MATH 793C - TOPOLOGY

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X – top. space

A <u>normal sequence</u> of open covers of X is a seq. $\mathcal{U}_1, \mathcal{U}_2, \ldots$ (of open covers of X) s.t. $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$ (it means if $U \in \mathcal{U}_{n+1}$, then there is $V \in \mathcal{U}_n$ s.t.

$$St(U, \mathcal{U}_{n+1}) := \bigcup \{U' \in \mathcal{U}_{n+1} : U' \cap U \neq \emptyset\} \subseteq V.$$

We say that a normal sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X is <u>compatible</u> (with the topology of X) iff for each $x \in X$ the set

$$\mathcal{B}_x = \{St(x, \mathcal{U}_n) : n \in \mathbb{N}\}$$

is a nbhd base at x, where

$$St(x, \mathcal{U}_n) = \bigcup \{U \in \mathcal{U}_n : x \in U\}.$$

Theorem 23.4.

A topological space X is pseudometrizable if and only if there exists a compatible normal sequence of open covers of X.

Proof

Suppose X is pseudometrizable. Let d be a pseudometric on X that is compatible with the topology of X.

For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the set of all open balls in X of radius $1/3n$ (with respect to d).

$U \in \mathcal{U}_n$ iff $U = \{y \in X : d(x,y) < 1/3n\}$ for some $x \in X$.

(a) $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a normal seq. of open covers of X

Let $n \in \mathbb{N}$ and $U \in \mathcal{U}_{n+1}$ $\quad U = B_d(x, 1/3^{n+1})$

$$St(U, \mathcal{U}_n) = \bigcup \{U' \in \mathcal{U}_{n+1} : U \cap U' \neq \emptyset\}.$$

Let $V = B_d\left(x, \frac{1}{3^n}\right) \in \mathcal{U}_n$.

If $U' = B_d\left(y, \frac{1}{3^{n+1}}\right) \in \mathcal{U}_{n+1}$ is such that $U \cap U' \neq \emptyset$

and $z \in U'$, $w \in U \cap U'$, then

$d(x, w) < \frac{1}{3^{n+1}}$, $d(y, w) < \frac{1}{3^{n+1}}$, $d(y, z) < \frac{1}{3^{n+1}}$

so $d(x, z) < \frac{1}{3^n}$. Hence $z \in V$.

We proved that $St(U, \mathcal{U}_{n+1}) \subseteq V$, so $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$.

(b) $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is compatible with the top. on X.

Let U be open in X and $x \in U$.

There is $\varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subseteq U$.

Let $n \in \mathbb{N}$ be s.t. $\frac{2}{3^n} < \varepsilon$.

$St(x, \mathcal{U}_n) \subseteq U$ since if $y \in St(x, \mathcal{U}_n)$ then

$y \in U'$ for some $U' \in \mathcal{U}_n$ s.t. $x \in U'$.

Then $d(x, y) < \frac{2}{3^n}$ so $y \in U$.

Now assume that $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a compatible normal sequence of open covers of X.

Define $t : X \times X \to [0,1]$ as follows:

Let $t(x,y) = 1/2^n$ where n is the largest element of $\mathbb{N}$ s.t. both x and y are in the same member of $\mathcal{U}_n$ (provided such n exists).

$t(x,y) = 0$ if x, y are in the same member of $\mathcal{U}_n$ for each $n \in \mathbb{N}$

$t(x,y) = 1$ if x, y are never in the same member of $\mathcal{U}_n$ for any $n \in \mathbb{N}$.

Note that x, y are in the same member of $\mathcal{U}_n$ iff $x \in St(y, \mathcal{U}_n)$ iff $y \in St(x, \mathcal{U}_n)$.

For $x, y \in X$, let $\mathcal{S}(x,y)$ be the set of all finite sequences $x_1, x_2, \ldots, x_n$ in X such that $x_1 = x$ and $x_n = y$.

Define

$$d(x,y) = \inf\left\{ \sum_{i=1}^{n-1} t(x_i, x_{i+1}) : (x_1, x_2, \dots, x_n) \in \mathcal{S}(x,y) \right\}$$

Then d is a pseudometric on X (exercise).

Let $\mathcal{V}_n$ be set of all open balls (with respect to d) of radius $1/2^n$ in X.

We show that for any $n \in \mathbb{N}$, we have

(a) $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{V}_n$.

(b) $\underbrace{\mathcal{V}_{n+1} \text{ is a refinement of } \mathcal{U}_n.}$

this means that for each $V \in \mathcal{V}_{n+1}$ there is $U \in \mathcal{U}_n$ s.t. $V \subseteq U$.

When (a) and (b) are proved then it follows that the topology induced by d is the same as the original topology of X.

If W is open in X in the original top. and $x \in W$, then there is $n \in \mathbb{N}$ s.t.

$$St(x, \mathcal{U}_n) \subseteq W$$

Since $\mathcal{V}_{n+1}$ is a cover, there is $V \in \mathcal{V}_{n+1}$ s.t. $x \in V$.

By (b) there is $U \in \mathcal{U}_n$ s.t. $V \subseteq U$.

Since $x \in U$, it follows that $U \subseteq W$.
Thus $V \subseteq W$

$$V = B_d\left(x, \frac{1}{2^{n+1}}\right)$$

Thus W is open in the top. induced by the pseudometric d.

Exercise. Prove that if W is open in the top. induced by d, then W is open in the original top.