MATH 793C — TOPOLOGY

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Theorem 23.4.
A topological space X is pseudometrizable if and only if there exists a compatible normal sequence of open covers of X.

Proof

We assume $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a compatible normal sequence of open covers. This sequence gives the function
$$t : X \times X \longrightarrow [0,1]$$

For $x, y \in X$, let $\mathcal{S}(x,y)$ be the set of all finite sequences $x_1, x_2, \ldots, x_n$ in X such that $x_1 = x$ and $x_n = y$.

$$d(x,y) = \inf\left\{ \sum_{i=1}^{n-1} t(x_i, x_{i+1}) : (x_1, x_2, \ldots, x_n) \in \mathcal{S}(x,y) \right\}$$

d is a pseudometric on X.

Let $\mathcal{V}_n$ be set of all open balls (with respect to d) of radius $1/2n$ in X.

To complete the proof it remains to show that the following conditions hold for each $n \in \mathbb{N}$.

(a) $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{V}_n$.

(b) $\mathcal{V}_{n+1}$ is a refinement of $\mathcal{U}_n$.

Proof of (a)

Let $U \in \mathcal{U}_{n+1}$ and $x \in U$. If $y \in U$, then

$$d(x,y) \leq t(x,y) \leq 1/2^{n+1} < 1/2^n$$

thus $y \in B_d(x, 1/2^n) \in \mathcal{V}_n$. Thus $U \subseteq B_d(x, 1/2^n)$.

Hence $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{V}_n$.

Proof of (b)

We show that

(*) if $x, y \in X$ and $d(x,y) < 1/2^n$ for some $n \in \mathbb{N}$, then some member of $\mathcal{U}_n$ contains ~~both~~ x and y.

Suppose (*) is proved.
Let $V \in \mathcal{V}_{n+1}$. Then $V = B_d(x, 1/2^{n+1})$ for some $x \in X$.
If $y \in V$, then $d(x,y) < 1/2^{n+1}$ so (*) implies that there is $U \in \mathcal{U}_{n+1}$ s.t. $x, y \in U$.

It follows that
$$V \subseteq St(x, \mathcal{U}_{n+1}) \subseteq U'$$
for some $U' \in \mathcal{U}_n$.

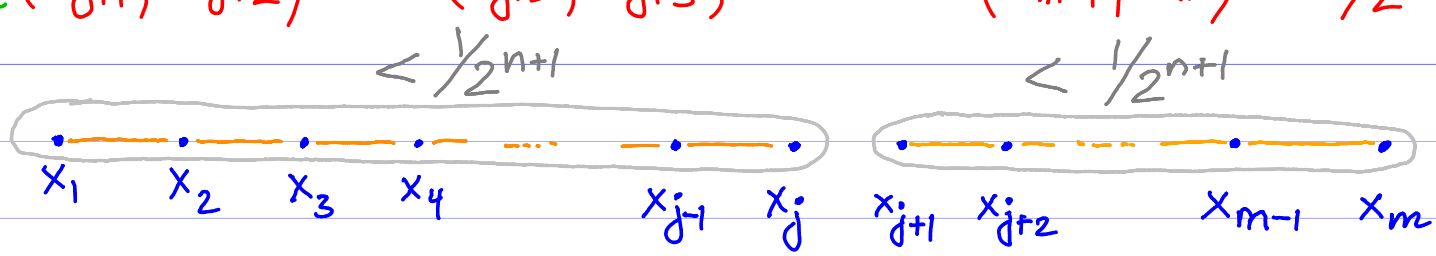
Thus $\mathcal{V}_{n+1}$ is a refinement of $\mathcal{U}_n$ and so ⓑ holds.

Now we prove ⊛.

Let $x, y \in X$ with $d(x,y) < 1/2^n$ for some $n \in \mathbb{N}$.

There is a sequence $(x_1, x_2, \dots, x_m) \in \mathcal{S}(x,y)$ s.t.
$$\sum_{i=1}^{m-1} t(x_i, x_{i+1}) < 1/2^n$$

$\triangle$ There is $j \in \{1, 2, \dots, m-1\}$ such that
$$t(x_1, x_2) + t(x_2, x_3) + \dots + t(x_{j-1}, x_j) < 1/2^{n+1}$$
and
$$t(x_{j+1}, x_{j+2}) + t(x_{j+2}, x_{j+3}) + \dots + t(x_{m-1}, x_m) < 1/2^{n+1}$$



If $m = 2$ then $j = 1$.

Otherwise, let $j \in \{1, 2, \ldots, m-1\}$ be the largest element s.t.

$$t(x_1, x_2) + \ldots + t(x_{j-1}, x_j) < \frac{1}{2^{n+1}}$$

If $j = m-1$, then (Δ) holds.
Assume $j < m-1$. Then

$$t(x_1, x_2) + \ldots + t(x_{j-1}, x_j) + t(x_j, x_{j+1}) \geq \frac{1}{2^{n+1}} .$$

so $t(x_{j+1}, x_{j+2}) + \ldots + t(x_{m-1}, x_m) < \frac{1}{2^{n+1}}$, and (Δ) holds.

Of course, $t(x_j, x_{j+1}) < \frac{1}{2^n}$. Thus

$$t(x_j, x_{j+1}) \leq \frac{1}{2^{n+1}}$$

(the values of t are in $\{1, 1/2, 1/4, 1/8, \ldots, 1/2^k, \ldots\} \cup \{0\}$

It follows that x_j and x_{j+1} belong to the same member U of $\mathcal{U}_{n+1}$.

If $m = 2$, the proof of $(*)$ is done since $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.

Assume $m > 2$ and that $(*)$ holds for smaller values of m

We have $d(x_1, x_j) < \frac{1}{2^{n+1}}$ and $d(x_{j+1}, x_m) < \frac{1}{2^{m+1}}$.

By the inductive hypothesis, it follows that x_1 and x_j belong to the same member U' of $\mathcal{U}_{n+1}$ and x_{j+1} and x_m belong to the same member U'' of $\mathcal{U}_{n+1}$.

Since $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$, there is $W \in \mathcal{U}_n$ such that

$$St(U, \mathcal{U}_{n+1}) \subseteq W.$$

Since $U' \cap U \neq \phi$ $(x_j \in U' \cap U)$ and $U'' \cap U \neq \phi$ $(x_{j+1} \in U'' \cap U)$

it follows that $U' \subseteq W$ and $U'' \subseteq W$.

In particular $x_1, x_m \in W$. Thus x, y are in the same member of $\mathcal{U}_n$.

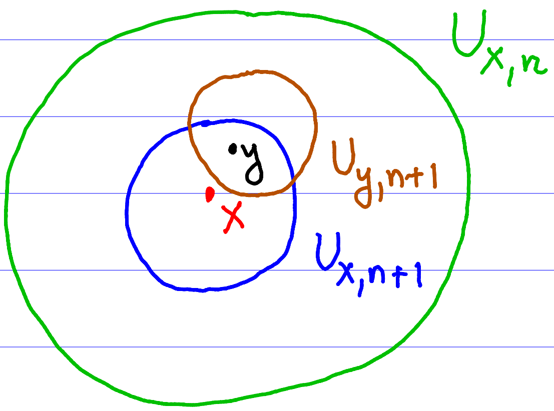
Theorem 23.5

A top. space X is pseudometrizable iff for each $x \in X$ there exists a countable nbhd base $\{U_{x,n} : n \in \mathbb{N}\}$ at x s.t. the following conditions hold for each $n \in \mathbb{N}$ and each $x, y \in X$:

(a) $y \in U_{x,n+1}$ implies that $U_{y,n+1} \subseteq U_{x,n}$

(b) $U_{y,n+1} \cap U_{x,n+1} \neq \emptyset$ implies that $y \in U_{x,n}$

(a)



(b)