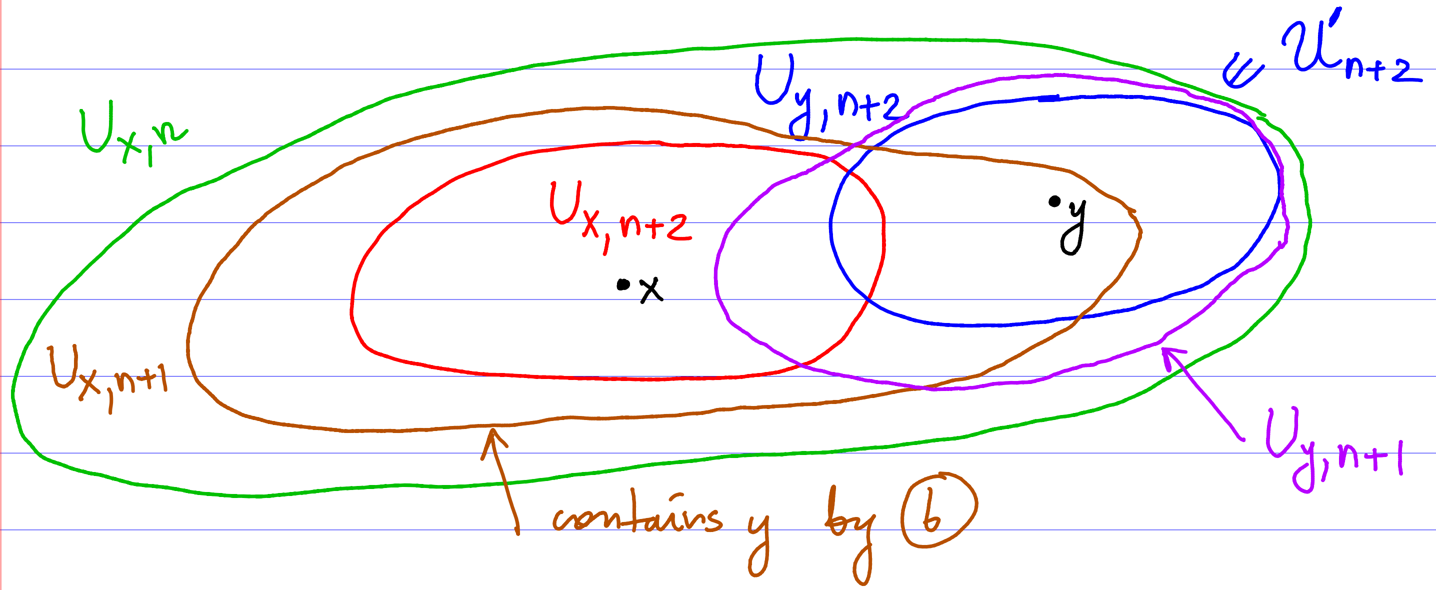
Then $St(x, \mathcal{V}_{n+1}) \subseteq V'$ for some $V' \in \mathcal{V}_n$.

$$V' = \text{int}\left(U_{x,2n}\right) \subseteq U.$$

We show that $St(U_{x,n+2}, \mathcal{U}'_{n+2}) \subseteq U_{x,n}$ $(*')$
for each $n \in \mathbb{N}$ and each $x \in X$.

note that $(*') \Rightarrow (*)$

$$\mathcal{U}'_{n+2} = \{U_{x,n+2} : x \in X\}$$



Since $y \in U_{x,n+1}$, (a) implies that $U_{y,n+1} \subseteq U_{x,n}$
Thus $U_{y,n+2} \subseteq U_{x,n}$ and $(*')$ is proved.