MATH 793C — TOPOLOGY

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<u>Theorem 23.1</u> (Urysohn's metrization theorem)
Let X be a T_0-space. The following are equivalent.

(a) X is regular and second countable
(b) X is separable and metrizable
(c) X is homeomorphic to a subspace of the Hilbert cube $[0,1]^{\mathbb{N}}$.

<u>Proof</u> (c) $\Rightarrow$ (b) $\Rightarrow$ (a) done

(a) $\Rightarrow$ (c)

<u>Recall</u>: Every second countable space is Lindelöff.

<u>Recall</u>: Every regular Lindelöf space is normal.

Assume X is regular and second countable. It follows that X is normal. Since X is T_0, it follows that X is Hausdorff.

Let $\mathcal{B}$ be a countable base for the topology on X. Let

$$\mathcal{A} = \{\langle U, V\rangle \in \mathcal{B} \times \mathcal{B} : \overline{U} \subseteq V\}$$

Then $\mathcal{A}$ is countable. Let

$$\mathcal{A} = \{\langle U_n, V_n\rangle : n \in \mathbb{N}\}$$

Since X is normal, for each $n \in \mathbb{N}$, there is continuous $f_n : X \to [0,1]$ s.t. $f_n[\overline{U_n}] \subseteq \{0\}$ and $f_n[X \setminus V_n] \subseteq \{1\}$.

Let $f : X \to [0,1]^{\mathbb{N}}$ be defined by

$$f(x) = (f_n(x))_{n \in \mathbb{N}}$$

<u>f is injective</u> since if $x, y \in X$ are distinct, then there is $n \in \mathbb{N}$ such that

$$x \in U_n \quad \text{and} \quad y \notin V_n$$

Then $f_n(x) = 0$ and $f_n(y) = 1$ so $f(x) \neq f(y)$.

<u>f is continuous</u> since $\pi_n \circ f = f_n$ is cont. for each $n \in \mathbb{N}$, where $\pi_n : X \to X_n$ is the

projection.

<u>f is closed</u>

Let $C \subseteq X$ be closed in X.
We show that $f[C]$ is closed in $f[X]$.
Let $y \in f[X] \setminus f[C]$. We show that

$$y \notin cl_{f[X]}(f[C]) \qquad \boxed{y = (y_n)_{n \in \mathbb{N}}}$$

There is $x \in X$ s.t. $y = f(x)$. $x \notin C$

There is $n \in \mathbb{N}$ s.t. $x \in U_n$ and $V_n \cap C = \emptyset$.

Then $y_n = f_n(x) = 0$ and $f_n[C] \subseteq \{1\}$.

$$W_n = [0, \tfrac{1}{2}) \text{ open in } [0,1]$$

$$W_m = [0,1] \text{ for } m \in \mathbb{N} \setminus \{n\}.$$

$$W = \left(\prod_{k \in \mathbb{N}} W_k\right) \cap f[X] \text{ is open in } f[X].$$

$y \in W \qquad f[C] \cap W = \emptyset$

Thus $y \notin cl_{f[X]}(f[C])$.

We have proved that $f : X \to f[X]$ is a homeomorphism.
Thus X is homeomorphic to a subspace of the Hilbert cube.

Corollary 23.2.
Let X be a compact metric space, Y be a Hausdorff space and $f : X \to Y$ be continuous. Then $f[X]$ is metrizable.

Example Let $X = [0,1]$
Let $Y = X/\approx$ be the quotient space

obtained by identifying points in $(0,1)$ to a single point.

$Y = \{\{0\}, (0,1), \{1\}\}$

$\pi : X \to Y$ identification map

$\pi(0) = \{0\}$, $\pi(x) = (0,1)$ for $x \in (0,1)$

and $\pi(1) = \{1\}$.

$U \subseteq Y$ is open in Y iff $\pi^{-1}[U]$ is open in X.

X is a compact metric space
$\pi : X \to Y$ is continuous and $Y = f[X]$.

Y is not T_1 so not metrizable

$\{(0,1)\}$ is not closed in Y.

Y is T_0.

$\{\{0\}, (0,1)\}$ is open in Y

$\{(0,1), \{1\}\}$ —— " ——

Problem Find a T_1-space Y, a compact metric space X and continuous $f : X \to Y$ s.t. $f[X]$ is not metrizable.