MATH 793C — TOPOLOGY

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<u>Problem</u> Find a T_1-space Y, a compact metric space X and continuous $f: X \to Y$ s.t. $f[X]$ is not metrizable.

Let $X = Y = [0,1]$, where X has the standard topology and Y has the cofinite topology ($A \subseteq Y$ is open in Y iff $A = \phi$ or $Y \setminus A$ is finite).

Let $f: X \to Y$ be the identity function $f(x) = x$ for each $x \in X$.

Then X is a compact metric space, f is continuous, Y is T_1, but $f[X] = Y$ is not metrizable.

Y is not Hausdorff.

Corollary 23.2.

Let X be a compact metric space, Y be a Hausdorff space and $f: X \to Y$ be continuous. Then $f[X]$ is metrizable.

Example

Let Y be a Hausdorff space that is not metrizable. For example $Y = [0,1]^{\mathbb{R}}$ with the product topology. Let X be equal to Y as a set, but let X have the discrete topology. Let $f: X \to Y$ be the identity function $f(x) = x$ for each $x \in X$. Then X is metrizable, f is continuous, but $f[X] = Y$ is not metrizable.

Proof of the corollary.

Recall

Every compact Hausdorff space is normal.
Continuous image of a compact space is compact.

Thus $f[X]$ is normal hence regular. To prove the corollary, it suffices to show that $f[X]$ is second countable.

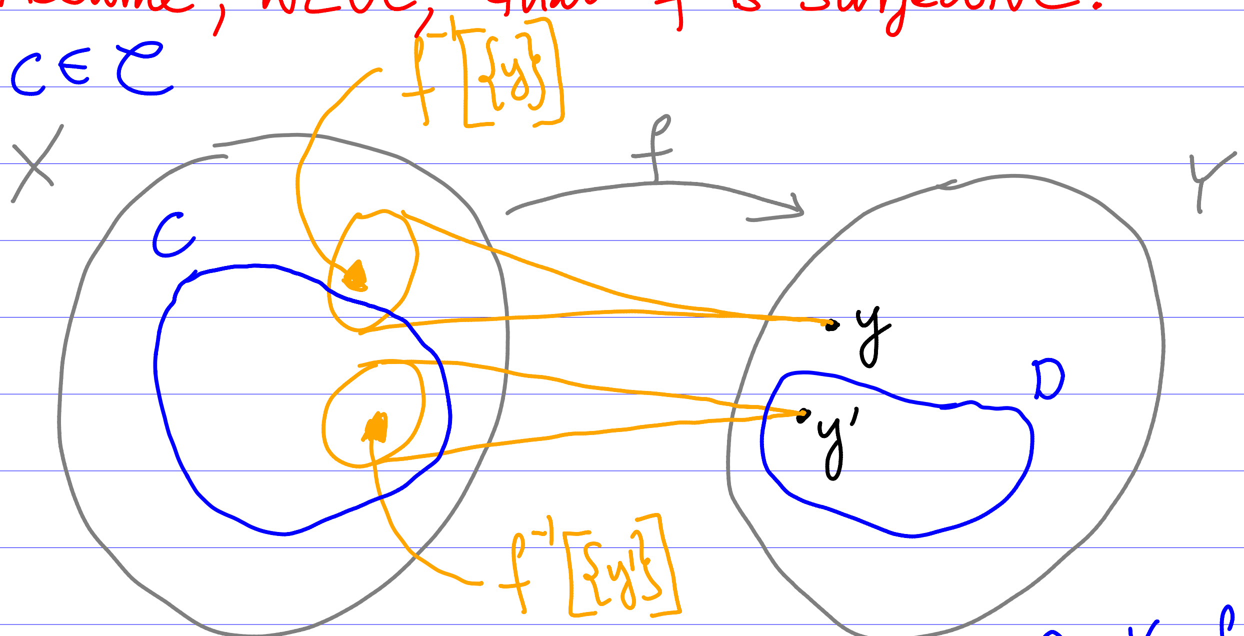
Recall

For metrizable spaces Z the following are equivalent:

1. Z is separable
2. Z is second countable
3. Z is Lindelöf

It follows that X is second countable.
Let $\mathcal{B}$ be a countable base for the top. on X.
Let $\mathcal{C} = \{ \bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B},\ \mathcal{B}' \text{ is finite} \}$.
Then $\mathcal{C}$ is also countable.
Assume, WLOG, that f is surjective.

$C \in \mathcal{C}$



$D = Y \setminus f[X \setminus C]$

$$\mathcal{D} = \{ \underbrace{Y \setminus f[X \setminus C]}_{\{ y \in Y :\ f^{-1}[\{y\}] \subseteq C \}} : C \in \mathcal{C} \}$$

each $D \in \mathcal{D}$ is open in Y

$\mathcal{D}$ is countable.

We show that $\mathcal{D}$ is a base for the top. on Y.
Let $U \subseteq Y$ be open in Y and $p \in U$.
Let $K = f^{-1}[\{p\}]$. Then K is compact.
For each $x \in K$, let $B_x \in \mathcal{B}$ be such that $x \in B_x$ and $B_x \subseteq f^{-1}[U]$. There is finite $K' \subseteq K$ s.t. $\mathcal{B}' = \{B_x : x \in K'\}$ covers K (since K is compact).
Let $C = \bigcup \mathcal{B}' \in \mathcal{C}$ and $D = Y \setminus f[X \setminus C] \in \mathcal{D}$.

$K \subseteq C$ so $p \in D$. $C \subseteq f^{-1}[U]$ so $D \subseteq U$.

if $y \in D$, then $y \notin f[X \setminus C]$ so $f^{-1}[\{y\}] \subseteq C$ so $y \in U$

X – top. space
A normal sequence of open covers of X is a seq. $\mathcal{U}_1, \mathcal{U}_2, \ldots$ (of open covers of X) s.t. $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$ (it means if $U \in \mathcal{U}_{n+1}$, then there is $V \in \mathcal{U}_n$ s.t.

$$St(U, \mathcal{U}_{n+1}) := \bigcup \{U' \in \mathcal{U}_{n+1} : U' \cap U \neq \emptyset\} \subseteq V.$$