Math 683 **Set Theory and Applications** Fall 2013

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Contents

1 ZFC Axioms.

Why do we need Axioms?

Set Property.

A set property $\varphi(x)$ is any expression build using the membership relation $\in$, equality $=$, logical connectives, quantifiers, some fixed sets (called parameters) and one free variable x. Given a set property $\varphi(x)$ and a set A, we say that A has this property if the expression becomes true when we substitute A for x in $\varphi(x)$.

Example 1.1. The following is a set property

$$\exists y \left(y \in x \wedge y \in \mathbb{R} \wedge \neg \left(y = 0 \right) \right).$$

A set A has this property if and only if it contains at least one nonzero real number. In set theory the number 0 is the same as the empty set $\varnothing$. Thus $\mathbb{R}$ and $\varnothing$ are parameters of that property.

We would like to form sets $\left\{ x : \varphi(x) \right\}$ for any set property $\varphi(x)$. However, this leads to a paradox.

Russell's Paradox.

Consider the property $x \notin x$ and let A be the set of all sets that have this property. Consider the question: "Is A a member of itself?" If "yes", then it is not true that $A \notin A$, a contradiction. If "no", then it is true that $A \notin A$, again a contradiction.

Solution — Axioms.

To avoid problems like the Russell's Paradox, we introduce axioms that the universe of sets is to satisfy. We will denote this universe by V. Every set is an element of V, but, as we will see later, V itself is not a set.

Classes.

We still want to be able to consider collections of the form $\left\{ x : \varphi(x) \right\}$ consisting of sets that have some fixed property $\varphi(x)$. We call such collections classes. Every set is a class since if A is a set then the class $\{x : x \in A\}$ (we use A as a parameter here) is equal to A. Classes that are not sets are called proper classes.

The Universe of Sets as a Directed Graph.

We can think of V as being a directed graph with a directed edge $u \to v$ joining u to v iff $u \in v$. We can think of the universe as being any directed graph that satisfies all the restrictions described by the axioms. Otherwise, it can be arbitrary.

Axiom of Extensionality.

- If the sets A and B have the same elements then they are equal.

If we think of the universe as a bipartite graph, then this axioms says that it is not possible to have two different vertices u and v such that the same vertices are joined to u as to v.

Axiom Schema of Separation (Comprehension).

Let $\varphi(x)$ be a set property. Then the following is an axiom of ZFC.

- For any set A the class $\{x : x \in A \wedge \varphi(x)\}$ is a set.

We will denote this set also as $\{x \in A : \varphi(x)\}$.

Remark 1.2. Now we can prove that the universe V is a proper class. Otherwise $\{x \in V : x \notin x\}$ would be a set and as for the Russell's Paradox we would get a contradiction.

"Axiom" of Paring.

- For any sets A and B, there is a set with A and B as elements and no other elements.

This "axiom" is unnecessary as it follows from the other axioms (to be shown later). Such a set will be denoted by $\{A, B\}$ or by $\{A\}$ if $A = B$.

Axiom of Union.

- For any set A the class $\bigcup A = \{x : \exists y \left(y \in A \wedge x \in y\right)\}$ is a set.

We call $\bigcup A$ the union of A.

Axiom of Power Set.

- For any set A the class $P(A) = \{x : x \subseteq A\}$ is a set.

The property $x \subseteq A$ expresses that x is a subset of A and is a set property since it can be formally stated as $\forall y \left(y \in x \Rightarrow y \in A\right)$.

We call $P(A)$ the power set of A.

Axiom of Infinity.

Natural Numbers.

In set theory, natural numbers are represented as sets as follows: 0 is the empty set $\varnothing$, 1 is the set $\{\varnothing\} = \{0\}$, 2 is the set $\{0, 1\} = \{\varnothing, \{\varnothing\}\}$ and so on. In general, $n+1$ is the set $\{0, 1, \ldots, n\}$. In particular, we have

$$n + 1 = n \cup \{n\} = \bigcup \{n, \{n\}\}.$$

If A is any set, then we will denote $A + 1 = A \cup \{A\}$. Now we are ready to state the axiom:

- There exists a set A such that $\varnothing \in A$ and if $B \in A$, then $B+1 \in A$.

Note that, in particular, the axiom says that there exists a set so the universe V is nonempty.

Axiom Schema of Replacement.

Ordered Pairs.

We want to define ordered pairs $\langle A, B\rangle$ of sets so that $\langle A, B\rangle = \left\langle A', B'\right\rangle$ if and only if $A = A'$ and $B = B'$. One possible way to do so is to define

$$\langle A, B\rangle = \{\{A, B\}, \{A\}\}.$$

Relations and Functions.

A relation is a class R whose elements are ordered pairs. The domain of a relation R is the class $\operatorname{dom}(R) = \left\{x : \exists y \left\langle x, y\right\rangle \in R\right\}$ and the range of R is the class $\operatorname{ran}(R) = \left\{y : \exists x \left\langle x, y\right\rangle \in R\right\}$. A function is a class F such that for any sets A, B, C if $\langle A, B\rangle \in F$ and $\langle A, C\rangle \in F$, then $B = C$. Thus for any $A \in \operatorname{dom}(F)$ there exists exactly one set B with $\langle A, B\rangle \in F$. We will write then $B = F(A)$. If A is a set, then

$$F\text{''}A = \{F(B) : B \in \operatorname{dom}(F) \wedge B \in A\}$$

is the image of A under F.

The Axiom Schema of Replacement.

Let F be a class that is a function.

- For every set A, the class $F\text{''}A$ is a set.

Remark 1.3. We can now prove that the Paring Axiom follows from the other axioms. Let A and B be arbitrary sets and C be a set that contains the empty set and a nonempty set. The existence of such C follows from the Axiom of Infinity. Consider the following class that is a function

$$F = \left\{x : x = \langle \varnothing, A\rangle \vee \exists y \left(x = \left\langle y, B\right\rangle \wedge y \neq \varnothing\right)\right\}.$$

Then $F\text{''}C = \{A, B\}$ and is a set by the Axiom of Replacement for F.

Axiom of Regularity.

- For every nonempty set A there exists $B \in A$ such that $B \cap A = \varnothing$.

This axiom implies that there are no infinite sequences $A_0 \ni A_1 \ni A_2 \ni \ldots$ of sets in V. In particular, the directed graph representing the universe of sets has no directed cycles and no loops (no set is an element of itself).

Axiom of Choice.

Choice Functions.

Let A be a set. A choice function for A is a function f with domain A such that $f(a) \in a$ for every $a \in A$.

Now we can state the Axiom of Choice.

- For every set A with $\varnothing \notin A$ there exists a choice function for A.

2 Cardinal Numbers.

Cardinals and their Ordering.

Equicardinality.

Let A and B be sets. We say that A and B have the same cardinality if there exists a bijection $A \to B$.

Assumptions about Cardinals.

Suppose $|\cdot|$ is a function (class) with domain equal to V (the class of all sets) such that if $\kappa = |A|$, then $|\kappa| = \kappa$ and $|A| = |B|$ for any sets A and B that have the same cardinality. A set is a cardinal number (or a cardinal) iff it belongs to the range of this function. It follows that $|A| = |B|$ if and only if A and B have the same cardinality. We assume also that if A is finite, then $|A|$ is the number of elements of A.

The existence of such a function will be shown later.

Ordering of Cardinals.

Definition. Let κ, λ be cardinals. We say that $\kappa \leq \lambda$ if there exists an injection $\kappa \to \lambda$. Note that it follows that for any sets A, B we have $|A| \leq |B|$ if and only if there exists an injection $A \to B$.

Remark. Obviously, $\leq$ is reflexive. It is also clear that $\leq$ is transitive, that is, for any cardinals κ, λ, μ if $\kappa \leq \lambda \leq \mu$, then $\kappa \leq \mu$.

The following theorem implies that it is a partial ordering of the class of cardinals.

Theorem 2.1 (Cantor-Bernstein)**.** *Let κ and λ be cardinals. If $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$.*

The following theorem implies that for every cardinal there exists a strictly larger cardinal.

Theorem 2.2 (Cantor)**.** *Let A be any set. Then $|P(A)| > |A|$.*

Cardinal Arithmetic.

Definition. Let κ and λ be cardinals. Then the sum $\kappa+\lambda$ is defined as $|\kappa\times\{0\}\cup\lambda\times\{1\}|$ (the cardinality of the disjoint union of κ and λ, the product $\kappa\lambda$ is defined as $|\kappa\times\lambda|$ and λ^κ as $\left|\{f : f : \kappa\to\lambda\}\right| = |{}^\kappa\lambda|$ (the cardinality of the set ${}^\kappa\lambda$ of all functions from κ to λ).

Remark. If A and B are any sets of cardinalities κ and λ respectively, then $A\times B$ has cardinality $\kappa\lambda$ and the set ${}^AB$ of all functions $A\to B$ has cardinality λ^κ. If A and B are moreover disjoint, then $A\cup B$ has cardinality $\kappa+\lambda$. Note that for each set A we have $|P(A)| = 2^{|A|}$. Cantor's Theorem implies that $2^\kappa > \kappa$ for any cardinal κ.

Lemma 2.3. *We have:*

1. *Addition and multiplication of cardinals is associative and commutative. Also multiplication is distributive with respect to addition.*
2. $(\kappa\lambda)^\mu = \kappa^\mu\lambda^\mu$.

   Proof. We need to show that there is a bijection $f : {}^\mu(\kappa\times\lambda)\to{}^\mu\kappa\times{}^\mu\lambda$. If $g:\mu\to\kappa\times\lambda$, then let $f(g) = \langle g_1, g_2\rangle$ where $g_1:\mu\to\kappa$ and $g_2:\mu\to\lambda$ are such that if $g(a) = \langle b, c\rangle$, then $g_1(a) = b$ and $g_2(a) = c$. Then f is a bijection since it has the inverse assigning to a pair $\langle g_1, g_2\rangle$ of functions $g_1:\mu\to\kappa$ and $g_2:\mu\to\lambda$ the function $g : {}^\mu(\kappa\times\lambda)\to{}^\mu\kappa\times{}^\mu\lambda$ such that if $g_1(a) = b$ and $g_2(a) = c$ then $g(a) = \langle b, c\rangle$. □

3. $\kappa^{\lambda+\mu} = \kappa^\lambda\kappa^\mu$.
4. $\left(\kappa^\lambda\right)^\mu = \kappa^{\lambda\mu}$.
5. *If $\kappa\le\lambda$, then $\kappa^\mu\le\lambda^\mu$.*
6. *If $0<\lambda\le\mu$, then $\kappa^\lambda\le\kappa^\mu$.*

Homework 1 (due 8/25).

Prove that for any cardinals κ,λ,μ we have $\kappa\left(\lambda+\mu\right) = \kappa\lambda+\kappa\mu$.

Homework 2 (due 8/28).

Prove that for any cardinals κ,λ,μ if $0<\lambda\le\mu$, then $\kappa^\lambda\le\kappa^\mu$.

Solution. If $\kappa=\emptyset$, then ${}^\lambda\kappa = {}^\mu\kappa = \emptyset$ so $\kappa^\lambda\le\kappa^\mu$. Assume that $\kappa\ne\emptyset$, let $d\in\kappa$ be a fixed element and let $f:\lambda\to\mu$ be an injection. We define an injection ${}^\lambda\kappa\to{}^\mu\kappa$ so that given $g:\lambda\to\kappa$ we assign to it the function $g':\mu\to\kappa$ defined as follows. If $a\in\mu$, then let $g'(a) = g\big(f^{-1}(a)\big)$ if $a\in\operatorname{ran}\big(f\big)$ and let $g'(a) = d$ otherwise. It remains to show that the function mapping g to g' is injective. Suppose $g_1, g_2:\lambda\to\kappa$ and $g_1\ne g_2$. Then $g_1(b)\ne g_2(b)$ for some $b\in\lambda$ implying that $g_1'\big(f(b)\big)\ne g_2'\big(f(b)\big)$ and so $g_1'\ne g_2'$. □

Remark. We will prove later then $\kappa\kappa=\kappa$ for any infinite cardinal κ. Before we can do that we need to introduce ordinal numbers.

3 Ordinal Numbers and Transfinite Induction.

Well-ordering.

Definition. A well-ordering $<$ on a set P is a linear ordering such that for every nonempty subset $A \subseteq P$ there exists the smallest element in A.

Lemma 3.1. *Let W be a set that is well-ordered by $<$ and $f : W \to W$ be a function that preserves the relation $<$. Then $f(x) \geq x$ for every $x \in W$.*

Initial Segment.

Definition. Let W be well-ordered by $<$. An initial segment of this ordering is a subset of W of the form

$$W(b) = \{a : a < b\}$$

for some $b \in W$.

Isomorphism of Well-ordered Sets.

Definition. An isomorphism of well-ordered sets is a bijection that preserves the order.

Lemma 3.2. *No well-ordered set is isomorphic to an initial segment of itself.*

Theorem 3.3. *Let W_1, W_2 be well-ordered sets. Then exactly one of the following three cases holds:*

1. *W_1 is isomorphic to W_2.*
2. *W_1 is isomorphic to an initial segment of W_2.*
3. *W_2 is isomorphic to an initial segment of W_1.*

Proof. Let F be the set of all ordered pairs $\langle a, b\rangle$ with $a \in W_1$ and $b \in W_2$ such that the initial segments $W_1(a)$ and $W_2(b)$ are isomorphic. Lemma 3.2 implies that for any $a \in W_1$ there is at most on $b \in W_2$ with $\langle a, b\rangle \in F$ and for each $b \in W_2$ there is at most one $a \in W_1$ with $\langle a, b\rangle \in F$. Thus F is an injective function. If $a' < a$ and $F(a) = b$, then $W_1\big(a'\big)$ is isomorphic to $W_2\big(b'\big)$ for some $b' < b$ implying that F is order preserving.

We claim that $\operatorname{dom}(F) = W_1$ or $\operatorname{ran}(F) = W_2$. Otherwise, let a be the smallest element in $W_1 \setminus \operatorname{dom}(F)$ and b be the smallest element in $W_2 \setminus \operatorname{ran}(F)$. Then $\operatorname{dom}(F) = W_1(a)$ and $\operatorname{ran}(F) = W_2(b)$ so $W_1(a)$ and $W_2(b)$ are isomorphic and $\langle a, b\rangle \in F$ which is a contradiction. □

Ordinals.

Transitive Sets.

Definition. A set A is transitive iff every element of A is a subset of A, that is, if $B \in A$ and $C \in B$, then $C \in A$.

Example. The set $\{\{\{\varnothing\}\}, \{\varnothing\}, \varnothing\}$ is transitive, but $\{\{\{\varnothing\}\}, \varnothing\}$ is not.

What is an Ordinal?

Definition. An ordinal is a transitive set A such that for any distinct $B, C \in A$ we have either $B \in C$ or $C \in B$.

We use greek letters to denote ordinals.

Example. The set $\{\{\{\varnothing\}\}, \{\varnothing\}, \varnothing\}$ is transitive, but is not an ordinal.

Remark. We will show later that any element of ω is an ordinal and ω itself is also an ordinal.

Homework 3 (due 8/30).

List all elements of the set $P(P(P(P(\varnothing))))$. Which of them are transitive, which are ordinals?

Solution. The transitive sets that are also ordinals are $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$. The transitive sets that are not ordinals are $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$ and $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$. The remaining elements are not transitive. □

Lemma 3.4. *Any ordinal is well-ordered by the relation $\in$.*

Proof. Let α be an ordinal. By the regularity axiom, we only need to show that the relation $\in$ restricted to α is transitive, or equivalently that every element of α is transitive. Let $a, b, c \in \alpha$ with $a \in b \in c$. We want to show that $a \in c$. Since $a \neq c$ and since α is an ordinal, we must have $a \in c$ or $c \in a$. As $c \in a$ is impossible by the regularity axiom, we have $a \in c$. □

Properties of Ordinals.

Lemma 3.5. *The following properties hold:*

1. *Any element of an ordinal is an ordinal.*

   Proof. Let α be an ordinal and let $b \in \alpha$. Lemma 3.4 implies that b is transitive. If $c, d \in b$ are distinct then by transitivity of α we have $c, d \in \alpha$ so either $c \in d$ or $d \in c$. □

2. *If β is a proper transitive subset of an ordinal α then $\beta \in \alpha$.*

   Proof. Since β is a proper subset of α, the difference $\alpha \setminus \beta$ is nonempty. Let γ be its smallest element (with respect to $\in$). Any element of γ belongs to α (by transitivity of α) hence to β (no element of γ can be in $\alpha \setminus \beta$ as γ is the smallest element of $\alpha \setminus \beta$). Thus $\gamma \subseteq \beta$. Any element δ of β must be in γ since otherwise $\gamma \in \delta$ (both γ and δ are in α) and consequently $\gamma \in \beta$ by transitivity of β. □

3. *If α and β are ordinals, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

   Proof. The intersection $\gamma = \alpha \cap \beta$ is an ordinal but not a member of itself so either $\gamma \notin \alpha$ or $\gamma \notin \beta$. Therefore $\gamma = \alpha$ or $\gamma = \beta$. (If γ is a proper subset of α, then $\gamma \in \alpha$. Similarly for β.) □

The Class of all Ordinals.

Let Ord be the class of all ordinal numbers. Since the elements of an ordinal are ordinals, Ord is a transitive class. If α and β are distinct ordinals then one of them is an element of the other. Thus Ord is not a set as it would be an ordinal otherwise, hence a member of itself. If $\alpha, \beta \in$ Ord, then we write $\alpha < \beta$ for $\alpha \in \beta$. Note that each ordinal is thus the set of all smaller ordinals.

Remark. Any nonempty class of ordinals has the smallest element.

Proof. Let C be a class of ordinals and $\alpha \in C$. If α is not the smallest ordinal in C, then there is the smallest ordinal β in the nonempty set $\alpha \cap C$. Since any ordinal smaller than β belongs to α, the ordinal β is the smallest in C. □

Homework 4 (due 9/4).

1. Let C be a nonempty class of ordinals. Prove that $\bigcap C$ is an ordinal and that $\bigcap C$ is the smallest element in C.

2. Let C be a nonempty set of ordinals. Prove that $\bigcup C$ is an ordinal and that $\bigcup C$ is the smallest element in the class
$$\left\{\alpha \in \mathrm{Ord} : \alpha \geq \beta \text{ for any } \beta \in C\right\}.$$

3. Prove that for each ordinal α, the set $\alpha + 1 = \alpha \cup \{\alpha\}$ is also an ordinal and that it is the smallest ordinal in the class $\left\{\beta \in \mathrm{Ord} : \beta > \alpha\right\}$.

Successor Ordinals and Limit Ordinals.

Definition. Any ordinal of the form $\alpha + 1$ for some ordinal α is called a successor ordinal. Other ordinals including 0 are called limit ordinals.

Remark. An ordinal α contains the largest element if and only if it is a successor ordinal. An ordinal α is equal to $\bigcup \alpha$ if and only if it is a limit ordinal.

Theorem 3.6. *Every well ordered set is order isomorphic to a unique ordinal.*

Proof. Let W be a well-ordered set. Let F be the set of all ordered pairs $\langle a, \alpha \rangle$ with $a \in W$ and α being an ordinal isomorphic to $W(a)$. Then F is an order preserving injective function with domain W (as in the proof of Theorem 3.3). By the axioms of replacement ran(F) is a set such that if $\alpha \in$ ran(F) then any ordinal smaller than α belongs to ran(F). Thus ran(F) is the smallest ordinal β in the nonempty class Ord $\setminus$ ran(F) and F is an isomorphism from W onto β.

Since for distinct ordinals one is an initial segment of the other, no different ordinals can be isomorphic. Thus we have uniqueness. □

Transfinite Induction and Recursion.

Transfinite Induction.

Theorem 3.7. *Let C be a class of ordinals such that for every ordinal α if $\alpha \subseteq C$ then $\alpha \in C$. Then $C = Ord$.*

Proof. Otherwise let α be the smallest element of $C \setminus \text{Ord}$. We have $\alpha \subseteq C$ so $\alpha \in C$ which is a contradiction. $\square$

Corollary 3.8. *Let C be a class of ordinals such that:*

1. $0 \in C$;
2. $\alpha + 1 \in C$ *for every* $\alpha \in C$;
3. *if β is a nonzero limit ordinal and $\gamma \in C$ for every $\gamma < \beta$, then $\beta \in C$.*

Then $C = Ord$.

Homework 5 (due 9/6).

Prove Corollary 3.8.

Transfinite Sequences.

A transfinite sequence is a function whose domain is an ordinal.

Transfinite Recursion.

Theorem 3.9. *Let G be a function whose domain is the class of all transfinite sequences. Then there exists a unique function F with domain Ord such that $F(\alpha) = G(F \restriction \alpha)$ for each ordinal α.*

Proof. Let C be the class of all ordinals β for which there exists a transfinite sequence $s_\beta = \langle a_\alpha : \alpha < \beta \rangle$ such that $a_\alpha = G\big(\langle a_\gamma : \gamma < \alpha \rangle\big)$ for every $\alpha < \beta$. Note that it follows by transfinite induction that if such s_β exists, then it is unique. We prove by transfinite induction that $C = \text{Ord}$. Then we define $F\big(\beta\big) = G\big(s_\beta\big)$ and use transfinite induction to prove the uniqueness of F. $\square$

How to Well-order any Set?

Theorem 3.10. *Every set can be well-ordered.*

Proof. Let W be a set, S be the set of all nonempty subsets of W and f be a choice function for S. Define the function G on the class of all transfinite sequences as follows. If $t = \langle a_\beta : \beta < \alpha \rangle$ is a transfinite sequence with all elements in W and $B = W \setminus \{a_\beta : \beta < \alpha\} \neq \emptyset$, then let $G(t) = f(B)$. Otherwise, let $G(t) = W$. Let F be the unique function with domain Ord so that $F(\alpha) = G(F \restriction \alpha)$ for every $\alpha \in \text{Ord}$ and let γ be the smallest ordinal with

$F(\gamma) = W$. Then F is a bijection $\gamma \to W$. We can now use F to transfer the well-ordering from γ to W.

The existence of the ordinal γ follows from the replacement axioms. If γ were not to exist, then F would be a bijection of Ord onto a subset of W. Thus the inverse F^{-1} would be a function with its domain being a set onto Ord, thus implying (using the replacement axioms) that Ord is a set, which is a contradiction. □

Remark. Assuming all the ZFC axiom except the axiom of choice we can prove that if every set can be well-ordered then the axiom of choice holds.

Proof. Let A be a set such that $\varnothing \notin A$. Let the set $S = \bigcup A$ be well-ordered. Define a choice function f for A so that for each $B \in A$, the value $f(B)$ is the smallest element in B. □

Zorn's Lemma.

Remark. Zorn's Lemma is also known as Kuratowski-Zorn Lemma. It was actually first proved by Kuratowski.

Theorem 3.11. *Let P be a partially ordered set such that every chain in P has an upper bound. Then there exists a maximal element in P.*

Proof. Let C be the set of all chains in P and for each chain $c \in C$ let A_c be the set of all upper bounds on c in $P \setminus c$. Let $S = \{A_c : c \in C \wedge A_c \neq \varnothing\}$ and f be a choice function for S. Define the function G on the class of all transfinite sequences as follows. If $t = \langle a_\beta : \beta < \alpha \rangle$ is a transfinite sequence such that the set $c = \{a_\beta : \beta < \alpha\}$ of its elements is a chain in P and $A_c \neq \varnothing$, then let $G(t) = f(A_c)$. Otherwise, let $G(t) = P$. Let F be the unique function with $F(\alpha) = G(F \restriction \alpha)$ for each ordinal α.

Using the replacement axiom as in the proof of Theorem 3.10 we conclude that there is the smallest ordinal γ such that $F(\gamma) = P$. Then $c = \{F(\alpha) : \alpha < \gamma\}$ is a chain with $A_c = \varnothing$. Let $b \in P$ be an upper bound on c. Then b is a maximal element since $b < d$ would imply that $d \in A_c$. □

Remark. Assuming Zorn's Lemma, we can also prove that ZF (which is ZFC without the axiom of choice) implies that every set can be well-ordered as follows.

Sketch of the Proof. Let A be a set. Consider the set of all ordered pairs $(B, <)$ with $B \subseteq A$ and $<$ being a well-ordering of B. Let $(B, <_B) \leq (C, <_C)$ when $B \subseteq C$ and $<_B \subseteq <_C$ (that is, when $<_C$ is an extension of $<_B$). Then we get a partial order with each chain having an upper bound so there is a maximal element $(D, <)$. It can then be seen that we must have $D = A$ so A can be well-ordered. □

Homework 6 (due 9/11).

Let A be a set and $S \subseteq P(A)$. We say that S has finite character iff for every subset B of A the following statements are equivalent:

1. B belongs to S;

2. every finite subset of B belongs to S.

Prove (using Zorn's Lemma) that if S has finite character and $C \in S$ then there exists $D \in S$ with $C \subseteq D$ and such that D is maximal in S. (D is maximal in S when there does not exists $E \in S$ with $D \subseteq E$ and $D \neq E$.)

Cardinal Numbers as Initial Ordinals.

Initial Ordinals.

Definition. An ordinal α is an initial ordinal iff there are no bijection $\alpha \to \beta$ for any ordinal $\beta < \alpha$.

Example. Any $n \in \omega$ is an initial ordinal. ω is also an initial ordinal. $\omega + 1$ is not initial as there is a bijection $\omega + 1 \to \omega$.

Remark. Note that for every ordinal α there exists an initial ordinal $\beta > \alpha$.

Proof. Let α be an ordinal. The set $P(\alpha)$ can be well-ordered so there is a bijection $P(\alpha) \to \gamma$ for some ordinal γ. Let β be the smallest ordinal for which there exists a bijection $P(\alpha) \to \beta$. Then β is an initial ordinal and no bijection maps $\alpha \to \beta$. Since there exists an injection $\alpha \to P(\alpha)$ there also exists an injection $\alpha \to \beta$. Thus no injection maps $\beta \to \alpha$ implying that $\alpha < \beta$. $\square$

Labeling the Initial Ordinals.

The smallest infinite initial ordinal is ω. It is denoted also by ω_0. Using transfinite recursion, we define ω_α to be the smallest infinite initial ordinal in $\mathrm{Ord} \setminus \left\{\omega_\beta : \beta < \alpha\right\}$.

Remark. It follows, using the axioms of replacement, that the class of initial ordinals is a proper class.

Definition of Cardinality and Cardinal Numbers.

Definition. For each set A, let the cardinality $|A|$ be the smallest ordinal α for which there exists a bijection $A \to \alpha$.

Remark. Note that $|A|$ is an initial ordinal and that $|\alpha| = \alpha$ for any initial ordinal α. Thus cardinal numbers are exactly the initial cardinals. It is clear that the cardinality function $|\cdot|$ as defined above has all the required properties stated in Section 2.

Labeling the Infinite Cardinals.

According to the definition above, cardinals are initial ordinals. However, when we think of an initial ordinal as a cardinal, we only care about the cardinality, not the well-ordering. For those purposes any definition of cardinals satisfying the required properties would be good. Using initial ordinals as cardinals is only one possible choice. When we want to to indicate that it is irrelevant that the cardinals are initial ordinals, we use the aleph notation and denote the cardinal corresponding to the initial ordinal ω_α by $\aleph_\alpha$.

4 The Real Numbers in Set Theory.

The Integers and Rationals.

Integers.

We define integers as equivalence classes of the equivalence relation $\sim$ on $\omega \times \omega$ given by $(i, j) \sim (k, m)$ iff $i - j = k - m$. We will identify the class containing $(i, 0)$ for $i \in \omega$ with i and denote the equivalence class containing $(0, i)$ with $i \in \omega \setminus \{0\}$ by $-i$. The set of all integers will be denoted by $\mathbb{Z}$.

Rational Numbers.

The rational numbers are the equivalence classes of the equivalence relation $\sim$ on the set $\mathbb{Z} \times (\omega \setminus \{0\})$ given by $(i, j) \sim (k, m)$ iff $im = jk$. The class containing (i, j) is denoted by i/j or by $\frac{i}{j}$. The set of rational numbers will be denoted $\mathbb{Q}$.

Theorem 4.1. *The set of rational numbers is countable.*

Ordering of Rational Numbers.

We define $\frac{i}{j} < \frac{k}{m}$ iff $im < kj$.

Dense Linear Orderings.

A linear ordering on a set P is dense iff for every $a, b \in P$ with $a < b$ there exists $c \in P$ with $a < c < b$.

Unbounded Linear Orderings.

A linear ordering on a set P is unbounded iff for every $a \in P$ there exist $b, c \in P$ with $b < a < c$.

Theorem 4.2. *The ordering of rational numbers is dense and unbounded.*

Theorem 4.3 (Cantor)**.** *Any two countable unbounded dense linearly ordered sets are order isomorphic.*

Proof. Let A and B be countable unbounded dense linearly ordered sets. Let $\langle a_i : i \in \omega \rangle$ be a bijection $\omega \to A$ and $\langle b_i : i \in \omega \rangle$ be a bijection $\omega \to B$. For each $i \in \omega$, we define an order preserving function f_i whose domain is a finite subset of A and range is a finite subset of B. (f_i is order preserving means that $a < a' \Rightarrow f_i(a) < f_i(a')$ for each $a, a' \in \operatorname{dom}(f_i)$.) Let $f_0 = (a_0, b_0)$. Suppose f_i is defined. If i is even, let j be the smallest element in ω such that $a_j \notin \operatorname{dom}(f_i)$. Since the ordering of B is dense and unbounded there exists $b \in B$ such that $f_{i+1} = f_i \cup \{(a_j, b)\}$ is order preserving. If i is odd, let j be the smallest element in ω such that $b_j \notin \operatorname{ran}(f_i)$. Since the ordering of A is dense and unbounded, there exists $a \in A$ such that $f_{i+1} = f_i \cup \{(a, b_j)\}$ is order preserving. Let $f = \bigcup_{i\in\omega} f_i$. Then f is an order preserving function with $\operatorname{dom}(f) \subseteq A$ and $\operatorname{ran}(f) \subseteq B$. Since $\operatorname{dom}(f)$ is infinite, for any $i \in \omega$, there exists $a_j \in \operatorname{dom}(f)$ with $j \geq i$ implying that $a_i \in \operatorname{dom}(f)$. Thus $\operatorname{dom}(f) = A$. Similarly, $\operatorname{ran}(f) = B$. □

Defining the Real Numbers.

Complete Linear Orderings.

Definition. A linear ordering is complete iff every nonempty subset with an upper bound has the least upper bound.

Homework 7 (due 9/18).

Prove that if a linear ordering is complete, then every nonempty subset with a lower bound has the greatest lower bound.

Dense Subsets of Linear Orderings.

Let P be a linearly ordered set and A be a subset of P. We say that A is dense in P iff for every $a, b \in P$ there exists $c \in A$ with $a < c < b$.

Theorem 4.4 (Cantor). *Any two complete unbounded linear orderings with a dense subset that is order isomorphic to $\mathbb{Q}$ are order isomorphic.*

Proof. Let P and Q be complete unbounded linearly ordered sets with dense subsets A and B, respectively, that are order isomorphic to $\mathbb{Q}$. Let $h : A \to B$ be an order isomorphism. We are going to extend h to an order isomorphism $g : P \to Q$. If $p \in P$, then let $A_p = \{a \in A : a \leq p\}$ and $B_p = \{h(a) : a \in A_p\}$. Since P is unbounded there exists $p' \in P$ with $p < p'$. Since A is dense in P, there exists $a' \in A$ with $p < a' < p'$. Then $h(a')$ is an upper bound on B_p so B_p is a nonempty bounded set. Let $g(p)$ be the least upper bound on B_p. It remains to show that g is order preserving and surjective. □

The Real Numbers.

Theorem 4.5. *Let P be a countable, dense, unbounded linearly ordered set. Then the ordering of P can be extended to an ordering on some set C containing P as a dense subset so that the resulting ordering is unbounded and complete.*

Proof. A cut in P is a partition (A_1, A_2) of P (with both A_1 and A_2 nonempty) such that $a < a'$ for any $a \in A_1$ and $a' \in A_2$. Let C be the set of all cuts of P such that A_2 does not have the smallest element and identify each element $p \in P$ with the cut $(A_{p,1}, A_{p,2})$ where $A_{p,1} = \{q \in P : q \leq p\}$ and $A_{p,2} = P \setminus A_{p,1}$. Order C so that $(A_1, A_2) \leq (A_1', A_2')$ iff $A_1 \subseteq A_1'$. It remains to show that the ordering of C is linear, unbounded and complete, and that P is a dense subset of C. □

Definition. The set $\mathbb{R}$ of real numbers is the unique (up to an order isomorphism) complete, unbounded, linearly ordered set containing $\mathbb{Q}$ as a dense subset.

Homework 8 (due 9/27).

Finish the proof of Theorem 4.5.

The Cardinality of the Set of Real Numbers.

Continuum.

Definition. The cardinality $2^{\aleph_0}$ of the set $P(\omega)$ is called continuum. It is also the cardinality of the set ${}^{\omega}2$ of all functions $\omega \to \{0,1\}$. It is denoted by $\mathfrak{c}$.

Lemma 4.6. *We have $\kappa + \omega = \kappa$ for any infinite cardinal κ.*

Theorem 4.7. *The set of real numbers has cardinality $\mathfrak{c}$.*

Proof. Let A be the set of all functions $\varphi : \omega \to \{0,1\}$ such that $\varphi^{-1}(\{1\})$ is infinite but not equal to ω. Then ${}^{\omega}2 \setminus A$ is countable so $|A| = \mathfrak{c}$. Let A be ordered so that $\varphi < \psi$ iff $\varphi(i) < \psi(i)$ for the smallest i at which φ and ψ take different values. Let B be the subset of A consisting of those $\varphi : \omega \to \{0,1\}$ for which $\varphi^{-1}(\{0\})$ is finite. Then A is an unbounded complete linearly ordered set with a countable dense subset B. Thus A is order isomorphic to $\mathbb{R}$ implying that $|\mathbb{R}| = \mathfrak{c}$. $\square$

Homework 9 (due 9/30).

Complete the proof of Theorem 4.7, that is, prove that the ordering on A as defined in the proof is complete and that B is a dense subset.

Continuum Hypothesis.

Continuum hypothesis (denoted CH) is the statement saying that every uncountable set has cardinality $\geq \mathfrak{c}$. Neither CH nor its negation can be proved using the ZFC axioms.

Another Attempt to Define the Set of Real Numbers.

Open Intervals.

Let P be a linearly ordered set. An open interval in P is any subset of P of any of the following forms:

1. $\varnothing$;
2. $(a,b) = \{c \in P : a < c < b\}$ with $a, b \in P$;
3. $(-\infty, a) = \{c \in P : c < a\}$ with $a \in P$;

4. $(a, \infty) = \{c \in P : a < c\}$ with $a \in P$;

5. P.

Countable Chain Condition.

Definition. Let P be a dense linearly ordered set. We say that P satisfies the countable chain condition (ccc) iff every disjoint set of open intervals in P is at most countable.

Remark. Note that $\mathbb{R}$ satisfies the ccc since any nonempty open interval contains a rational number and the set $\mathbb{Q}$ is countable.

Suslin's Problem.

Problem. Let P be a complete dense unbounded linearly ordered set that satisfies the ccc. Is P order isomorphic to $\mathbb{R}$?

Remark. Suslin's problem is undecidable in ZFC. That is neither it or its negation can be proved using the ZFC axioms.

The Standard Topology on the Set of Real Numbers.

Topology.

Definition. Let A be a set. A topology on A is a family $\mathcal{O}$ of subsets of A such that $A, \varnothing \in \mathcal{O}$ and $\mathcal{O}$ is closed under finite intersections and arbitrary unions.

Homework 10 (due 10/2).

Let P be a linearly ordered set and $\mathcal{O}$ be the family of all unions of open intervals in P. Prove that $\mathcal{O}$ is a topology on P.

Open and Closed Subsets.

Definition. Given a topology $\mathcal{O}$ on a set A, we say that the elements of $\mathcal{O}$ are the open subsets of A and that the complements of the sets in $\mathcal{O}$ are the closed subsets of A.

Open Sets of Real Numbers.

Definition. A set of real numbers is open iff it is the union of a nonempty family of open intervals.

Remark. Note that if follows from Homework 10 that the above definition gives a topology on $\mathbb{R}$.

Cardinality Questions Concerning the Standard Topology on $\mathbb{R}$.

Remark. Note that every nonempty open set of real numbers has cardinality $\mathfrak{c}$. We will show later that any uncountable closed subset of $\mathbb{R}$ has cardinality $\mathfrak{c}$. This result can interpreted as saying that the CH holds for closed sets of real numbers.

Lemma 4.8. *The cardinality of the set of all open subsets of $\mathbb{R}$ is $\mathfrak{c}$. In particular, there exists a subset of $\mathbb{R}$ that is not open.*

Proof. Let $\mathscr{A}$ be the family of all nonempty open intervals in $\mathbb{R}$ with rational endpoints. Then every open set in $\mathbb{R}$ is a union of some subfamily of $\mathscr{A}$. Moreover, the function from the standard topology $\mathscr{O}$ on $\mathbb{R}$ to the power set $P(\mathscr{A})$ assigning to $B \in \mathscr{O}$ the set of all $A \in \mathscr{A}$ that are subsets of B is an injection. $\square$

Isolated Points.

Let $a \in A$ and $A \subseteq \mathbb{R}$. We say that a is an isolated point of A iff there exists an open set B in $\mathbb{R}$ such that $B \cap A = \{a\}$.

Perfect Sets.

A subset of $\mathbb{R}$ is perfect iff it is nonempty, closed and has no isolated points.

Theorem 4.9. *Any perfect set of real numbers has cardinality $\mathfrak{c}$.*

Proof. Let $P \subseteq \mathbb{R}$ be perfect. Clearly, $|P| \leq \mathfrak{c}$. It remains to show that $|P| \geq \mathfrak{c}$. We will define an injection ${}^{\omega}2 \to P$.

Without loss of generality we can assume that P is bounded (contained in some closed interval $[a, b]$). For each finite 0-1 sequence s we define a perfect subset P_s of P such that both $P_{s^\frown 0}$ and $P_{s^\frown 1}$ are disjoint subsets of P_s. ($s^\frown i$ denotes the sequence obtained from s by adjoining i at the end.) If s is the empty sequence then let $P_s = P$. Assume that s is any 0-1 sequence and that P_s has been defined. Since P_s is nonempty and has no isolated points it contains two different points a and b. Let $A = \left[a_1, a_2\right]$ and $B = \left[b_1, b_2\right]$ be disjoint closed intervals such that $a_1 < a < a_2$ and $b_1 < b < b_2$. Let $P_{s^\frown 0}$ be the intersection $P_s \cap A$ and $P_{s^\frown 1} = P_s \cap B$.

Given $\varphi \in {}^{\omega}2$, note that the intersection $P_\varphi = \bigcap_{n \in \omega} P_{\varphi \restriction n}$ is nonempty and that assigning to φ any point in P_φ we get an injection. Thus the proof is complete. $\square$

Uncountable Closed Sets of Real Numbers.

Theorem 4.10 (Cantor-Bendixson). *Any uncountable closed set A of real numbers contains a perfect subset A' such that $A \setminus A'$ is at most countable.*

Proof. Let A be any set of real numbers. By transfinite recursion define nonempty subsets A_α of A for $\alpha < \beta$ so that A_α is the set of all isolated points of $A \setminus \bigcup_{\gamma < \alpha} A_\gamma$ and $A' = A \setminus \bigcup_{\alpha < \beta} A_\alpha$ has no isolated points. If A is closed, then A' is also closed so it is either perfect or empty.

We claim that $\bigcup_{\alpha<\beta} A_\alpha$ is at most countable. The claim implies that if A is closed and uncountable, then A' is perfect. It remains to prove the claim.

For each $a \in A_\alpha$ take an open interval around a containing no points from $A \setminus \bigcup_{\gamma<\alpha} A_\gamma$ except a. From that interval select one rational number $< a$ and one $> a$. That defines a function from $\bigcup_{\alpha<\beta} A_\alpha$ into the countable set $\mathbb{Q}\times\mathbb{Q}$. It remains to show that such a function is injective. $\square$

Remark. The ordinal β in the proof above is called the Cantor-Bendixson rank of the set A. The proof above implies that the Cantor-Bendixson rank of any subset of $\mathbb{R}$ is an ordinal $\beta < \omega_1$. It can be proved that for any ordinal $\beta < \omega_1$ there exists a subset $A \subseteq \mathbb{Q}$ with Cantor-Bendixson rank equal to β.

Corollary 4.11. *Any uncountable closed set of real numbers has cardinality $\mathfrak{c}$.*

Homework 11 (due 10/4).

Complete the proof of Theorem 4.10 by showing that the function $\bigcup_{\alpha<\beta} A_\alpha \to \mathbb{Q}\times\mathbb{Q}$ defined there is injective.

5 More on Cardinal Arithmetic.

Products of Infinite Cardinals.

The Canonical Ordering of $\alpha \times \alpha$.

Definition. Let α be an ordinal. If $(\beta,\gamma),(\xi,\zeta) \in \alpha\times\alpha$, then we define $(\beta,\gamma) < (\xi,\zeta)$ iff one of the following conditions holds:

1. $\max(\beta,\gamma) < \max(\xi,\zeta)$;
2. $\max(\beta,\gamma) = \max(\xi,\zeta)$ and $\beta < \xi$;
3. $\max(\beta,\gamma) = \max(\xi,\zeta)$, $\beta = \xi$ and $\gamma < \zeta$.

Lemma 5.1. *Let α be an ordinal. The canonical ordering of α is a well-ordering.*

Proof. Clearly the ordering is a linear ordering. If A is a nonempty subset of $\alpha\times\alpha$, then $A' = \{\max(\beta,\gamma) : (\beta,\gamma) \in A\}$ is a nonempty subset of α so it has the smallest element δ. If

$$A'' = \{\beta \in \alpha : (\exists \gamma \in \alpha)(\beta,\gamma) \in A \land \max(\beta,\gamma) = \delta\},$$

then A'' is a nonempty subset of α so it has the smallest element β'. If

$$A''' = \{\gamma \in \alpha : (\beta',\gamma) \in A \land \max(\beta',\gamma) = \delta\},$$

then A''' is a nonempty subset of α so it has the smallest element γ'. Then (β',γ') is the smallest element in A. $\square$

Squares of Infinite Cardinals.

Theorem 5.2. *If κ is an infinite cardinal, then $\kappa \cdot \kappa = \kappa$.*

Proof. Let α be the unique ordinal so that there exists an order isomorphism $\varphi : \kappa \times \kappa \to \alpha$ where $\kappa \times \kappa$ has the canonical well-ordering. We will show that $\alpha = \kappa$. We use transfinite induction. Suppose, by way of contradiction, that κ is the smallest infinite cardinal for which the equation fails. Then $\alpha > \kappa$.

Let $(\beta, \gamma) \in \kappa \times \kappa$ be such that $\varphi(\beta, \gamma) = \kappa$. Then there exists $\delta < \kappa$ such that $\beta, \gamma < \delta$ so $\varphi(\delta, \delta) > \varphi(\beta, \gamma) = \kappa$ implying that $|\delta \times \delta| \geq \kappa$. But if $\lambda = |\delta|$, then $\lambda < \kappa$ so $|\delta \times \delta| = \lambda \cdot \lambda = \lambda < \kappa$ which gives a contradiction. □

Products and Sums of Infinite Cardinals.

Corollary 5.3. *We have:*

1. *If κ, λ are nonzero cardinals and at least one of them is infinite, then $\kappa \cdot \lambda = \max(\kappa, \lambda)$.*

2. *If κ, λ are cardinals and at least one of them is infinite, then $\kappa + \lambda = \max(\kappa, \lambda)$.*

Cofinality.

Continuous Functions.

Definition. Suppose we have topologies on sets A and B and $f : A \to B$. We say that f is continuous iff $f^{-1}(C)$ is open for every open subset C of B. If $a \in A$, then f is continuous at a iff for every open subset B' of B that contains $f(a)$ there exists an open subset A' of A containing a such that $f"A' \subseteq B'$.

Standard Topology on an Ordinal.

Definition. If α is an ordinal then the standard topology on α is the topology induced by the ordering of α.

Limits of Transfinite Sequences.

Definition. Let A be a set with a fixed topology and $(a_\beta : \beta < \alpha)$ be a transfinite sequence in A with α being a limit ordinal. If $b \in A$, then we say that $b = \lim_{\beta \to \alpha} a_\beta$ iff the function $f : \alpha + 1 \to A$ such that $f(\beta) = a_\beta$ for $\beta < \alpha$ and $f(\alpha) = b$ is continuous at α. Explicitly, $b = \lim_{\beta \to \alpha} a_\beta$ if for every open neighborhood B of b there exists $\gamma < \alpha$ with $a_\delta \in B$ for every δ such that $\gamma < \delta < \alpha$.

Cofinal Transfinite Sequences.

Definition. Let α, γ be nonzero limit ordinals and $\{\xi_\beta : \beta < \alpha\}$ be a transfinite sequence of ordinals smaller than γ. We say that the sequence is cofinal in γ iff $\gamma = \lim_{\beta \to \alpha} \xi_\beta$. Explicitly, the sequence is cofinal in γ iff for every ordinal $\delta < \gamma$ there exists an ordinal $\zeta < \alpha$ such that $\delta < \xi_\beta$ for every β with $\zeta < \beta < \alpha$.

Homework 12 (due 10/16).

Let α and γ be nonzero limit ordinals. Prove that a sequence $(\xi_\beta : \beta < \alpha)$ in γ is cofinal in γ iff the extension $(\xi_\beta : \beta < \alpha + 1)$ of it to a sequence of length $\alpha + 1$ in $\gamma + 1$ obtained by assigning γ to α (setting $\xi_\alpha = \gamma$) is continuous at α.

Cofinality of an Ordinal.

Definition. Let γ be a nonzero limit ordinal. The cofinality of γ is the smallest nonzero limit ordinal α for which there exists a cofinal transfinite sequence in γ of length α. The cofinality of γ is denoted by $\operatorname{cf}\gamma$.

Remark. Note that $\operatorname{cf}\gamma$ exists for any nonzero limit ordinal γ and that $\operatorname{cf}\gamma \leq \gamma$.

The Induced Non-decreasing Sequence.

Definition. Let $(\xi_\beta : \beta < \alpha)$ be a sequence of ordinals. The induced sequence $(\xi'_\beta : \beta < \alpha)$ is defined so that ξ'_β is the least upper bound on $\{\xi_\delta : \delta \leq \beta\}$. Note that if the original sequence is a sequence in γ, then the induced sequence is a sequence in $\gamma + 1$ and it is non-decreasing.

Remark. If γ is a limit ordinal and $(\xi_\beta : \beta < \operatorname{cf}\gamma)$ is cofinal in γ, then the induced non-decreasing sequence is also cofinal in γ.

The Cofinality is a Cardinal.

Lemma 5.4. *For every limit ordinal γ, the cofinality of γ is an infinite cardinal.*

Proof. Let $\alpha = \operatorname{cf}\gamma$ and let $(\xi_\beta : \beta < \alpha)$ be cofinal in γ. Suppose α is not a cardinal. Then there exists a bijection $\varphi : \delta \to \alpha$ for some ordinal $\delta < \alpha$. Then the induced non-decreasing sequence of the sequence $(\xi_{\varphi(\zeta)} : \zeta < \delta)$ or its initial segment is cofinal in γ. That contradict the assumption that $\alpha = \operatorname{cf}\gamma$. Thus α is a cardinal. Since α is a nonzero limit ordinal, α is an infinite cardinal. $\square$

The Successor Cardinals and Limit Cardinals.

Definition. If κ is a cardinal, then let κ^+ be the smallest cardinal that is larger than κ. Any cardinal of the form κ^+ is a successor cardinal. Any nonzero cardinal that is not a successor cardinal is called a limit cardinal.

Remark. Note that if $\kappa = \aleph_\alpha$, then $\kappa^+ = \aleph_{\alpha+1}$.

Regular and Singular Cardinals.

Definition. A cardinal κ is regular iff it is infinite and $\operatorname{cf}\kappa = \kappa$. Any other infinite cardinal is called a singular cardinal.

Example. The cardinal $\aleph_0$ is regular. The cardinal $\aleph_\omega$ has cofinality ω so it is singular.

Remark. For every cardinal κ there exists a singular cardinal larger than κ. If $\kappa = \aleph_\alpha$, then $\aleph_{\alpha+\omega}$ has cofinality ω so it is singular.

Lemma 5.5. *For any infinite cardinal κ, the cardinal κ^+ is regular.*

Proof. Suppose, by way of contradiction, that $\operatorname{cf}\kappa^+ = \lambda \leq \kappa$ and let $\left(\xi_\beta : \beta < \lambda\right)$ be cofinal in κ^+. Then $\kappa^+ = \bigcup_{\beta<\lambda} \xi_\beta$. For each $\beta < \lambda$, we have $\xi_\beta < \kappa^+$ implying that $\left|\xi_\beta\right| \leq \kappa$. Thus

$$\kappa^+ = \left|\bigcup_{\beta<\lambda} \xi_\beta\right| \leq \lambda \cdot \kappa \leq \kappa \cdot \kappa = \kappa$$

which is a contradiction. □

Weakly Inaccessible Cardinals.

Definition. A cardinal is weakly inaccessible iff it is an uncountable limit regular cardinal.

Remark. It is consistent with ZFC that weakly inaccessible cardinals do not exist.

Cofinalities are Regular Cardinals.

Lemma 5.6. *Let γ be a limit ordinal and $\kappa = \operatorname{cf}\gamma$. Then κ is a regular cardinal.*

Proof. Let $\lambda = \operatorname{cf}\kappa$. If $\left(\nu_\beta : \beta < \kappa\right)$ is cofinal in γ and $\left(\xi_\zeta : \zeta < \lambda\right)$ is cofinal in κ, then $\left(\nu_{\xi_\zeta} : \zeta < \lambda\right)$ is cofinal in γ so $\lambda = \kappa$. □

The Gimel Function.

Definition. The gimel function is defined for all infinite cardinals and assigns $\kappa^{\operatorname{cf}\kappa}$ to the cardinal κ. Its value at κ is also denoted by $\gimel_\kappa$.

Remark. If κ is regular then $\gimel_\kappa = \kappa^\kappa = 2^\kappa$.

Theorem 5.7. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.*

Proof. Let $\mathscr{A} = \left\{f_\alpha : \alpha < \kappa\right\} \subseteq {}^{\operatorname{cf}\kappa}\kappa$. It suffices to show that there exists $g \in {}^{\operatorname{cf}\kappa}\kappa \smallsetminus \mathscr{A}$. Let $\left(\xi_\beta : \beta < \operatorname{cf}\kappa\right)$ be cofinal in κ. If $\beta < \operatorname{cf}\kappa$, then

$$\left|\left\{f_\alpha\left(\beta\right) : \alpha < \xi_\beta\right\}\right| \leq \left|\xi_\beta\right| < \kappa$$

so the set $\kappa \smallsetminus \left\{f_\alpha\left(\beta\right) : \alpha < \xi_\beta\right\}$ is nonempty. Define the value of g at β so that

$$g\left(\beta\right) \in \kappa \smallsetminus \left\{f_\alpha\left(\beta\right) : \alpha < \xi_\beta\right\}.$$

If $\alpha < \kappa$, then there exists $\beta < \operatorname{cf}\kappa$ so that $\xi_\beta > \alpha$. Then $f_\alpha\left(\beta\right) \neq g\left(\beta\right)$ implying that $g \neq f_\alpha$. Thus the obtained function g is as required. □

The Cofinality of the Cardinal Powers.

Corollary 5.8. *$\mathrm{cf}\,\kappa^\lambda > \lambda$ for any cardinal $\kappa \geq 2$ and any infinite cardinal λ. Thus $\mathrm{cf}\,\mathfrak{c} > \aleph_0$. In particular $\mathfrak{c} \neq \aleph_\omega$.*

Proof. We have $\mathfrak{c} = 2^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_0} = \mathfrak{c}^{\aleph_0}$ thus if $\mathrm{cf}\,\mathfrak{c} = \aleph_0$ then $\mathfrak{c}^{\mathrm{cf}\,\mathfrak{c}} = \mathfrak{c}$ which is a contradiction. In general if $\mathrm{cf}\,\kappa^\lambda \leq \lambda$, then

$$\kappa^\lambda = \left(\kappa^\lambda\right)^\lambda \geq \left(\kappa^\lambda\right)^{\mathrm{cf}\,\kappa^\lambda}$$

which is a contradiction. □

Remark. If $\aleph_\alpha$ has uncountable cofinality, then it is consistent with ZFC that $\mathfrak{c} = \aleph_\alpha$.

Infinite Sums and Products of Cardinals.

Infinite Sums of Cardinals.

Definition. If κ_i is a cardinal for every $i \in I$, then $\sum_{i\in I} \kappa_i$ is the cardinality of the set $\bigcup_{i\in I} \left(\kappa_i \times \{i\}\right)$.

Remark. If A_i is a set of cardinality κ_i for each $i \in I$ and the sets A_i and A_j are disjoint for any $i \neq j$ from I, then

$$\left|\bigcup_{i\in I} A_i\right| = \sum_{i\in I} \kappa_i.$$

Lemma 5.9. *If λ is an infinite cardinal, κ_α is a positive cardinal for every $\alpha < \lambda$ and $\kappa = \sup\left\{\kappa_\alpha : \alpha < \lambda\right\}$, then*

$$\sum_{\alpha<\lambda} \kappa_\alpha = \lambda \cdot \kappa = \max(\lambda, \kappa).$$

Proof. Let $\mu = \sum_{\alpha<\lambda} \kappa_\alpha$. Clearly $\mu \leq \lambda \cdot \kappa$. Since $\kappa_\alpha \geq 1$ for each $\alpha < \lambda$ it follows that $\mu \geq \lambda$. Since $\mu \geq \kappa_\alpha$ for each $\alpha < \lambda$ it follows that $\mu \geq \kappa$. Thus $\mu \geq \max\{\kappa, \lambda\} = \kappa \cdot \lambda$. □

Infinite Products of Cardinals.

Definition. If κ_i is a cardinal for every $i \in I$, then $\prod_{i\in I} \kappa_i$ is the cardinality of the set of all functions $f : I \to \bigcup_{i\in I} \kappa_i$ such that $f(i) \in \kappa_i$ for each $i \in I$.

Remark. If A_i is a set of cardinality κ_i for each $i \in I$, then $\prod_{i\in I} \kappa_i$ is equal to the cardinality of the set of all functions $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for each $i \in I$.

Lemma 5.10. *If λ is an infinite cardinal, $\left(\kappa_\alpha : \alpha < \lambda\right)$ is a non-decreasing sequence of positive cardinals and $\kappa = \sup\left\{\kappa_\alpha : \alpha < \lambda\right\}$, then*

$$\prod_{\alpha<\lambda} \kappa_\alpha = \kappa^\lambda.$$

Proof. Clearly $\prod_{\alpha<\lambda} \kappa_\alpha \leq \kappa^\lambda$. Using a bijection $\lambda \times \lambda \to \lambda$ partition λ into λ subsets $\{A_\beta : \beta < \lambda\}$ with each A_β having cardinality λ. Then $\prod_{\alpha \in A_\beta} \kappa_\alpha$ is greater or equal to κ_γ for every $\gamma \in A_\beta$. Since the sequence $(\kappa_\alpha : \alpha < \lambda)$ is non-decreasing, we have $\sup\{\kappa_\gamma : \gamma \in A_\beta\} = \kappa$ for each $\beta < \lambda$ implying that $\prod_{\alpha \in A_\beta} \kappa_\alpha \geq \kappa$. Thus

$$\prod_{\alpha<\lambda} \kappa_\alpha = \prod_{\beta<\lambda} \prod_{\alpha \in A_\beta} \kappa_\alpha \geq \kappa^\lambda.$$

□

König's Theorem.

Theorem 5.11. *If $\kappa_i < \lambda_i$ for each $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.*

Proof. Let $i \in I$ be fixed. Let A_i be a subset of cardinality κ_i of the set of all functions $f : I \to \bigcup_{i \in I} \lambda_i$ such that $f(i) \in \lambda_i$ and let $B_i = \{f(i) : f \in A_i\}$. Then $|B_i| \leq \kappa_i$ so there exists $a_i \in \lambda_i \setminus B_i$. The function assigning a_i to i for each $i \in I$ does not belong to $\bigcup_{i \in I} A_i$. □

Remark. Let I be a cardinal κ. If $\kappa_\alpha = 1$ and $\lambda_\alpha = 2$ for each $\alpha \in \kappa$, then it follows from König's Theorem that $\kappa < 2^\kappa$.

Homework 13 (due 10/21).

Let κ be an infinite cardinal. Taking $(\kappa_\alpha : \alpha < \operatorname{cf}\kappa)$ to be cofinal in κ and suitable $(\lambda_\alpha : \alpha < \operatorname{cf}\kappa)$, show that König's Theorem implies that $\kappa^{\operatorname{cf}\kappa} > \kappa$.

Cardinal Powers.

The Continuum Function.

Definition. The continuum function is the function that assigns 2^κ to κ for each cardinal κ.

Remark. It can be proved that the only restrictions in ZFC on the values of the continuum function at regular cardinals are:

1. $2^\kappa \leq 2^\lambda$ for $\kappa < \lambda$.

2. $\operatorname{cf} 2^\kappa > \kappa$ for each κ.

The Beth Function.

Definition. The beth function is defined for all ordinals by transfinite recursion. The value at $\alpha \in \mathrm{Ord}$ is denoted by $\beth_\alpha$ with $\beth_0 = \aleph_0$, $\beth_{\beta+1} = 2^{\beth_\beta}$ and $\beth_\gamma = \sup\{\beth_\beta : \beta < \gamma\}$ for a nonzero limit ordinal γ.

The Generalized Continuum Hypothesis.

The Generalized Continuum Hypothesis (denoted GCH) states that $2^\kappa = \kappa^+$ for any infinite cardinal κ or equivalently that $\beth_\alpha = \aleph_\alpha$ for every ordinal α.

Remark. The GCH is independent from ZFC.

The Continuum Function at Limit Cardinals.

Definition. If κ is a limit cardinal, then let

$$2^{<\kappa} = \sup\left\{2^\lambda : \lambda \in \text{Card and } \lambda < \kappa\right\}.$$

Lemma 5.12. *If κ is a limit cardinal, then $2^\kappa = (2^{<\kappa})^{cf\,\kappa}$.*

Proof. We have $2^\kappa = (2^\kappa)^\kappa \geq (2^{<\kappa})^{\mathrm{cf}\,\kappa}$ so it remains to show that $(2^{<\kappa})^{\mathrm{cf}\,\kappa} \geq 2^\kappa$. Let $\left(\kappa_\beta : \beta < \mathrm{cf}\,\kappa\right)$ be cofinal in κ. Since κ is a limit cardinal, we can assume that all κ_β are cardinals. We have $\left|\kappa_\beta\right| < \kappa$ so there are at most $2^{<\kappa}$ functions $\kappa_\beta \to \{0,1\}$. To a function $\kappa \to \{0,1\}$ assign the sequence of its restrictions to each κ_β for $\beta < \mathrm{cf}\,\kappa$. This assignment is an injection and the cardinality of the set of such sequences is at most $(2^{<\kappa})^{\mathrm{cf}\,\kappa}$. Thus $(2^{<\kappa})^{\mathrm{cf}\,\kappa} \geq 2^\kappa$. □

Remark. If κ is a regular limit cardinal, then the statement of the lemma is just a special case of the simple observation that $\mu^\kappa = 2^\kappa$ for any cardinal μ with $2 \leq \mu \leq 2^\kappa$. The lemma becomes interesting when κ is singular.

Exponentials of Limit Cardinals.

Definition. If κ is a limit cardinal, then let

$$\sup{(<\kappa)}^\lambda = \sup\left\{\mu^\lambda : \mu \in \text{Card and } \mu < \kappa\right\}.$$

Lemma 5.13. *If κ is a limit cardinal then*

1. $\kappa^\lambda = \sup{(<\kappa)}^\lambda$ *when* $\lambda < cf\,\kappa$.
2. $\kappa^\lambda = \left(\sup{(<\kappa)}^\lambda\right)^{cf\,\kappa}$ *when* $\lambda \geq cf\,\kappa$.

Proof. Assume that $\lambda < \mathrm{cf}\,\kappa$. Clearly $\kappa^\lambda \geq \sup{(<\kappa)}^\lambda$. Since $\lambda < \mathrm{cf}\,\kappa$, any function $\lambda \to \kappa$ is a function $\lambda \to \alpha$ for some ordinal $\alpha < \kappa$. For any $\alpha < \kappa$, we have

$$\left|{}^\lambda\alpha\right| = |\alpha|^\lambda \leq \sup{(<\kappa)}^\lambda.$$

Thus

$$\kappa^\lambda \leq \left|\bigcup_{\alpha<\kappa} {}^\lambda\alpha\right| \leq \kappa \cdot \sup{(<\kappa)}^\lambda = \sup{(<\kappa)}^\lambda.$$

Assume that $\lambda \geq \operatorname{cf}\kappa$. Then

$$\left(\sup(<\kappa)^\lambda\right)^{\operatorname{cf}\kappa} \leq \left(\kappa^\lambda\right)^{\operatorname{cf}\kappa} = \kappa^\lambda.$$

Let $\left(\kappa_\alpha : \alpha < \operatorname{cf}\kappa\right)$ be cofinal in κ. Since κ is a limit cardinal, we can choose κ_α to be a cardinal for each $\alpha < \operatorname{cf}\kappa$. Assign to a function $\varphi : \lambda \to \kappa$ the sequence $\left(\varphi_\alpha : \alpha < \operatorname{cf}\kappa\right)$ defined by

$$\varphi_\alpha(\beta) = \begin{cases} \varphi(\beta) & \text{if } \varphi(\beta) \leq \kappa_\alpha \\ \kappa_\alpha & \text{otherwise.} \end{cases}$$

Such an assignment is an injection and for each $\alpha < \operatorname{cf}\kappa$ we have at most $\sup(<\kappa)^\lambda$ of such restrictions so $\kappa^\lambda \leq \left(\sup(<\kappa)^\lambda\right)^{\operatorname{cf}\kappa}$. □

The Continuum Function at Singular Cardinals.

Definition. Let κ be a limit cardinal. We say that the continuum function stabilizes below κ if $2^{<\kappa} = 2^\lambda$ for some $\lambda < \kappa$.

Theorem 5.14. *If κ is a singular cardinal and the continuum function stabilizes below κ, then $2^\kappa = 2^{<\kappa}$.*

Proof. Let $\lambda < \kappa$ be such that $2^{<\kappa} = 2^\lambda$ and $\mu = \max(\operatorname{cf}\kappa, \lambda)$. Then $\operatorname{cf}\kappa \leq \mu < \kappa$ since κ is singular and $2^\mu = 2^{<\kappa}$ since $\mu \geq \lambda$. Thus

$$2^\kappa = \left(2^{<\kappa}\right)^{\operatorname{cf}\kappa} = (2^\mu)^{\operatorname{cf}\kappa} = 2^{\mu\cdot\operatorname{cf}\kappa} = 2^\mu.$$

□

Lemma 5.15. *If $\left(\xi_\beta : \beta < \alpha\right)$ is a sequence that is cofinal in γ, then $\operatorname{cf}\gamma = \operatorname{cf}\alpha$.*

Proof. If $\left(\beta_\delta : \delta < \operatorname{cf}\alpha\right)$ is cofinal in α, then $\left(\xi_{\beta_\delta} : \delta < \operatorname{cf}\alpha\right)$ is cofinal in γ so $\operatorname{cf}\gamma \leq \operatorname{cf}\alpha$.

Assume first that $\left(\xi_\beta : \beta < \alpha\right)$ is non-decreasing. Then no proper initial part of it is cofinal in γ. If $\left(\zeta_\nu : \nu < \operatorname{cf}\gamma\right)$ is cofinal in γ then defining β_ν to be the smallest ordinal with $\xi_{\beta_\nu} > \zeta_\nu$ we get a sequence $\left(\beta_\nu : \nu < \operatorname{cf}\gamma\right)$ that is cofinal in α. Thus $\operatorname{cf}\alpha \leq \operatorname{cf}\gamma$.

In general, given any sequence $\left(\xi_\beta : \beta < \alpha\right)$ cofinal in γ, let $\left(\xi'_\beta : \beta < \alpha\right)$ be defined by

$$\xi'_\beta = \min\left\{\xi_\delta : \beta \leq \delta < \alpha\right\}.$$

Then $\left(\xi'_\beta : \beta < \alpha\right)$ is non-decreasing and cofinal in γ implying that $\operatorname{cf}\alpha = \operatorname{cf}\gamma$. □

Theorem 5.16. *If κ is a limit cardinal and the continuum function does not stabilize below κ, then $2^\kappa = \mu^{\operatorname{cf}\mu}$, where $\mu = 2^{<\kappa}$.*

Proof. The sequence $\left(2^\lambda : \lambda < \kappa\right)$ is non-decreasing and cofinal in μ implying that $\operatorname{cf}\mu = \operatorname{cf}\kappa$. We already know that $2^\kappa = \mu^{\operatorname{cf}\kappa}$. □

Remark. The theorem holds for any limit cardinal κ, but it is only interesting when κ is singular.

The Singular Cardinal Hypothesis.

The Singular Cardinal Hypothesis (denoted SCH) says that if κ is singular and $2^{\mathrm{cf}\,\kappa} < \kappa$ then $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$.

Remark. Note the $2^{\mathrm{cf}\,\kappa}$ can't be equal κ since if $2^{\mathrm{cf}\,\kappa} = \left(2^{\mathrm{cf}\,\kappa}\right)^{\mathrm{cf}\,\kappa}$ and $\kappa < \kappa^{\mathrm{cf}\,\kappa}$. If $2^{\mathrm{cf}\,\kappa} > \kappa$ then $\kappa^{\mathrm{cf}\,\kappa} = 2^{\mathrm{cf}\,\kappa}$. The GCH implies the SCH.

Proof. Assume the GCH. Since $\kappa^{\mathrm{cf}\,\kappa} > \kappa$, it follows that $\kappa^{\mathrm{cf}\,\kappa} \geq \kappa^+$. Since $\mathrm{cf}\,\kappa \leq \kappa$, it follows that $\kappa^{\mathrm{cf}\,\kappa} \leq \kappa^\kappa = 2^\kappa = \kappa^+$. Thus $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$. □

Proposition 5.17. *If the SCH holds then the continuum function is determined by its values on regular cardinals. If κ is a singular cardinal, then $2^\kappa = 2^{<\kappa}$ if the continuum function stabilizes below κ and $2^\kappa = (2^{<\kappa})^+$ otherwise.*

Proof. We have proved already that if there exists $\lambda < \kappa$ with $2^\lambda = 2^{<\kappa}$, then $2^\kappa = 2^{<\kappa}$. Otherwise, let $\mu = 2^{<\kappa}$. Since $\mathrm{cf}\,\mu = \mathrm{cf}\,\kappa < \kappa$, it follows that $2^{\mathrm{cf}\,\mu} = 2^{\mathrm{cf}\,\kappa} < \mu$ so $2^\kappa = \mu^{\mathrm{cf}\,\mu} = \mu^+$ by SCH. □

Proposition 5.18. *Assume that the SCH holds and that κ, λ are infinite cardinals with $\kappa > 2^\lambda$. If $\lambda < \mathrm{cf}\,\kappa$ then $\kappa^\lambda = \kappa$ and otherwise $\kappa^\lambda = \kappa^+$.*

Proof. Let λ be fixed. We use transfinite induction on κ. If $\kappa = \mu^+$ then $\lambda < \mathrm{cf}\,\kappa$. The inductive hypothesis tells us that $\mu^\lambda \leq \kappa$. Thus

$$\kappa^\lambda = \left|\bigcup_{\alpha<\kappa} {}^\lambda\alpha\right| \leq \kappa \cdot \left|{}^\lambda\mu\right| = \kappa$$

and since clearly $\kappa^\lambda \geq \kappa$ we conclude that $\kappa^\lambda = \kappa$.

Now assume that κ is a limit cardinal. The inductive hypothesis implies that $\sup{(<\kappa)}^\lambda = \kappa$. If $\lambda < \mathrm{cf}\,\kappa$ then

$$\kappa^\lambda = \sup{(<\kappa)}^\lambda = \kappa.$$

If $\lambda \geq \mathrm{cf}\,\kappa$ then $2^{\mathrm{cf}\,\kappa} \leq 2^\lambda < \kappa$ so

$$\kappa^\lambda = \left(\sup{(<\kappa)}^\lambda\right)^{\mathrm{cf}\,\kappa} = \kappa^{\mathrm{cf}\,\kappa} = \kappa^+.$$

□

6 Introduction to Descriptive Set Theory.

Baire Category Theorem.

Dense Subsets of a Topological Space.

Definition. Let X be a topological space (X is a set and we have a topology on X). A subset A of X is dense in X iff every nonempty open subset of X contains a point from A.

Remark. If $\mathscr{B}$ is a base of the topology on X (each open subset of X is the union of some subfamily of $\mathscr{B}$), then it suffices to consider only the open sets from $\mathscr{B}$.

Subspace Topology.

Definition. If X is a topological space and $Y \subseteq X$ then Y is a topological space with a subset A of Y being open iff it is the intersection of Y with an open subset of X.

Theorem 6.1. *The intersection of a countable family of dense open subsets of $\mathbb{R}$ is dense in $\mathbb{R}$. If C is a closed subset of $\mathbb{R}$ then the intersection of a countable family of dense open subsets of C is dense in C.*

Corollary 6.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}^+$ the limit $\lim_{n\to\infty} f(nx)$ equals 0. Then $\lim_{x\to\infty} f(x) = 0$.*

Proof. By way of contradiction, suppose that $\lim_{x\to\infty} f(x) \neq 0$. Then there exist a sequence $\left(a_n : n < \omega\right)$ diverging to ∞, a sequence $\left(r_n : n < \omega\right)$ of positive real numbers and $\varepsilon > 0$ such that the values of f on the open interval $A_n = \left(a_n - r_n, a_n + r_n\right)$ have absolute values at least ε for each n. For each $m \in \omega$ the set

$$B_m = \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \frac{a}{k} : a \in A_n \right\}$$

is dense and open in $\mathbb{R}^+$ so the set $B = \bigcap_{m\in\omega} B_m$ is dense hence nonempty. If x belongs to B then $\lim_{n\to\infty} f(nx) \neq 0$ which is a contradiction. $\square$

Homework 14 (due 11/11).

Prove that for each $m \in \omega$, the set B_m in the proof of Corollary 6.2 is dense in $\mathbb{R}$.

Corollary 6.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for every $x \in \mathbb{R}$ there exists a positive integer k_x with the property that the k-th derivative of f at x is equal zero for every $k \geq k_x$. Then f is a polynomial.*

Proof. Suppose, by way of contradiction, that f is not a polynomial. Let A be the subset of $\mathbb{R}$ so that $a \in A$ iff there exists an open interval around a where the k-th derivative of f is zero for some k. Then the set $P = \mathbb{R} \setminus A$ is perfect. For each $m \in \omega$, let B_m be the subset of P so that $b \in B_m$ iff the k-th derivative of f is nonzero at b for some $k \geq m$. Then B_m is open and dense in P for each $m \in \omega$, implying that $B = \bigcap_{m\in\omega} B_m$ is dense in P. In particular $B \neq \emptyset$, which is a contradiction. $\square$

Homework 15 (due 11/13).

Prove that the set P as defined in the proof of Corollary 6.3 is perfect.

Homework 16 (due 11/15).

Prove that the set B_m as defined in the proof of Corollary 6.3 is dense in P for every $m \in \omega$.

Homework 17 (due 11/18).

Prove that if C is a closed subset of $\mathbb{R}$ then the intersection of a countable family of dense open subsets of C is dense in C.

The Baire Space.

Definition. The Baire space, denoted $\mathcal{N}$ is the set ${}^{\omega}\omega$ of all sequences $\left(a_n : n < \omega\right)$ of the elements of ω with the product topology, where ω has the discrete topology. Explicitly, let $\mathcal{B}$ be the family of subsets of $\mathcal{N}$ such that $B \in \mathcal{B}$ iff there exists $m \in \omega$ and a sequence $\left(b_n : n < m\right)$ such that $\left(a_n : n < \omega\right) \in B$ iff $a_n = b_n$ for each $n \leq m$. The family $\mathcal{B}$ is a base of the topology on $\mathcal{N}$, that is, a subset of $\mathcal{N}$ is open iff it is the union of some subfamily of $\mathcal{B}$.

Theorem 6.4. *The Baire space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, that is, there exists a bijection $\mathcal{N} \to \mathbb{R} \setminus \mathbb{Q}$ which is continuous and whose inverse is continuous.*

Proof. Let $\varphi : \omega \to \mathbb{Q}$ be a bijection and let $\left(a_i : i < \omega\right)$ be such that $a_i = \varphi(i)$. To each $f \in \mathcal{N}$ we assign an element in $\mathbb{R} \setminus \mathbb{Q}$ as follows.

Let $i_0 = 0$ and $i_1 > i_0$ be the smallest integer with $a_{i_1} > a_{i_0}$ and let I_0 be the open interval $\left(a_{i_0}, a_{i_1}\right)$. Let $i_2 > i_1$ be the smallest integer with $a_{i_2} < a_{i_0}$ and let $I_1 = \left(a_{i_2}, a_{i_0}\right)$. Let $i_3 > i_2$ be the smallest integer with $a_{i_3} > a_{i_1}$ and $I_2 = \left(a_{i_1}, a_{i_3}\right)$. Continuing like that we get a sequence $\left(I_i : i < \omega\right)$ of disjoint open intervals in $\mathbb{R}$.

Let $I_i = (a, b)$ be one of these intervals. Let i_0 be the smallest integer with $a_{i_0} \in I_i$. Let $i_1 > i_0$ be the smallest integer with $a_{i_0} < a_{i_1} < b$ and let $I_{i\,0}$ be the open interval $\left(a_{i_0}, a_{i_1}\right)$. Let $i_2 > i_1$ be the smallest integer with $a < a_{i_2} < a_{i_0}$ and let $I_{i\,1} = \left(a_{i_2}, a_{i_0}\right)$. Let $i_3 > i_2$ be the smallest integer with $a_{i_1} < a_{i_3} < b$ and let $I_{i\,2} = \left(a_{i_1}, a_{i_3}\right)$. Continuing like that we get a sequence $\left(I_{ij} : j < \omega\right)$ of disjoint open intervals in I_i.

Repeating this again and again, for each finite sequence s of elements of ω we get an open interval I_s. If $f \in \mathcal{N}$, then the intersection $\bigcap_{n\in\omega} I_{f\restriction n}$ consists of a single element of $\mathbb{R}\setminus\mathbb{Q}$ that we assign to f. It remains to show that this assignment is a homeomorphism. $\square$

Trees.

Definition. Let $\Omega = {}^{<\omega}\omega$ be the set of all finite sequences of nonnegative integers. A tree T is a subset of Ω that is closed under restrictions of the domain to initial segments, that is T is such that if $s \in T \cap {}^{n}\omega$ then $s\restriction m \in T$ for every $m \leq n$.

A tree T is pruned iff for every $s \in T$ and every $n \geq \operatorname{dom}(s)$ there exists $t \in T \cap {}^{n}\omega$ such that $s = t\restriction\operatorname{dom}(s)$.

Remark. Let $A \subseteq \mathcal{N}$ be a subset of the Baire space $\mathcal{N}$. Then the set

$$T_A = \left\{\varphi\restriction n : \varphi \in A,\ n \in \omega\right\}$$

is a pruned tree.

Definition. If T is a tree then let

$$[T] = \{\varphi \in \mathcal{N} : \varphi \restriction n \in T \text{ for all } n \in \omega\}$$

be the set of all infinite paths through T.

Remark. For any tree T the set $[T]$ is a closed subset of $\mathcal{N}$. If T is pruned, then $T_{[T]} = T$.

Homework 18 (due 11/20).

Let $A \subseteq \mathcal{N}$. Prove that $[T_A]$ is the closure of A in $\mathcal{N}$. In particular, A is closed if and only if $A = [T_A]$.

Definition. A tree T is perfect iff it is nonempty and for every $s \in T$ there exist $n < m < \omega$ and $t_1, t_2 \in T \cap {}^m\omega$ such that $t_1 \neq t_2$ and

$$s = t_1 \restriction n = t_2 \restriction n.$$

In particular, every perfect tree is pruned.

Homework 19 (due 11/22).

Prove that a pruned tree T is perfect iff $[T]$ is a perfect subset of $\mathcal{N}$.

Polish Spaces.

Metric on a Set.

Let X be a set. A metric on X is a function $\rho : X \times X \to \mathbb{R}$ such that

1. $\rho(a,b) = \rho(b,a)$ for every $a, b \in X$,
2. $\rho(a,b) \geq 0$ for every $a, b \in X$,
3. $\rho(a,b) = 0$ iff $a = b$,
4. $\rho(a,b) + \rho(b,c) \geq \rho(a,c)$.

Open Balls.

Let ρ be a metric on a set X. If $a \in X$ and $r > 0$ is a real number, then

$$B(a,r) = \{b \in X : \rho(a,b) < r\}$$

is the open ball with center a and radius r.

The Topology Induced by a Metric.

Let ρ be a metric on a set X. The topology induced by ρ consists of all unions of families of open balls in X.

Remark. If τ is the topology on X induced by a metric ρ on X then we also say that τ is compatible with ρ.

Metrizable Spaces.

A topological space is metrizable iff there exists a metric compatible with the topology.

Cauchy Sequences.

Let ρ be a metric on a set X. A sequence $\left(a_i : i < \omega\right)$ of elements of X is Cauchy iff for every $\varepsilon > 0$ there exists $n < \omega$ such that $\rho\left(a_i, a_j\right) < \varepsilon$ for every $i, j \geq n$.

Convergent Sequences.

Let ρ be a metric on a set X. A sequence $\left(a_i : i < \omega\right)$ of elements of X converges to $b \in X$ iff for every $\varepsilon > 0$ there exists $n < \omega$ such that $\rho\left(a_i, b\right) < \varepsilon$ for every $i \geq n$. A sequence is convergent if there exists $b \in X$ so that it converges to b.

Remark. Every convergent sequence is Cauchy.

Complete Metric Space.

A metric space is complete iff every Cauchy sequence is convergent.

Completely Metrizable Spaces.

A topological space is completely metrizable iff there exists a metric compatible with the topology in which the space is complete.

Remark. The open interval $(0, 1)$ is not complete under the standard metric, but it is completely metrizable.

Separable Spaces.

A topological space is separable iff there exists a dense countable subset.

Example. The real numbers $\mathbb{R}$ is a separable topological space. The Baire space $\mathscr{N}$ is also a separable topological space.

Polish Spaces.

A Polish space is a topological space that is separable and completely metrizable.

Example. The real numbers $\mathbb{R}$ is a Polish space.

Remark. The Baire space $\mathcal{N}$ is a Polish space. To see that $\mathcal{N}$ is completely metrizable, let ρ be the metric on $\mathcal{N}$ defined as follows. If $f, g \in \mathcal{N}$ then let $\rho\big(f, g\big) = 1/(n+1)$ where n is the smallest element in ω so that $f(n) \neq g(n)$ is such n exists and $\rho\big(f, g\big) = 0$ otherwise.

Homework 20 (due 12/4).

Prove that the metric ρ defined above is complete ($\mathcal{N}$ is a complete metric space under it).

Theorem 6.5. *Let X be topological space. Then X is separable and metrizable iff it is homeomorphic to a subspace of ${}^{\omega}\mathbb{R}$ (with the product topology). Moreover, X is Polish iff it is homeomorphic to a closed subspace of ${}^{\omega}\mathbb{R}$.*

Remark. The proof of 6.5 will be omitted this semester. See the book "Classical Descriptive Set Theory" by Kechris for a proof.

Theorem 6.6. *Every Polish space is a continuous image of the Baire space.*

Proof. Let ρ be a metric on X so that X becomes a complete metric space under it and let A be a countable dense subset of X. Let $\big(a_i : i \in \omega\big)$ be an enumeration of A and let $\big(B_j : j \in \omega\big)$ be an enumeration of the following countable set of open balls

$$\mathscr{B} = \left\{B\big(a_i, 1/(n+1)\big) : i \in \omega, n \in \omega\right\}.$$

We are going to define a continuous function from $\mathcal{N}$ onto X.

If $f \in \mathcal{N}$, then assign to f the limit of the Cauchy sequence $\big(c_{j_t} : t \in \omega\big)$, where c_{j_t} is the center of the ball B_{j_t} and the sequence $\big(j_t : t \in \omega\big)$ is defined as follows. Let $j_0 = f(0)$. Suppose that $j_t \in \omega$ has been defined. Consider the increasing sequence $j'_0 < j'_1 < \dots$ consisting of all the elements of ω for which $B_{j'_q} \subsetneq B_{j_t}$. Let $j_{t+1} = j'_{f(t+1)}$. $\square$

Borel Subsets.

Definition. Let X be a set. A σ-algebra on X is a family of subsets of X that contains the empty set $\varnothing$ and is closed under countable unions and taking complements.

Remark. If $\left\{\mathscr{A}_i : i \in I\right\}$ is a family of σ-algebras on X for each $i \in I$, then $\bigcap_{i\in I} \mathscr{A}_i$ is also a σ-algebra on X. It follows that for any family $\mathscr{A}$ of subsets of X there exists the smallest σ-algebra on X containing $\mathscr{A}$.

Definition. Let X be a topological space. The Borel subsets of X are the elements of the smallest σ-algebra on X containing the open subsets of X. In particular, the countable unions of closed subsets of X are called F_σ-subsets of X and countable intersections of open subsets of X are called the G_δ-subsets of X.

Theorem 6.7. *Let X be a completely metrizable topological space. Any G_δ-subset of X is completely metrizable in the subspace topology.*

Corollary 6.8. *The Baire space is completely metrizable. Since it is also separable, it is Polish.*

Proof. Each singleton in $\mathbb{R}$ is closed so $\mathbb{Q}$ is an F_σ-subset of $\mathbb{R}$. Thus $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ-subset of $\mathbb{R}$. Thus $\mathbb{R} \setminus \mathbb{Q}$ is completely metrizable. Since the Baire space $\mathcal{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ it is also completely metrizable. $\square$

The Borel Hierarchy.

Definition. Let X be a topological space. For every ordinal $\alpha \geq 1$ we define the families $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of subsets of X by transfinite induction as follows:

1. $\mathbf{\Sigma}^0_1$ is the family of open subsets of X and $\mathbf{\Pi}^0_1$ is the family of closed subsets of X.
2. $A \in \mathbf{\Sigma}^0_\alpha$ iff A is a countable union of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}_\beta$ and $A \in \mathbf{\Pi}^0_\alpha$ iff A is a countable intersection of sets in $\bigcup_{\beta<\alpha} \mathbf{\Sigma}_\beta$.

Remark. $\mathbf{\Sigma}^0_2$ is the family of all F_σ-subsets of X and $\mathbf{\Pi}^0_2$ is the family of all G_δ-subsets of X.

Theorem 6.9. *Let X be a Polish space. Then*

$$\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta \cap \mathbf{\Pi}^0_\beta$$

for every $\alpha < \beta$. Moreover

$$\mathbf{\Sigma}^0_{\omega_1} = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha = \mathbf{\Pi}^0_{\omega_1}$$

and the above set is equal to the family of all Borel subsets of X. If X is uncountable, then $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Pi}^0_\alpha$ for every $\alpha < \omega_1$.

Analytic Sets.

Definition. A subset of a Polish space is analytic iff it is a continuous image of the Baire space.

Remark. Any Borel subset of a Polish space is analytic. There exists an analytic subset of the Baire space $\mathcal{N}$ that is not Borel.

Theorem 6.10. *Let X be a Polish space and $\Omega = {}^{<\omega}\omega$ be the set of all finite sequences of the element of ω. A subset A of X is analytic iff there exists a function f from Ω to the family of closed subsets of X such that*

$$A = \bigcup_{\varphi \in \mathcal{N}} \bigcap_{n \in \omega} f\left(\varphi \restriction n\right).$$

Theorem 6.11 (Suslin)**.** *If A is an analytic subset of a Polish space X and $X \setminus A$ is also analytic, then A is Borel.*