MATH 793C — TOPOLOGY

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Example

$\mathbb{R}^{\mathbb{N}}$ is metrizable — Fréchet space

$[0,1]^{\mathbb{N}}$ is metrizable — Hilbert cube

Theorem 22.5 (Lebesgue covering lemma)
Let $\{U_1, \dots, U_n\}$ be a finite open cover of a compact metric space X. There is $\delta > 0$ s.t. for every $A \subseteq X$ of diameter $< \delta$, there is $i \in \{1, \dots, n\}$ s.t. $A \subseteq U_i$.

Proof

open ball cent. at x of radius ε_x

For each $x \in X$, let $\varepsilon_x > 0$ s.t. $B(x, \varepsilon_x) \subseteq U_i$ for some $i \in \{1, \dots, n\}$.

Let $B_x = B(x, \frac{\varepsilon_x}{2})$

$\mathcal{U} = \{B_x : x \in X\}$ is an open cover of X.

Since X is compact there is finite $X' \subseteq X$ s.t.

$\mathcal{U}' = \{B_x : x \in X'\}$ covers X.

Let $\delta = \min\left\{\frac{\varepsilon_x}{2} : x \in X'\right\} > 0$

Let $A \subseteq X$ with $\operatorname{diam}(A) < \delta$.

Let $y \in A$.

There is $x \in X'$ s.t. $y \in B_x$

Let d be the metric of X.

Let $i \in \{1, \dots, n\}$ be such that $B(x, \varepsilon_x) \subseteq U_i$.

For each $z \in A$, $d(y,z) < \delta \leq \frac{\varepsilon_x}{2}$

$$d(y,x) < \frac{\varepsilon_x}{2}$$

$$d(z,x) < \varepsilon_x \qquad z \in B(x, \varepsilon_x) \subseteq U_i$$

Thus $A \subseteq U_i$.

23. Metrization.

Example

$X = \mathbb{R}$ with topology $\tau = \{(x, \infty) : x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a T_0-space but not a T_1-space

Theorem 23.1 (Urysohn's metrization theorem)

Let X be a T_0-space. The following are equivalent.

(a) X is regular and second countable
(b) X is separable and metrizable
(c) X is homeomorphic to a subspace of the Hilbert cube $[0,1]^{\mathbb{N}}$.

Proof

(c) $\Rightarrow$ (b)

Recall that

If A is a nonempty set s.t. $|A| \leq |\mathbb{R}|$ and X_α is a separable top. space for each $\alpha \in A$. (X_α has a countable dense subset)
Then $\prod_{\alpha\in A} X_\alpha$ is separable.

Theorem 16.4 (c) (assumming X_α is Hausdorff for each $\alpha \in A$).

(b) $\Rightarrow$ (a) clear.

(a) $\Rightarrow$ (c)

First we show that X is Hausdorff.
Let $x, y \in X$ be distinct.
WLOG, there is closed C in X s.t. $x \in C$ and $y \notin C$.
Since X is regular, there are disjoint open sets U, V s.t. $C \subseteq U$ and $y \in V$.
Then $x \in U$ and $y \in V$. Thus X is Hausdorff.

Recall: Every second countable space is Lindelöff.

Recall: Every regular Lindelöf space is normal.