CHAPTER 6

AN OBSERVATION ON INTERSECTION DIGRAPHS OF CONVEX SETS IN THE PLANE

Given a finite family of sets, its *intersection graph* has a vertex corresponding to each set, with edges between vertices corresponding to non-disjoint sets. The notion of intersection graphs is well studied—see an issue of *Discrete Mathematics* [20] which is dedicated to papers on this subject. Maehara [45] introduced and studied a class of *intersection digraphs*; this notion was later generalized by Sen, Das, Roy and West [50]. Let $D = (V, E)$ be a digraph and $\{(S_v, T_v) : v \in V\}$ be a family of ordered pairs of sets. Sen, Das, Roy and West define D to be the intersection digraph of this family if $E = \{\overrightarrow{uv} : S_u \cap T_v \neq \emptyset\}$. Note that this definition allows loops in our digraph.

By assigning to a vertex of a graph the set of edges incident with it, it is easy to see that every graph is an intersection graph of finite sets. Let the *intersection number* $i\#(G)$ of a graph G be the minimum size of a set U such that G is the intersection graph of subsets of U. Erdös, Goodman and Pósa [24] proved that $i\#(G)$ is equal to the minimum number of complete subgraphs needed to cover all its edges and that

$$i\#(G) \leq \lfloor n^2/4 \rfloor \tag{1}$$

for an n-vertex graph G. Equality in (1) is achieved by the complete bipartite graph $G = K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$.

An analogous construction shows also that every digraph is an intersection digraph of finite sets. To get a representation of a digraph as an intersection digraph, it is enough to assign to a vertex v a pair of subsets (S_v, T_v) of the edge-set, where S_v is the set of edges having their 'starting point' at v and T_v is the set of edges with their 'terminal point' at v. By analogy to graphs, Sen, Das, Roy and West [50] define the *intersection number* $i\#(D)$ of a digraph D as the minimum size of a set U such that D is the intersection digraph of ordered pairs of subsets of U. They also define a *generalized complete bipartite subgraph* (GBS) of a digraph D as a subdigraph whose vertex-set can be expressed as $X \cup Y$ (X and Y need not be disjoint) and whose edge-set is equal to $\{\overrightarrow{xy} : x \in X,\ y \in Y\}$. An easy result of Sen, Das, Roy and West characterizes the intersection number $i\#(D)$ as the minimum number of GBS's required to cover the edges of D. They also give the best possible upper bound on the intersection number of digraphs:

$$i\#(D) \leq n,$$

for an n-vertex digraph D.

Given a family of sets, a natural question to ask about intersection graphs and digraphs is whether all graphs (all digraphs) are intersection graphs (digraphs) of sets (ordered pairs of sets) from this family. Of special interest are intersection graphs and digraphs where the sets are required to be convex sets in the Euclidean space. If the space is one-dimensional then we get interval graphs, characterized in [28], [29], [44], and interval digraphs, characterized in [50]. In three dimensions all graphs and digraphs can be represented. With two-dimensional convex sets not all graphs can be obtained. Wegner [59] gave an example of a graph which is not an intersection graph of convex sets in the plane. The graph is obtained from K_5 by subdividing each edge. For digraphs Sen, Das, Roy and West [50] observed that an analogue of Wegner's counter-example fails and posed the question whether every digraph is the intersection digraph of ordered pairs of convex sets in the plane. In this brief chapter we present a simple observation allowing us to give a positive answer to this question.

Theorem 6.1. *Let $D = (V, E)$ be a digraph. Then there is a family $\mathcal{A} = \{(S_v, T_v) : v \in V\}$ of pairs of convex sets in $\mathbb{R}^2$ such that D is the intersection digraph of $\mathcal{A}$.*

Proof. Set $n = |V|$. Let $A \subset \mathbb{R}^2$ be a set of n points on a circle, and let $f : V \to A$ be a bijection. Set, for each $v \in V$,

$$S_v = \{f(v)\},$$
$$T_v = \mathrm{conv}(\{f(u) : \overrightarrow{uv} \in E\}),$$

where for $B \subset \mathbb{R}^2$, $\mathrm{conv}(B)$ is the convex hull of B, that is the smallest convex set containing B. So, for each vertex v of D, the 'source set' of v contains one element of A, namely the one that corresponds to v under f, and the 'terminal set' of v is the convex hull of the elements of A corresponding to all predecessors of v. It is easy to see that T_v does not contain any other elements of A. So $S_u \cap T_v \neq \emptyset$ if and only if $f(u) \in T_v$, which holds if and only if $\overrightarrow{uv} \in E$. Therefore D is the intersection digraph of the family $\{(S_v, T_v) : v \in V\}$. Of course for each $v \in V$, the sets S_v and T_v are convex subsets of $\mathbb{R}^2$ so the theorem is proved. $\square$