CHAPTER 5

REMARKS ON A GENERALIZATION OF RADON'S THEOREM

The well-known theorem of Radon [36] says that, for any $A \subset \mathbb{R}^n$ satisfying $|A| \geq n+2$, there are disjoint subsets B and C of A such that their convex hulls have nonempty intersection. Since, for any $A \subset \mathbb{R}^n$ satisfying $|A| = n+2$ the convex hull of A is the image of the closure of an $(n+1)$-dimensional simplex under a linear map, Radon's theorem is an immediate corollary to the following theorem. The terms used in this chapter are defined in Chapter 4.

Theorem 5.1. *Let $\Delta \subset \mathbb{R}^{n+1}$ be an $(n+1)$-dimensional simplex and let K be the simplicial complex containing all faces of Δ. If $f : |K| \to \mathbb{R}^n$ is a linear map, then there are two disjoint faces Δ_1, Δ_2 of Δ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.*

Thus the following theorem of Bajmóczy and Bárány [6] can be thought of as a generalization of Radon's theorem.

Theorem 5.2. *Let Δ and K be as in Theorem 5.1. If $f : |K| \to \mathbb{R}^n$ is a continuous map, then there are two disjoint faces Δ_1, Δ_2 of Δ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.*

Bajmóczy and Bárány use the following antipodal theorem of Borsuk and Ulam [13] in their proof.

Theorem 5.3. *For any continuous map $h : \mathbb{S}^n \to \mathbb{R}^n$ there exists $x \in \mathbb{S}^n$ with $h(x) = h(-x)$.*

Theorem 5.2 follows immediately from Theorem 5.3 and the following theorem.

Theorem 5.4. *Let Δ and K be as in Theorem 5.1. There exists a continuous map $g : \mathbb{S}^n \to |K|$ such that for every $x \in \mathbb{S}^n$ the supports of $g(x)$ and $g(-x)$ are disjoint.*

In this brief chapter we are going to give a new very simple proof of Theorem 5.4. We present in it an explicit construction of the function g.

Proof of Theorem 5.4. Assume that $\Delta = (x_0, \ldots, x_{n+1})$. Let K_1 be the simplicial complex with $\{x_0, \ldots, x_{n+1}\}$ as its set of vertices and all proper faces of Δ as its simplices. Let K_2 be the barycentric subdivision of K_1. Let $\omega : |K_2| \to |K_2|$ be a free Z_2-action defined as follows. If $T \subset \{x_0, \ldots, x_{n+1}\}$ is the skeleton of a simplex σ of K_1 and c_σ is the barycentre of σ, then let

$$\omega(c_\sigma) = c_{\sigma'} \,,$$

where σ' is the simplex of K_1 whose skeleton T' is the complement of T, that is

$$T' = \{x_0, \ldots, x_{n+1}\} \setminus T.$$

Thus, we have defined ω on the vertices of K_2. Let us extend ω linearly to $|K_2|$, that is for any $x \in (c_{\sigma_1}, \ldots, c_{\sigma_r}) \in K_2$ having the following barycentric representation

$$x = \sum_{i=1}^{r} \mu_i c_{\sigma_i},$$

let

$$\omega\,(x) = \sum_{i=1}^{r} \mu_i \omega(c_{\sigma_i}).$$

Clearly, ω is well defined and there is a homeomorphism $f : \mathbb{S}^n \to |K_2|$ which is equivariant with respect to the antipodal map on $\mathbb{S}^n$ and ω on $|K_2|$, that is such that for every $x \in \mathbb{S}^n$ the following equality holds:

$$f(-x) = \omega(f(x)).$$

Therefore, to prove our theorem, it is enough to show the existence of a continuous map $h : |K_2| \to |K|$ such that for every $x \in |K_2|$ the supports of $h(x)$ and $h(\omega(x))$ are disjoint.

Let K_3 be the barycentric subdivision of K_2. We shall define h on the vertices of K_3 first. Let d_A be the barycentre of the simplex $A = (c_{\sigma_1}, \ldots, c_{\sigma_r})$ of K_2. Since A is a simplex of K_2, we can assume that σ_i is a proper face of σ_{i+1}, $i = 1, \ldots, r-1$. Define

$$h(d_A) = c_{\sigma_1}.$$

Now, let us extend h linearly to $|K_3| = |K_2|$, that is for $x \in (d_{A_1}, \ldots, d_{A_s}) \in K_3$ with the following barycentric representation

$$x = \sum_{i=1}^{s} \mu_i d_{A_i},$$

let

$$h(x) = \sum_{i=1}^{s} \mu_i h(d_{A_i}).$$

Now, we shall show that for every $x \in |K_2|$ the supports of $h(x)$ and $h(\omega(x))$ in K are disjoint. Note first that if d_A is the barycentre of a simplex $A = (c_{\sigma_1}, \ldots, c_{\sigma_r})$, then

$$\omega(d_A) = \omega\left(\frac{1}{r}\sum_{i=1}^{r} c_{\sigma_i}\right) = \frac{1}{r}\sum_{i=1}^{r} \omega(c_{\sigma_i}) = d_B$$

where

$$B = (\omega(c_{\sigma_1}), \ldots, \omega(c_{\sigma_r})).$$

For $x \in |K_2|$, let

$$\{A_1, \ldots, A_r\}$$

be the support of x in K_3 and

$$\{B_1, \ldots, B_r\}$$

be the support of $\omega(x)$ in K_3 where $B_i = \omega(A_i)$, $i = 1, \ldots, r$. Let

$$\{\sigma_{i,1}, \ldots, \sigma_{i,s_i}\}$$

be the skeleton of A_i, $i = 1, \ldots, r$, where $\sigma_{i,j}$ is a proper face of $\sigma_{i,j+1}$, $j = 1, \ldots, s_i - 1$. Now let

$$\{\sigma'_{i,1}, \ldots, \sigma'_{i,s_i}\}$$

be the skeleton of B_i, $i = 1, \ldots, r$. Since the skeleton of $\sigma'_{i,j}$ is the complement of the skeleton of $\sigma_{i,j}$, the simplex $\sigma'_{i,j+1}$ is a proper face of $\sigma'_{i,j}$, for all $i = 1, \ldots, r$ and $j = 1, \ldots, s_i - 1$.

Since $h(A_i) = \sigma_{i,1}$, $i = 1, \ldots, r$, the support of $h(x)$ in K_2 is the set

$$\{\sigma_{1,1}, \sigma_{2,1}, \ldots, \sigma_{r,1}\},$$

and since $h(B_i) = \sigma'_{i,s_i}$, $i = 1, \ldots, r$, the support of $h(\omega(x))$ in K_2 is the set

$$\{\sigma'_{1,s_1}, \sigma'_{2,s_2}, \ldots, \sigma'_{r,s_r}\}.$$

We can assume that A_i is a proper face of A_{i+1}, $i = 1, \ldots, r$. Then $\sigma_{i+1,1}$ is a (not necessarily proper) face of $\sigma_{i,1}$, $i = 1, \ldots, r$, and thus the support of $h(x)$ in Δ^{n+1} is the skeleton of $\sigma_{1,1}$. Since A_i is a proper face of A_{i+1}, the simplex B_i is a proper face of B_{i+1}, $i = 1, \ldots, r$. Therefore, $\sigma'_{i+1,s_{i+1}}$ is a face of σ'_{i,s_i}, $i = 1, \ldots, r$, and thus the support of $h(\omega(x))$ in K is the skeleton of σ'_{1,s_1}. Now recall that the skeleton of σ'_{1,s_1} is a complement of the skeleton of σ_{1,s_1}. But the skeleton of $\sigma_{1,1}$ is contained in the skeleton of σ_{1,s_1} so the supports of $h(x)$ and $h(\omega(x))$ in K are disjoint, and the theorem is proved. □