

**LONG INDUCED CYCLES IN THE HYPERCUBE  
AND COLOURINGS OF GRAPHS**

**Jerzy Marian Wojciechowski**

**Trinity College**

A dissertation submitted for the degree of  
Doctor of Philosophy  
of the University of Cambridge

April 1990

## **Declaration**

No part of this dissertation is derived from any other source, except where explicitly stated otherwise.

Chapter 2 presents joint work with Y. Kohayakawa. The rest of this dissertation is my own unaided work.

## **Acknowledgement**

I would like to thank my research supervisor, Dr Béla Bollobás, for his continual help, advice and encouragement that were far beyond the call of duty. It has been a privilege and a pleasure to work under his supervision.

## CONTENTS

INTRODUCTION	4
1. A LOWER BOUND FOR SNAKE-IN-THE-BOX CODES	10
2. ON SMALL GRAPHS WITH HIGHLY IMPERFECT POWERS	29
3. EQUITABLE LABELLINGS OF CYCLES	50
4. SPLITTING NECKLACES AND A GENERALIZATION OF THE BORSUK-ULAM ANTIPODAL THEOREM	69
5. REMARKS ON A GENERALIZATION OF RADON'S THEOREM	94
6. AN OBSERVATION ON INTERSECTION DIGRAPHS OF CONVEX SETS IN THE PLANE	98
REFERENCES	101

## INTRODUCTION

Cycles are very simple combinatorial structures, yet there are many interesting problems concerning them. The major part of this dissertation is concerned with problems about cycles.

One very natural question about cycles goes as follows: given a graph  $G$ , what is the length of the longest induced cycle in  $G$ ? In Chapter 1 we deal with this question when  $G$  is the  $d$ -dimensional hypercube (also called the cube). Since the vertices of the  $d$ -dimensional cube can be considered as  $d$ -tuples of binary digits, a long induced cycle in the  $d$ -dimensional cube can be applied as a type of error-checking code as explained in the introduction to Chapter 1 (see also Kautz [37]). The main result of Chapter 1 is a very natural explicit construction of an induced cycle of length  $(9/64)2^d$  in the  $d$ -dimensional cube.

The first lower bound for the maximal possible length of an induced cycle in the  $d$ -dimensional cube was given by Kautz [37]. He proved that such cycles can have length greater than  $\lambda\sqrt{2^d}$ , where  $\lambda$  is a constant. This bound was later improved many times leading eventually to the bound given by Danzer and Klee [15], who proved the lower bound  $2^{d+1}/d$  when  $d$  is a power of 2, and  $(7/4)2^d/(d-1)$  for all  $d \geq 5$ . The best upper bound at present is  $2^{d-1}(1 - 2/(d^2 - d + 2))$  for  $d \geq 7$ , given by Solov'jeva [54].

We improve the bound of Danzer and Klee by giving a construction that can be outlined as follows. We regard the  $d$ -dimensional cube  $I[d]$  as the  $(d-5)$ -dimensional cube with vertices being copies of the 5-dimensional cube. Our induced cycle visits every vertex of the  $(d-5)$ -dimensional cube exactly once, thus it is an ‘expansion’ of a Hamiltonian cycle in  $I[d-5]$ . The Hamiltonian cycle

in  $I[d-5]$  is constructed by induction in a way that allows us to give an exact and simple characterization of the pairs of vertices of  $I[d-5]$  which are connected by an edge but are not consecutive in the cycle. This is crucial since for each such pair  $(x, y)$  the ‘expansion’ of the Hamiltonian cycle in  $I[d-5]$  to an induced cycle in  $I[d]$  cannot use the same vertex of  $I[5]$  in both of its copies corresponding to the vertices  $x$  and  $y$ .

Having constructed the Hamiltonian cycle in  $I[d-5]$  and having characterized the ‘bad’ edges of  $I[d-5]$ , we embed a certain subdivision of the complete bipartite graph  $K_{2,5}$  into  $I[5]$ . To construct our ‘expansion’ of the Hamiltonian cycle, we use at each copy of  $I[5]$  one of the paths obtained as the images of the edges of  $K_{2,5}$ .

In Chapter 2, which presents joint work with Yoshiharu Kohayakawa, we consider the following colouring problem. Let an integer  $s \geq 1$  and a graph  $G$  be given. Let us denote by  $\chi_s(G)$  the smallest integer  $\chi$  for which there exists a vertex-colouring of  $G$  with  $\chi$  colours such that any two distinct vertices of the same colour are at distance greater than  $s$ . Note that  $\chi_1(G)$  is the usual chromatic number of  $G$ , and hence  $\chi_s(G)$  is a very natural generalization of  $\chi_1(G)$ . Let us denote by  $\omega_s(G)$  the maximal cardinality of a subset of the vertices of  $G$  with diameter at most  $s$ . Clearly  $\chi_s(G) \geq \omega_s(G)$ . For  $s \geq 1$  and  $h \geq 0$  set  $\gamma_s(G) = \chi_s(G) - \omega_s(G)$  and

$$\nu_s(h) = \max \{n \in \mathbb{N} : \text{for any graph } G, |G| < n \text{ implies } \gamma_s(G) < h\}.$$

Gionfriddo [30] has given estimates for  $\nu_s(h)$ . We improve the recent bound  $\nu_2(h) \leq 6h$  ( $h \geq 3$ ) of Gionfriddo and Milici [31] to  $\nu_2(h) \leq 5h$  ( $h \geq 3$ ). More generally, we give the following tight bounds for arbitrary  $s \geq 1$  and large enough  $h$ :

$$2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2} \leq \nu_s(h) \leq 2h + h^{1-\epsilon_s},$$

where  $\epsilon_s > 0$  depends only on  $s$ . The upper bound is proved entirely by constructive methods.

In Chapter 3 we consider a problem concerning colourings of cycles. Before we state the problem let us present some background. In 1963 Ringel [47] conjectured that for any natural number  $n$  and any tree  $T$  with  $n$  edges, the complete graph  $K_{2n+1}$  could be decomposed into  $2n + 1$  subgraphs isomorphic to the tree  $T$ . Later Kotzig (reported by Rosa [48]) strengthened Ringel’s conjecture by adding a condition of cyclic symmetry on the decomposition. This Ringel-Kotzig conjecture remains open, and so does its weaker version due to Ringel.

In connection with the Ringel-Kotzig conjecture, Rosa [48] studied four classes of labellings of graphs, i.e. assignments of natural numbers to their vertices and edges satisfying the condition that the label of an edge is the absolute value of the difference of the labels of its end-points. Showing that one of Rosa’s classes of labellings could be used to label all trees would prove the Ringel-Kotzig conjecture. The smallest class of Rosa’s labellings for which it is still unknown whether they can be used to label all trees is the class of  $\beta$ -labellings, also called graceful labellings. The condition for a labelling of a graph with  $n$  edges to be graceful is that the labels of its vertices should be distinct elements of the set  $[0, n] \subset \mathbb{N}$  and that the labels of its edges should be distinct elements of  $[1, n] \subset \mathbb{N}$ .

Since the conjecture whether all trees can be labelled gracefully has proved to be very difficult, Bloom [10] defined an analogous notion, namely that of minimally  $k$ -equitable labellings of graphs, where  $k$  is a natural number. A labelling of a graph on  $n$  vertices is *minimally  $k$ -equitable* if the labels of vertices are distinct elements of  $[1, n] \subset \mathbb{N}$  and every edge label occurs either  $k$ -times or does not occur at all. Thus for trees graceful labellings are essentially equivalent to minimally 1-equitable labellings. Bloom was mainly interested in minimally  $k$ -equitable labellings of cycles. The obvious necessary condition for the cycle  $C_n$  to have a minimally  $k$ -equitable labelling is that  $k$  should be a proper divisor of  $n$  (i.e. different from 1 and  $n$ ).

Bloom [10] has asked whether this simple necessary divisibility condition is in fact sufficient. In Chapter 3 we answer Bloom’s question in the positive. The

proof we give is constructive. We consider three cases;  $k$  odd,  $k \equiv 2 \pmod{4}$ , and  $k \equiv 0 \pmod{4}$ . In each case the proof goes by induction on  $m = n/k$ . When performing our construction we look into the problem from a different point of view. Instead of trying to label the vertices of the cycle  $C_n$ , we try to build a cycle on the vertex-set  $\{1, 2, \dots, n\}$  using edges of  $k$  different lengths, where the *length* of an edge is the absolute value of its end-points. This approach proves to be very useful.

We start from a simple observation that for  $k$  odd and  $n = 2k$ , it is enough to connect  $i$  with  $i + k$ ,  $i = 1, 2, \dots, k$  and  $j$  with  $j + 1$ ,  $j = 1, 3, 5, \dots, 2k - 1$ . We give a similar but a bit more complicated construction for  $n = mk$ ,  $m = 3, 4, 5$ . Then we apply induction. In the inductive step we subdivide edges of a certain length in such a way that we get edges of two different lengths ( $k$  edges of each) and also different from the lengths of other edges. In each of the two remaining cases the proof is analogous; they differ only in the first step of the construction.

In Chapter 4 we consider a problem concerned with an ‘opened’ coloured cycle, i.e. a coloured path. Assume that the vertices of a path  $N$  are coloured with the integers  $1, 2, \dots, t$ . We shall call such a path  $N$  an *opened  $t$ -coloured necklace*. Suppose we want to cut only a small number of edges of our necklace and use the obtained segments to partition the set of vertices of  $N$  into  $k$  classes such that, for each colour  $i$ , the vertices of colour  $i$  are partitioned evenly between them. Let us call such a partition a  *$k$ -splitting* and let its *size* be the minimal number of cuts required to obtain it. The problem of calculating the size of a  $k$ -splitting has some applications to VLSI circuit design, as noted by Bhatt and Leiserson [9] and Bhatt and Leighton [8].

If the vertices of each colour are consecutive in  $N$ , then for any  $k$ -splitting of  $N$ , each segment of vertices of one colour must be cut at  $k - 1$  points at least. Thus any  $k$ -splitting of  $N$  has size at least  $t(k - 1)$ . Goldberg and West [34] proved that this trivial lower bound is also an upper bound for 2-splittings, and they posed a question about the general case of arbitrary  $k$ . Alon and West [5] conjectured

that  $t(k-1)$  is an upper bound on the size of  $k$ -splittings for any  $k$  and  $t$ . Alon [4] proved this conjecture. His proof uses many techniques from algebraic topology. In Chapter 4 we present a different, more combinatorial, proof of Alon's result using a theorem from algebraic topology only as a starting point. In our proof, the main tool is a new very natural generalization of the Borsuk-Ulam antipodal theorem which says that for any continuous map  $h : \mathbb{S}^m \rightarrow \mathbb{R}^m$ , there is a point  $x \in \mathbb{S}^m$  such that  $h(x) = h(-x)$ .

To formulate our generalization we first define a generalization  $\mathbb{S}_p^{m(p-1)}$  of the  $m$ -dimensional  $\ell_1$ -sphere  $\mathbb{S}^m = \mathbb{S}_2^m$ , for any prime number  $p$ . If we think of the set  $\mathbb{R}$  of reals as two half-lines with a common end-point 0, then its natural generalization is the set  $\mathbb{R}_{+,p}$  of  $p$  half-lines having a common end-point (we denote it also by 0). In analogy to  $\mathbb{S}_2^m$  being the set of points of  $\mathbb{R}^{m+1}$  which are at distance 1 from the point  $(0, 0, \dots, 0) \in \mathbb{R}^{m+1}$ , we define  $\mathbb{S}_p^{m(p-1)}$  as the set of points of  $\mathbb{R}_{+,p}^{m(p-1)+1} = (\mathbb{R}_{+,p})^{m(p-1)+1}$  which are at distance 1 from  $(0, 0, \dots, 0) \in \mathbb{R}_{+,p}^{m(p-1)+1}$ .

As an analogue of the antipodal map on  $\mathbb{S}_2^m$ , we have a very natural free  $\mathbb{Z}_p$ -action  $\omega$  on  $\mathbb{S}_p^{m(p-1)}$ . Note that the antipodal map swaps the two half-lines in every coordinate of  $x \in \mathbb{S}_2^m$ ; in the general case we define  $\omega$  to permute the half-lines cyclicly in every coordinate. Our generalization of the Borsuk-Ulam antipodal theorem says that for any continuous map

$$h : \mathbb{S}_p^{m(p-1)} \rightarrow \mathbb{R}^m$$

there is a point  $x \in \mathbb{S}_p^{m(p-1)}$  such that

$$h(x) = h(\omega(x)) = \dots = h(\omega^{p-1}(x)).$$

This theorem easily implies Alon's result.

In Chapter 5 we again consider a problem connected with the Borsuk-Ulam antipodal theorem. Bajmóczy and Bárány [6] proved that if  $\Delta$  is the closure of an  $(n+1)$ -dimensional simplex and  $f : \Delta \rightarrow \mathbb{R}^n$  is a continuous map, then



there are two disjoint faces of  $\Delta$  whose images intersect. Since the Borsuk-Ulam theorem says that for any continuous map  $h : \mathbb{S}^n \rightarrow \mathbb{R}^n$  there exists  $x \in \mathbb{S}^n$  with  $h(x) = h(-x)$ , to prove the Bajmóczy-Bárány theorem it is enough to show that there is a continuous map  $g : \mathbb{S}^n \rightarrow \Delta$  such that for every  $x \in \mathbb{S}^n$  the supports of  $g(x)$  and  $g(-x)$  are disjoint. In Chapter 5 we give a very natural construction of such a function  $g$ .

Finally, in Chapter 6 we present a simple observation allowing us to give a positive answer to a question posed by Sen, Das, Roy and West [50]. They asked whether each digraph can be represented as an intersection digraph of convex sets in two dimensional Euclidean space. Sen, Das, Roy and West defined intersection digraphs as digraphs with ordered pairs of sets assigned to vertices, where  $\overrightarrow{uv}$  is a directed edge when the ‘source set’ of  $u$  intersects the ‘terminal set’ of  $v$ .