MATH 793C — TOPOLOGY

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Theorem 22.3

A – nonempty set

M_α – nonempty top. space for each $\alpha \in A$

$M = \prod_{\alpha \in A} M_\alpha$

The following are equivalent:

① Each M_α is metrizable and $\{\alpha \in A : |M_\alpha| \geq 2\}$ is countable.

② M is metrizable.

Proof

② $\Rightarrow$ ① done

① $\Rightarrow$ ②

WLOG, we can assume A is countable and $|M_\alpha| \geq 2$ for each $\alpha \in A$.

Let d_α be a compatible metric for M_α s.t. $|d_\alpha(x,y)| \leq 1$ for each $x, y \in M_\alpha$.

Assume $A \subseteq \mathbb{N}$.

Define $d : M \times M \to [0, \infty)$ by

$$d(x,y) = \sum_{\alpha \in A} \frac{d_\alpha(x_\alpha, y_\alpha)}{2^\alpha},$$

where $x = (x_\alpha)_{\alpha \in A}$ and $y = (y_\alpha)_{\alpha \in A}$.

It is easy to show that d is a metric on M.

We show that d induces the product topology on M.
Let $U \subseteq M$ be open in the product topology.
Let $x \in U$, $x = (x_\alpha)_{\alpha \in A}$.

There is finite $A' \subseteq A$ and open V_α in M_α for each $\alpha \in A'$ s.t.

$$x \in \prod_{\alpha \in A} V_\alpha \subseteq U \quad \left(x_\alpha \in V_\alpha \text{ for } \alpha \in A' \right)$$

where $V_\alpha = M_\alpha$ for $\alpha \in A \setminus A'$.

For each $\alpha \in A'$ there is $\varepsilon_\alpha > 0$ s.t.

$$B(x_\alpha, \varepsilon_\alpha) \subseteq V_\alpha$$

open ball centered at x_α with radius ε_α

Let $\varepsilon = \min\left\{ \frac{\varepsilon_\alpha}{2^\alpha} : \alpha \in A' \right\} > 0$

Let $V = B(x, \varepsilon)$. $\quad x \in V$

We show that $V \subseteq U$.

Assume that $z \in V$, $\quad z = (z_\alpha)_{\alpha \in A}$

$$\sum_{\alpha \in A} \frac{d(x_\alpha, z_\alpha)}{2^\alpha} < \varepsilon$$

Then $\frac{d(x_\alpha, z_\alpha)}{2^\alpha} < \varepsilon$ for each $\alpha \in A'$.

Thus $d(x_\alpha, z_\alpha) < \varepsilon_\alpha$ for each $\alpha \in A'$.

so $x \in U$.

Now let $U = B(x, \varepsilon)$, $\varepsilon > 0$, $x = (x_\alpha)_{\alpha \in A}$.

Let $m \in \mathbb{N}$ s.t. $\sum_{\alpha = m}^{\infty} \frac{1}{2^\alpha} < \frac{\varepsilon}{2}$.

Let $A' = \{\alpha \in A : \alpha < m\}$ — finite.

For each $\alpha \in A'$ let ε_α be such that

$$\sum_{\alpha \in A'} \frac{\varepsilon_\alpha}{2^\alpha} < \frac{\varepsilon}{2}$$

Let $V_\alpha = B(x_\alpha, \varepsilon_\alpha)$ for $\alpha \in A'$

$V_\alpha = M_\alpha$ for $\alpha \in A \setminus A'$

Then $\prod_{\alpha \in A} V_\alpha \subseteq U$.