

# MATH 793C

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# TOPOLOGY

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## Theorem 22.3

$A$  - nonempty set

$M_\alpha$  - nonempty top. space for each  $\alpha \in A$

$$M = \prod_{\alpha \in A} M_\alpha$$

The following are equivalent:

- ① Each  $M_\alpha$  is metrizable and  $\{\alpha \in A : |M_\alpha| \geq 2\}$  is countable.
- ②  $M$  is metrizable.

Proof

②  $\Rightarrow$  ① done

①  $\Rightarrow$  ②

WLOG, we can assume  $A$  is countable and  $|M_\alpha| \geq 2$  for each  $\alpha \in A$ .

Let  $d_\alpha$  be a compatible metric for  $M_\alpha$  s.t.  
 $|d_\alpha(x, y)| \leq 1$  for each  $x, y \in M_\alpha$ .

Assume  $A \subseteq \mathbb{N}$ .

Define  $d: M \times M \rightarrow [0, \infty)$  by

$$d(x, y) = \sum_{\alpha \in A} \frac{d_{\alpha}(x_{\alpha}, y_{\alpha})}{2^{\alpha}},$$

where  $x = (x_{\alpha})_{\alpha \in A}$  and  $y = (y_{\alpha})_{\alpha \in A}$ .

It is easy to show that  $d$  is a metric on  $M$ .

We show that  $d$  induces the product topology on  $M$ .

Let  $U \subseteq M$  be open in the product topology.

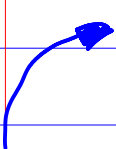
Let  $x \in U$ ,  $x = (x_{\alpha})_{\alpha \in A}$ .

There is finite  $A' \subseteq A$  and open  $V_{\alpha}$  in  $M_{\alpha}$  for each  $\alpha \in A'$  s.t.

$$x \in \prod_{\alpha \in A} V_{\alpha} \subseteq U \quad \left( x_{\alpha} \in V_{\alpha} \text{ for } \alpha \in A' \right)$$

where  $V_{\alpha} = M_{\alpha}$  for  $\alpha \in A \setminus A'$ .

For each  $\alpha \in A'$  there is  $\varepsilon_{\alpha} > 0$  s.t.


$$B(x_{\alpha}, \varepsilon_{\alpha}) \subseteq V_{\alpha}$$

open ball centered at  $x_{\alpha}$  with radius  $\varepsilon_{\alpha}$

$$\text{Let } \varepsilon = \min \left\{ \frac{\varepsilon_\alpha}{2^\alpha} : \alpha \in A' \right\} > 0$$

$$\text{Let } V = B(x, \varepsilon). \quad x \in V$$

We show that  $V \subseteq U$ .

Assume that  $z \in V$ ,  $z = (z_\alpha)_{\alpha \in A}$

$$\sum_{\alpha \in A} \frac{d(x_\alpha, z_\alpha)}{2^\alpha} < \varepsilon$$

Then  $\frac{d(x_\alpha, z_\alpha)}{2^\alpha} < \varepsilon$  for each  $\alpha \in A'$ .

Thus  $d(x_\alpha, z_\alpha) < \varepsilon_\alpha$  for each  $\alpha \in A'$ .

so  $x \in U$ .

Now let  $U = B(x, \varepsilon)$ ,  $\varepsilon > 0$ ,  $x = (x_\alpha)_{\alpha \in A}$ .

Let  $m \in \mathbb{N}$  s.t.  $\sum_{\alpha=m}^{\infty} \frac{1}{2^\alpha} < \frac{\varepsilon}{2}$ .

Let  $A' = \{\alpha \in A : \alpha < m\}$  - finite.

For each  $\alpha \in A'$  let  $\varepsilon_\alpha$  be such that

$$\sum_{\alpha \in A'} \frac{\varepsilon_\alpha}{2^\alpha} < \frac{\varepsilon}{2}$$

Let  $V_\alpha = B(x_\alpha, \varepsilon_\alpha)$  for  $\alpha \in A'$

$V_\alpha = M_\alpha$  for  $\alpha \in A \setminus A'$

Then  $\bigcap_{\alpha \in A} V_\alpha \subseteq U$ .