MATH 793C — TOPOLOGY

Jerzy Wojciechowski Spring 2020

CLASS 1 January 13

Metrizable spaces

Equivalent metrics

Two metrics on the same space X are equivalent iff they induce the same topology on X.

Theorem 22.2

Every metric is equivalent to a bounded metric.

There is $n \in \mathbb{N}$ s.t. the values of the metric are in $[0, n]$.

Proof

Let ρ be a metric on X. Define $\rho_1, \rho_2 : X \times X \to [0, \infty)$

$$\rho_1(x,y) = \min\{1, \rho(x,y)\}$$

$$\rho_2(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$$

It can be proved that both ρ_1 and ρ_2 are eq. to ρ.

Theorem 22.3

A – nonempty set

M_α – nonempty top. space

$$M = \prod_{\alpha \in A} M_\alpha$$

The following are equivalent:

① Each M_α is metrizable and $\{\alpha \in A : |M_\alpha| \geq 2\}$ is countable.

② M is metrizable.

Proof

② $\Rightarrow$ ① Assume M is metrizable.

Let $x_\alpha \in M_\alpha$ be fixed for each $\alpha \in A$

Let $f_\alpha : M_\alpha \longrightarrow M$ def. by

$f_\alpha(y) = (z_\alpha)_{\alpha \in A}$ $\quad z_\beta = x_\beta$ for each $\beta \in A \setminus \{\alpha\}$.

$z_\alpha = y$.

Then f_α is an embedding (homeomorphism onto $f_\alpha[M_\alpha]$)

Each $f_\alpha[M_\alpha]$ is a subspace of M so is metrizable.

Thus each M_α is metrizable.

Suppose, BWOC, $A' = \{\alpha \in A : |M_\alpha| \geq 2\}$ is uncountable. WLOG we can assume $A' = A$.

Let $x = (x_\alpha)_{\alpha \in A} \in M$.

Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable nbhd base at x.

For each $\alpha \in A$, let U_α be an open nbhd of x_α in M_α s.t. $U_\alpha \neq M_\alpha$, and let

$$V_\alpha = \prod_{\beta \in A} U'_\beta \quad \text{where} \quad \begin{array}{l} U'_\alpha = U_\alpha \text{ and} \\ U'_\beta = M_\beta \text{ for } \beta \neq \alpha \end{array}$$

V_α is an open nbhd of x in M.

For each $i \in \mathbb{N}$, there is finite $A_i \in A$ s.t.

$$S_i = \prod_{\alpha \in A} S_\alpha \subseteq B_i \quad \text{where} \quad \begin{array}{l} S_\alpha = \{x_\alpha\} \text{ for } \alpha \in A_i \\ S_\alpha = M_\alpha \text{ for } \alpha \in A \setminus A_i \end{array}$$

$\bigcup_{i \in \mathbb{N}} A_i$ is countable so there is $\gamma \in A \setminus \bigcup_{i \in \mathbb{N}} A_i$

$$T = \prod_{\alpha \in A} T_\alpha \subseteq \bigcap_{i \in \mathbb{N}} B_i \quad \text{where} \quad \begin{array}{l} T_\alpha = \{x_\alpha\} \text{ for } \alpha \neq \gamma \\ T_\gamma = M_\gamma \end{array}$$

But $T \not\subseteq V_\gamma$ so $B_i \not\subseteq V_\gamma$ for any $i \in \mathbb{N}$.
contradiction.

① $\Longrightarrow$ ② Assume A is countable $A \subseteq \mathbb{N}$.
Let d_α be a compatible metric for M_α (bounded by 1)
define $d : M \times M \longrightarrow [0, \infty)$

$$d(x, y) = \sum_{\alpha \in \mathbb{N}} \frac{d_\alpha(x_\alpha, y_\alpha)}{2^\alpha}$$