CHAPTER 4

SPLITTING NECKLACES AND A GENERALIZATION OF THE BORSUK-ULAM ANTIPODAL THEOREM

§4.1. Introduction

Let t be a natural number. An *opened t-coloured necklace* is a sequence of elements (*beads*) from the integer segment $[1, t]$. Let N be an opened t-coloured necklace. A *splitting* of N is a partition $N_1 \cup N_2 \cup \ldots \cup N_\ell$ of the set of beads of N such that for every colour i, $1 \leq i \leq t$, the beads of colour i are spread evenly between the sets N_j, i.e. all of the sets N_j contain the same number of beads of colour i. A splitting of N which is a partition into k sets is called a *k-splitting*. The *size* of the splitting of N is the minimal number of cutpoints of N needed to partition it into segments preserved by the splitting.

Note that if the beads of each colour are consecutive in N, then any k-splitting cuts each segment of one colour beads at $k-1$ points at least, and hence has size at least $t(k-1)$. The following natural question arises: is this trivial lower bound also an upper bound? In other words, if N is an opened t-coloured necklace admitting a k-splitting, does N have a k-splitting of size $t(k-1)$? Somewhat surprisingly the answer to this question is 'yes'.

Let us now briefly describe the history of this problem. Bhatt and Leiserson [9] and Bhatt and Leighton [8] pointed out that this problem has some applications to VLSI circuit design. Goldberg and West [34] proved that for every t, an opened

t-coloured necklace admitting a 2-splitting has a 2-splitting of size t. They also raised the question about the general upper bound for k-splittings. Alon and West [5] gave a very short proof of the above upper bound for 2-splittings using the Borsuk-Ulam antipodal theorem; they also conjectured that $t(k-1)$ is an upper bound for k-splittings. Alon [4] proved the $t(k-1)$ upper bound for k-splittings using involved methods of algebraic topology. In this chapter we are going to give another proof of Alon's result. Our proof will be more elementary and will use a classical result of algebraic topology (Lemma 4.10) only as a starting point; after that the argument will be purely combinatorial.

Theorem 4.1. (N. Alon [4]) *Every necklace with ka_i beads of colour i, $1 \leq i \leq t$, has a k-splitting of size at most $t(k-1)$.*

To prove Theorem 4.1 we shall formulate and prove a new, very natural generalization of the Borsuk-Ulam antipodal theorem. From this generalization we shall immediately obtain a continuous version of Theorem 4.1 implying, as in Alon [4], Theorem 4.1 itself.

To formulate our generalization of the Borsuk-Ulam antipodal theorem, we must introduce some more terminology. Let $\mathbb{R}_+$ be the metric space of nonnegative reals with the natural metric. Given a natural number n, let $\mathbb{R}_{+,n}$ be obtained by taking the product of $\mathbb{R}_+$ with the integer segment $[0, n-1] \subset \mathbb{N}$ and identifying the points $(0,0)$, $(0,1)$, …, $(0, n-1)$ to a single point denoted 0. The metric μ on $\mathbb{R}_{+,n}$ is defined as follows:

$$\mu\big((x,i),(y,i)\big) = |x-y|$$

and

$$\mu\big((x,i),(y,j)\big) = x+y$$

for $x, y \in \mathbb{R}_+$, $0 \leq i, j \leq n-1$, and $i \neq j$. Thus $\mathbb{R}_{+,n}$ is the union of n half-lines with a common endpoint and equipped with the natural metric.

Given a natural number m, let $\mathbb{R}^m_{+,n}$ be the product

$$\underbrace{\mathbb{R}_{+,n} \times \mathbb{R}_{+,n} \times \ldots \times \mathbb{R}_{+,n}}_{m \text{ times}}$$

with the metric μ defined by

$$\mu\Big(\big(x_1, x_2, \ldots, x_m\big), \big(y_1, y_2, \ldots, y_m\big)\Big) = \sum_{i=1}^{m} \mu\big(x_i, y_i\big).$$

Let $\mathbb{O}$ be the point $(0, 0, \ldots, 0) \in \mathbb{R}^m_{+,n}$, and let $\mathbb{S}^{m-1}_n$ be the unit sphere in $\mathbb{R}^m_{+,n}$ with the centre at $\mathbb{O}$, i.e. let

$$\mathbb{S}^{m-1}_n = \left\{x \in \mathbb{R}^m_{+,n} : \mu(x, \mathbb{O}) = 1\right\}.$$

Let $\eta : [0, n-1] \to [0, n-1]$ be the function of taking the cyclic successor, i.e. let $\eta(i) = (i+1) \bmod n$, $i = 0, 1, \ldots, n-1$. Let $\omega : \mathbb{S}^{m-1}_n \to \mathbb{S}^{m-1}_n$ be defined by

$$\omega\Big(\big(x_1, i_1\big), \big(x_2, i_2\big), \ldots, \big(x_m, i_m\big)\Big) = \Big(\big(x_1, \eta(i_1)\big), \big(x_2, \eta(i_2)\big), \ldots, \big(x_m, \eta(i_m)\big)\Big).$$

We are now ready to state our generalization of the Borsuk-Ulam's theorem.

Theorem 4.2. *If p is a prime and m is any natural number, then for any continuous map*

$$h : \mathbb{S}^{m(p-1)}_p \to \mathbb{R}^m$$

there exists an $x \in \mathbb{S}^{m(p-1)}_p$ such that

$$h(x) = h\big(\omega(x)\big) = \ldots = h\big(\omega^{p-1}(x)\big).$$

Note that for $p = 2$, $\mathbb{S}^{m(p-1)}_p$ is naturally homeomorphic to $\mathbb{S}^m$, the ℓ_1-sphere in $\mathbb{R}^{m+1}$, with the map ω on $\mathbb{S}^m_2$ corresponding to the antipodal map on $\mathbb{S}^m$. Thus if $p = 2$, Theorem 4.2 is a reformulation of the Borsuk-Ulam antipodal theorem. In Section 4.4 (Lemma 4.12), we shall give another description of $\mathbb{S}^{m(p-1)}_p$ by defining a triangulation of it.

The rest of this chapter is partitioned as follows. In Section 4.2, we prove Theorem 4.1 using Theorem 4.2; in Section 4.3, we prove the main lemma needed in the proof of Theorem 4.2, whose proof is given in Section 4.4.

§4.2. Continuous Splittings

In this section we shall prove Theorem 4.3, which easily implies Theorem 4.1, and is in fact a continuous version of it. We shall show that Theorem 4.3 follows immediately from Theorem 4.2. Now, let us introduce the terminology needed to formulate Theorem 4.3. Let $I = [0,1]$ be the real unit interval. An *interval m-colouring* is a function from I to the integer segment $[1,m]$ such that the set of points mapped to i, $1 \leq i \leq m$, is (Lebesgue) measurable. A *k-splitting of size r* of such a colouring is a partition $I = F_1 \cup \ldots \cup F_k$ satisfying the following conditions:

(i) There is a sequence of numbers $0 = y_0 \leq y_1 \leq \ldots \leq y_r \leq y_{r+1} = 1$ such that for each of the segments (y_i, y_{i+1}), $0 \leq i \leq m$, and each of the sets F_j, $1 \leq j \leq k$, (y_i, y_{i+1}) is either contained in F_j or is disjoint from it.

(ii) The measure of the set of points mapped to i, $1 \leq i \leq m$, which are contained in F_j, $1 \leq j \leq k$, is precisely $1/k$ of the total measure of the points of the colour i.

Theorem 4.3. (Alon [4]) *If p is a prime number, then every interval m-colouring has a p-splitting of size $m(p-1)$.*

The proof of this result given by Alon uses a generalization of the Borsuk-Ulam antipodal theorem due to Bárány, Shlosman and Szücs [7], and another topological result of Bárány, Shlosman and Szücs ([7] Statement A′). We shall show that our new generalization of the Borsuk-Ulam antipodal theorem is strong enough to imply Theorem 4.3 immediately.

Proof of Theorem 4.3. Let $f : I \to [1,m]$ be an interval m-colouring. We shall define a continuous map $h : \mathbb{S}_p^{m(p-1)} \to \mathbb{R}^m$ and apply Theorem 4.2. Let $q = m(p-1)+1$. Given

$$x = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_q) \in \mathbb{S}_p^{m(p-1)}$$

where

$$\overline{x}_i = (x_i, k_i),$$

$i = 1, 2, \ldots, q$, $x_i \in \mathbb{R}_+$, $0 \leq k_i \leq p-1$, let

$$I = F_0^{(x)} \cup F_1^{(x)} \cup \ldots \cup F_{p-1}^{(x)}$$

be a splitting of size $m(p-1)$ of f defined as follows. Let $0 = y_0 \leq y_1 \leq \ldots \leq y_{q-1} \leq y_q = 1$ be the sequence or reals satisfying

$$y_i - y_{i-1} = x_i,$$

for $i = 1, \ldots, q$. Note that

$$\sum_{i=1}^{q} x_i = 1.$$

Let

$$J_s^{(x)} = \{i : 1 \leq i \leq q,\ k_i = s\},$$

and

$$F_s^{(x)} = \bigcup_{i \in J_s^{(x)}} (y_{i+1}, y_i),$$

$s = 0, 1, \ldots, p-1$. In other words the partition $I = F_0^{(x)} \cup F_1^{(x)} \cup \ldots \cup F_{p-1}^{(x)}$ is obtained by cutting I into consecutive segments of lengths $x_1, x_2, \ldots, x_q$ and putting the i-th segment into the set $F_{k_i}^{(x)}$. Let $h(x) = (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m$ be such that r_i, $1 \leq i \leq m$, is the measure of the set of points contained in $F_0^{(x)}$ which are mapped to i by f. Clearly h is continuous.

By Theorem 4.2, there exists $x \in \mathbb{S}_p^{m(p-1)}$ such that

$$h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x)). \tag{1}$$

We claim that the partition $I = F_0^{(x)} \cup F_1^{(x)} \cup \ldots \cup F_{p-1}^{(x)}$ is a p-splitting of f. To prove the claim we shall show that

$$h(\omega^j(x)) = (r_1^{(j)}, r_2^{(j)}, \ldots, r_m^{(j)}),$$

$0 \leq j \leq p-1$, where $r_i^{(j)}$, $1 \leq i \leq m$, is the measure of the set of points contained in $F_{\eta^{-j}(0)}^{(x)}$ which are mapped to i by f. This will finish the proof of the theorem since it follows from (1) that, for $1 \leq i \leq m$, we have

$$r_i^{(0)} = r_i^{(1)} = \ldots = r_i^{(p-1)}.$$

Note that for $j = 0, 1, \ldots, p-1$ we have

$$\omega^j(x) = \left(\overline{\overline{x}}_1, \overline{\overline{x}}_2, \ldots, \overline{\overline{x}}_q\right)$$

where

$$\overline{\overline{x}}_i = \left(x_i, \eta^j(k_i)\right),$$

$i = 1, 2, \ldots, q$. Thus

$$\begin{aligned} J_s^{(\omega^j(x))} &= \left\{i : 1 \le i \le q,\ \eta^j(k_i) = s\right\} \\ &= \left\{i : 1 \le i \le q,\ k_i = \eta^{-j}(s)\right\} = J^{(x)}_{\eta^{-j}(s)}. \end{aligned}$$

Therefore

$$F_0^{(\omega^j(x))} = \bigcup_{k \in J^{(y)}_{\eta^{-j}(0)}} (y_{k+1}, y_k) = F^{(x)}_{\eta^{-j}(0)},$$

and

$$h(\omega^j(x)) = (r_1^{(j)}, r_2^{(j)}, \ldots, r_m^{(j)}) \in \mathbb{R}^m$$

where r_i, $1 \le i \le m$, is the measure of the set of points contained in $F^{(x)}_{\eta^{-j}(0)}$ which are mapped to i by f. This completes our proof. □

Note that in Theorem 4.3 we assume that p is prime. Unlike in the case of Theorem 4.2 this assumption is not essential. We are now going to present Alon's proofs that Theorem 4.3 implies its generalized version, Corollary 4.4, and that Corollary 4.4 implies Theorem 4.1.

Corollary 4.4. (Alon [4]) *For any natural numbers k and m, every interval m-colouring has a k-splitting of size $m(k-1)$.*

Proof. (Alon [4]) We shall use induction on the number of prime factors of k. If k is prime then Corollary 4.4 follows from Theorem 4.3. Let $k = k_1 k_2$ where $k_1, k_2 \neq 1$, and assume that every interval m-colouring has a k'-splitting of size $m(k'-1)$ for any integer k' having less primes in its factorization than k has.

Let $f : I \to [1, m]$ be an interval m-colouring. We shall show that f has a k-splitting of size $k(m-1)$. By our induction assumption f has a k_1-splitting $I = F_1 \cup F_2 \cup \ldots \cup F_{k_1}$ of size $m(k_1 - 1)$. By point (i) of the definition of splittings for interval colourings there is a sequence of numbers $0 = y_0 \leq y_1 \leq \ldots \leq y_{m(k_1-1)} \leq y_{m(k_1-1)+1} = 1$ such that for each of the segments $I_i = (y_i, y_{i+1})$, $0 \leq i \leq m(k_1 - 1)$, and each of the sets F_j, $1 \leq j \leq k_1$, I_i is either contained in F_j or is disjoint from it. Clearly we can assume that all I_i are nonempty since otherwise we can change our sequence of numbers y_j by deleting repeating ones, and adding new.

For $j = 1, 2, \ldots, k_1$, let $f_j : I \to [1, m]$ be the interval m-colouring obtained as follows. Let us place the intervals I_i contained in F_j next to each other getting an interval A_j, and let $\alpha_j : I \to A_j$ be the affine map taking 0 to the smaller endpoint of A_j and 1 to its bigger endpoint. Now set $f_j = f \circ \alpha_j$. By the inductive assumption there is a k_2-splitting of f_j of size $m(k_2 - 1)$, $j = 1, 2, \ldots, k_1$. Transforming these k_2-splittings into partitions of F_j, for $j = 1, 2, \ldots, k_1$, we get a partition of I into $k = k_1 k_2$ sets which is a k-splitting of f of size

$$m(k_1 - 1) + k_1(m(k_2 - 1)) = m(k - 1).$$

Thus the proof of the theorem is complete. □

Proof of Theorem 4.1. (Alon [4]) Let $f : I \to [1, t]$ be the interval t-colouring obtained by partitioning I into $s = k \sum_{i=1}^{t} a_i$ segments of equal length (called in the future by *small segments*) and colouring the i-th small segment with the colour of the i-th bead of the necklace. By Corollary 4.4 there is a k-splitting $I = F_1 \cup F_2 \cup \ldots \cup F_k$ of size $t(k-1)$ of f. This splitting can be transformed into a k-splitting of size $t(k-1)$ of the necklace provided that the cuts do not occur inside the small segments. We shall show by induction on the number of this 'bad' cuts that the k-splitting of size $t(k-1)$ of f can be transformed into a k-splitting of size $t(k-1)$ of f without any 'bad' cuts.

If there are no 'bad' cuts then we are done. Assume that there are $k > 0$ 'bad' cuts and that the result holds for any number $k' < k$ of 'bad' cuts. Let i, $1 \leq i \leq t$, be a colour such that the number of 'bad' cuts occuring inside small segments of colour i is positive. Let us define a multigraph with $\{F_j : 1 \leq j \leq k\}$ as the vertex set and (F_j, F_ℓ) being an edge if there is a 'bad' cut occuring inside a small segment of colour i and between a segment contained in F_j and a segment contained in F_ℓ. Since for every j, $1 \leq j \leq k$, the measure of points of colour i contained in F_j is a multiple of the length of a small segment, there are no vertices of degree 1 in our multigraph. Therefore it contains a cycle. By shifting the cuts corresponding to this cycle along the small segments in which they occur, we can decrease the number of 'bad' cuts at least by 1 getting again a k-splitting of f of size $t(k-1)$. This completes the proof of the induction step, and hence the proof of the theorem. $\square$

§4.3. The Main Lemma

Our aim in this section is to prove Lemma 4.11 from which we shall deduce Theorem 4.2 in the next section. First, let us introduce some more terminology. If $x_0, x_1, \ldots, x_k$ are points in $\mathbb{R}^m$ such that $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is a linearly independent set of k vectors in $\mathbb{R}^m$, then we say that these points are *affinely independent*. Let $0 \leq k \leq m$, and $x_0, x_1, \ldots, x_k$ be affinely independent points in $\mathbb{R}^m$. The *k-simplex* $\Delta = (x_0, x_1, \ldots, x_k)$ is the following subset of $\mathbb{R}^m$:

$$\left\{ x = \sum_{i=0}^{k} \mu_i x_i : \sum_{i=0}^{k} \mu_i = 1,\ \mu_i > 0 \right\}. \tag{2}$$

Since the points $x_0, x_1, \ldots, x_k$ are affinely independent, the reals μ_i, $0 \leq i \leq k$, are uniquely determined by x and $x_0, x_1, \ldots, x_k$. We shall call the sum in (2) the barycentric representation of x with respect to $(x_0, x_1, \ldots, x_k)$. The points $x_0, \ldots, x_k$ are the *vertices* of Δ; the *skeleton* of Δ is the set of all its vertices, and k is the dimension of Δ. A simplex Δ_1 is a *face* of a simplex Δ_2 if the skeleton of

Δ_1 is a subset of the skeleton of Δ_2.

A *simplicial complex* K is a finite set of disjoint simplices such that every face of every simplex of K is also a simplex of K. The *body* $|K|$ of the simplicial complex K is the union of all its simplices; the complex K is then also called a *simplicial decomposition* of $|K|$.

If $\{x_1, x_2, \ldots, x_k\}$ is the set of vertices of the simplicial complex K and $x \in |K|$, then there are unique reals $\mu_1, \mu_2, \ldots, \mu_k$ such that

$$x = \sum_{i=1}^{k} \mu_i x_i, \tag{3}$$

where $\mu_i \geq 0$ for every $i = 1, 2, \ldots, k$,

$$\sum_{i=1}^{k} \mu_i = 1,$$

and the set $\{x_i : \mu_i > 0\}$ is a simplex of K. We shall call the sum (3) the *barycentric representation* of x with respect to K, or just the barycentric representation of x if the complex is clear from the context.

The simplicial complex K' is a *subcomplex* of the simplicial complex K if the set of simplices of K' is a subset of the set of simplices of K, in particular the set of vertices of K' is a subset of the set of vertices of K.

Let ω be a continuous function from a subset X of $\mathbb{R}^m$ to itself, and k be a natural number. We shall say that ω is a *Z_k-action* if the set $\{\omega^0, \omega, \omega^2, \ldots, \omega^{k-1}\}$, where ω^0 is the identity map on X, is a k-element cyclic group under composition. We shall also say that such an action is *free* if for every $x \in X$ all the elements x, $\omega(x)$, $\omega^2(x), \ldots, \omega^{k-1}(x)$ are different.

Let $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ be the ℓ_1-norm on $\mathbb{R}^m$, namely for $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, let

$$\|x\| = \sum_{i=1}^{m} |x_i|.$$

Let

$$\mathbb{B}^m = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$$

be the m-*dimensional unit ball* and let

$$\mathbb{S}^m = \left\{x \in \mathbb{R}^{m+1} : \|x\| = 1\right\}$$

be the m-*dimensional unit sphere*.

Let p be a fixed prime number. For each natural number n, we are going to define a simplicial complex $X_{n,p}$ such that $|X_{n,p}|$ is homeomorphic to the topological space obtained by identifying the boundaries of p disjoint copies of the ball $\mathbb{B}^{(p-1)n}$. Also, each of the complexes $X_{n,p}$ will be equipped with a free Z_p-action ω. We shall prove that for any continuous map $h : |X_{n,p}| \to \mathbb{R}^n$, there exists an $x \in |X_{n,p}|$ such that $h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x))$.

Before we define the family of complexes $X_{n,p}$, let us define the family of complexes $Y_{n,p}$ in $\mathbb{R}^{(p-1)n}$. For a given positive integer n and $i = 1, \ldots, n$, let

$$x^0_{n,i} = (\underbrace{0, 0, \ldots, 0}_{(p-1)(i-1)}, \underbrace{-1, -1, \ldots, -1}_{p-1}, \underbrace{0, 0, \ldots, 0}_{(p-1)(n-i)}) \in \mathbb{R}^{(p-1)n},$$

and

$$x^j_{n,i} = (\underbrace{0, 0, \ldots, 0}_{(p-1)(i-1)}, \underbrace{0, 0, \ldots, 0}_{j-1}, 1, \underbrace{0, 0, \ldots, 0}_{p-j-1}, \underbrace{0, 0, \ldots, 0}_{(p-1)(n-i)}) \in \mathbb{R}^{(p-1)n},$$

for $j = 1, 2, \ldots, p-1$. Set

$$T_{n,i} = \left\{x^j_{n,i} : j = 0, 1, \ldots, p-1\right\}$$

and let $\Delta_{n,i}$ be the simplex with the skeleton $T_{n,i}$, $i = 1, \ldots, n$, and let

$$\bigcup_{i=1}^{n} T_{n,i}$$

be the set of vertices of $Y_{n,p}$. Let T be the skeleton of a simplex of $Y_{n,p}$ if and only if for every $i = 1, \ldots, n$ we have

$$|T \cap T_{n,i}| \leq p - 1. \tag{4}$$

The elements of $Y_{n,p}$ are indeed simplices since for any set T satisfying (4), the elements of T are affinely independent.

Our aim now is to show that $Y_{n,p}$ is a simplicial decomposition of a subset of $\mathbb{R}^{(p-1)n}$ which is homeomorphic to the sphere $\mathbb{S}^{(p-1)n-1}$. Let us first prove the following lemma.

Lemma 4.5. *$Y_{n,p}$ is a simplicial complex.*

Proof. To prove that $Y_{n,p}$ is a simplicial complex it is enough to show that the simplices of $Y_{n,p}$ are pairwise disjoint. Let Δ_1 and Δ_2 be a pair of distinct simplices of $Y_{n,p}$, and suppose that there is an $a \in \Delta_1 \cap \Delta_2$. Let T_1 and T_2 be the skeletons of Δ_1 and Δ_2 respectively. As $a \in \Delta_1$ we have

$$a = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_{i,j} x_{n,i}^{j} \tag{5}$$

where $\mu_{i,j} \geq 0$, $1 \leq i \leq n$, $0 \leq j \leq p-1$,

$$\sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_{i,j} = 1,$$

and

$$T_1 = \left\{ x_{n,i}^{j} : 1 \leq i \leq n,\ 0 \leq j \leq p-1,\ \mu_{i,j} > 0 \right\}.$$

Analogously, as $a \in \Delta_2$ we have

$$a = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu'_{i,j} x_{n,i}^{j} \tag{6}$$

where $\mu'_{i,j} \geq 0$, $1 \leq i \leq n$, $0 \leq j \leq p-1$,

$$\sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu'_{i,j} = 1,$$

and

$$T_2 = \left\{ x_{n,i}^{j} : 1 \leq i \leq n,\ 0 \leq j \leq p-1,\ \mu'_{i,j} > 0 \right\}.$$

Thus, (5) and (6) are the barycentric representations of a with respect to Δ_1 and Δ_2 respectively. Since $\Delta_1 \neq \Delta_2$, we have $T_1 \neq T_2$, and thus there are i_0 and j_0, $1 \leq i_0 \leq n$, $0 \leq j_0 \leq p-1$, such that $\mu_{i_0,j_0} \neq \mu'_{i_0,j_0}$.

Assume $a = (a_1, a_2, \ldots, a_{(p-1)n}) \in \mathbb{R}^{(p-1)n}$, and let

$$b = (b_1, \ldots, b_{p-1}) = (a_{(p-1)(i_0-1)+1}, a_{(p-1)(i_0-1)+2}, \ldots, a_{(p-1)i_0})$$

be the image of a under the projection onto the i_0-th component of $\mathbb{R}^{(p-1)n} = \mathbb{R}^{p-1} \times \ldots \times \mathbb{R}^{p-1}$. We have

$$b = \sum_{j=0}^{p-1} \mu_{i_0,j} x_{n,i_0}^j = \sum_{j=0}^{p-1} \mu'_{i_0,j} x_{n,i_0}^j.$$

We shall obtain a contradiction by showing that $\mu_{i_0,j} = \mu'_{i_0,j}$, $0 \leq j \leq p-1$.

By the definition of $Y_{n,p}$, not all of $\mu_{i_0,j}$, $0 \leq j \leq p-1$, can be positive since Δ_1 is a simplex of $Y_{n,p}$, and hence

$$\mu_{i_0,0} = -\min\{0, b_1, b_2, \ldots, b_{p-1}\},$$

and

$$\mu_{i_0,j} = b_j + \mu_{i_0,0},$$

for $j = 1, \ldots, p-1$. Analogously, since Δ_2 is a simplex of $Y_{n,p}$, not all of $\mu'_{i_0,j}$, $0 \leq j \leq p-1$, can be positive and we have

$$\mu'_{i_0,0} = -\min\{0, b_1, b_2, \ldots, b_{p-1}\},$$

and

$$\mu'_{i_0,j} = b_j + \mu'_{i_0,0},$$

for $j = 1, \ldots, p-1$. Thus $\mu_{i_0,j} = \mu'_{i_0,j}$, $j = 0, \ldots, p-1$, as required. □

Let $X_{n,p}$ be the subcomplex of $Y_{n+1,p}$ such that T is the skeleton of a simplex of $X_{n,p}$ if and only if

$$|T \cap T_{n+1,n+1}| \leq 1.$$

Now, we are going to prove that $|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$ which implies that $|X_{n,p}|$ is homeomorphic to the topological space obtained by identifying the boundaries of p disjoint copies of the ball $\mathbb{B}^{(p-1)n}$.

In the proof we shall need the following two lemmas. Let K be a simplicial complex and let x be a vertex of K. We say that K is an *x-cone* if for every simplex Δ of K with skeleton T, say, $T \cup \{x\}$ is also the skeleton of a simplex of K. Furthermore, for an x-cone K let K' be the simplicial complex such that Δ is a simplex of K' if Δ is a simplex of K and x is not a vertex of Δ. Then, we shall say that K is an *x-cone over K'*. Lemmas 4.6 and 4.7 clearly hold.

Lemma 4.6. *If K is an x-cone over K', and $|K'|$ is homeomorphic to the sphere $\mathbb{S}^k$ or to the ball $\mathbb{B}^k$, then $|K|$ is homeomorphic to $\mathbb{B}^{k+1}$.* □

Lemma 4.7. *Let K_1 and K_2 be simplicial complexes such that $|K_1|$ and $|K_2|$ are both homeomorphic to the ball $\mathbb{B}^{k+1}$, $K_1 \cup K_2$ is a simplicial complex and $|K_1 \cap K_2|$ is homeomorphic to the sphere $\mathbb{S}^k$. Then $|K_1 \cup K_2|$ is homeomorphic to the sphere $\mathbb{S}^{k+1}$.* □

We can now prove the following lemma.

Lemma 4.8. *$|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$.*

Proof. We shall use induction on n. For $n = 1$, $|Y_{n,p}|$ is the boundary of a $(p-1)$-dimensional simplex so $Y_{n,p}$ is homeomorphic to $\mathbb{S}^{p-2}$.

Given $n \geq 1$, assume that $|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$. Let $Y_{n,p}^{(\alpha)}$, $\alpha = 0, 1, \ldots, p-1$, and $\overline{Y}_{n,p}^{(\alpha)}$, $\alpha = 0, 1, \ldots, p-2$, be subcomplexes of $Y_{n+1,p}$ defined as follows. Let

$$\left\{x_{n+1,i}^j : i = 1, \ldots, n,\ j = 0, \ldots, p-1\right\} \cup \left\{x_{n+1,n+1}^j : j = 0, \ldots, \alpha\right\}$$

be the set of vertices of both $Y_{n,p}^{(\alpha)}$ and $\overline{Y}_{n,p}^{(\alpha)}$. Let T be the skeleton of a simplex of $Y_{n,p}^{(\alpha)}$ if and only if

$$\left|T \cap \left\{x_{n+1,n+1}^j : j = 0, \ldots, \alpha\right\}\right| \leq \alpha,$$

and Δ be a simplex of $\overline{Y}_{n,p}^{(\alpha)}$ if and only if Δ is a simplex of $Y_{n+1,p}$. Note that $Y_{n,p}^{(p-1)} = Y_{n+1,p}$. We shall show that $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to the ball $\mathbb{B}^{(p-1)n+\alpha}$, $\alpha = 0, \ldots, p-2$, and $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to the sphere $\mathbb{S}^{(p-1)n+\alpha-1}$, $\alpha =$

$0, \ldots, p-1$, thus in particular that $|Y_{n+1,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)(n+1)-1}$. We shall use induction on α.

Let us consider the case $\alpha = 0$. Clearly, $\overline{Y}_{n,p}^{(0)}$ is an $x_{n+1,n+1}^{0}$-cone over $Y_{n,p}$. Hence, by Lemma 4.6, $|\overline{Y}_{n,p}^{(0)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n}$ since $|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$. $|Y_{n,p}^{(0)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$ since $Y_{n,p}^{(0)} = Y_{n,p}$.

Given α, $0 \le \alpha \le p-3$, assume that $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$. Clearly, $\overline{Y}_{n,p}^{(\alpha+1)}$ is an $x_{n+1,n+1}^{\alpha+1}$-cone over $\overline{Y}_{n,p}^{(\alpha)}$. Hence, by Lemma 4.6, $|\overline{Y}_{n,p}^{(\alpha+1)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha+1}$ since $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$. Thus, we get that $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$ for all $\alpha = 0, \ldots, p-2$.

Now, given α, $0 \le \alpha \le p-2$, assume that $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$. Let K be a subcomplex of $Y_{n,p}^{(\alpha+1)}$ with the same set of vertices and such that T is the skeleton of a simplex of K if and only if

$$\left| T \cap \left\{ x_{n+1,n+1}^{j} : j = 0, \ldots, \alpha \right\} \right| \le \alpha.$$

We claim that $|K|$ is homeomorphic to the ball $\mathbb{B}^{(p-1)n+\alpha}$. Indeed, K is an $x_{n+1,n+1}^{\alpha+1}$-cone over $Y_{n,p}^{(\alpha)}$. Thus, by Lemma 4.6, $|K|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$ since $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$.

Now, observe that

$$Y_{n,p}^{(\alpha+1)} = \overline{Y}_{n,p}^{(\alpha)} \cup K,$$

and

$$\overline{Y}_{n,p}^{(\alpha)} \cap K = Y_{n,p}^{(\alpha)}.$$

Thus, by Lemma 4.7, $|Y_{n,p}^{(\alpha+1)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha}$ since $|\overline{Y}_{n,p}^{(\alpha)}|$ and $|K|$ are both homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$ and $|\overline{Y}_{n,p}^{(\alpha)} \cap K|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$. Therefore, $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$ for all $\alpha = 0, \ldots, p-1$ and the lemma is proved. □

By using Lemmas 4.6 and 4.7, it is straightforward to verify that $|X_{n,p}|$ is homeomorphic to the topological space obtained by identifying the boundaries of p disjoint copies of the ball $\mathbb{B}^{(p-1)n}$.

Let us now define a free Z_p-action ω_n on the complex $Y_{n,p}$. Assume that $y \in |Y_{n,p}|$ has the following barycentric representation:

$$y = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_i^j x_{n,i}^j.$$

Then set

$$\omega_n(y) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_i^{(j+1) \bmod p} x_{n,i}^j.$$

Note that if $x_{n,i}^j$ is a vertex of $Y_{n,p}$, then

$$\omega_n(x_{n,i}^j) = x_{n,i}^{(j-1) \bmod p}.$$

The map ω_n is clearly a Z_p-action; moreover we have the following lemma.

Lemma 4.9. *The map ω_n is a free action.*

Proof. Since p is a prime, it is enough to show that $\omega_n(y) \neq y$ for all $y \in |Y_{n,p}|$. Suppose there is a $y \in |Y_{n,p}|$ such that $\omega_n(y) = y$. Let T be the skeleton of the simplex Δ containing y, and let $T_{n,i}$ have a nonempty intersection with T. By the definition of $Y_{n,p}$, $T \cap T_{n,i}$ has at most $p-1$ elements. Since p is prime, $\omega(T) \cap T_{n,i} = \omega(T \cap T_{n,i}) \neq T \cap T_{n,i}$. Hence the simplices of $Y_{n,p}$ containing y and $\omega(y)$ are different. This contradiction completes the proof of the lemma. □

Note that ω_{n+1} restricted to the complex $X_{n,p}$ is a Z_p-free action on $X_{n,p}$. In the sequel, we shall drop the subscript from ω_n when the domain is clear from the context.

Let M_1 and M_2 be two metric spaces and let $\alpha_1, \alpha_2 : M_1 \to M_2$ be continuous maps. If

$$H : M_1 \times [0,1] \to M_2$$

is a continuous map such that

$$H(x, 0) = \alpha_1(x)$$

and

$$H(x, 1) = \alpha_2(x),$$

for all $x \in M_1$, then we say that H is a *homotopy* from α_1 to α_2. If there is a homotopy from α_1 to a constant map, then we say that α_1 is *null homotopic*. If θ is a free action on the sphere $\mathbb{S}^k$ and α is a map from $\mathbb{S}^k$ to $\mathbb{S}^k$, then we say that α is *equivariant* with respect to θ if $\alpha \circ \theta = \theta \circ \alpha$. The following lemma ([43] Theorem 8.3, p.42, and [7] Lemma 2) will be needed in the proof of the main result of this section, Lemma 4.11.

Lemma 4.10. *Suppose that $k \geq 1$, $p \geq 2$, and we are given a free Z_p-action on the sphere $\mathbb{S}^k$. Then there is no equivariant map $\alpha : \mathbb{S}^k \to \mathbb{S}^k$ which is null homotopic.* □

The following lemma is analogous to the generalization of the Borsuk-Ulam antipodal theorem due to Bárány, Shlosman and Szücs. The difference is in the definition of the action ω, and the proof given here is more elementary as well.

Lemma 4.11. *For any continuous map $h : |X_{n,p}| \to \mathbb{R}^n$, there exists an $x \in |X_{n,p}|$ such that $h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x))$.*

Proof. Suppose there is a continuous map $h : |X_{n,p}| \to \mathbb{R}^n$ such that for no $x \in |X_{n,p}|$ we have $h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x))$. We shall get a contradiction with Lemmas 4.8 and 4.10 by obtaining a map $\alpha : |Y_{n,p}| \to |Y_{n,p}|$ equivariant with respect to ω and null homotopic.

Let us first define a map $f : |X_{n,p}| \to |Y_{n,p}|$. For $x \in |X_{n,p}|$, assume

$$h(x) = (r_1^0, \ldots, r_n^0),$$
$$h(\omega(x)) = (r_1^1, \ldots, r_n^1),$$
$$\vdots$$
$$h(\omega^{p-1}(x)) = (r_1^{p-1}, \ldots, r_n^{p-1}).$$

For $i = 1, \ldots, n$, set $r_i = \min\{r_i^0, \ldots, r_i^{p-1}\}$ and let

$$r = \sum_{i=1}^{n} \sum_{j=0}^{p-1} (r_i^j - r_i).$$

By our assumption about h, $r > 0$ and hence we can set $s_i^j = (r_i^j - r_i)/r$. Let $f(x)$ be defined as follows:

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^j x_{n,i}^j.$$

Since for all i and j, $1 \leq i \leq n$, $0 \leq j \leq p-1$, we have $s_i^j \geq 0$ and

$$\sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^j = 1,$$

to show that $f(x) \in |Y_{n,p}|$ it is clearly enough to show that

$$T = \{x_{n,i}^j : 1 \leq i \leq n,\ 0 \leq j \leq p-1,\ s_i^j > 0\}$$

is the skeleton of a simplex of $Y_{n,p}$. But we indeed have that $|T \cap T_{n,i}| \leq p-1$ for every $i = 1, 2, \ldots, n$, since r_i is one of $r_i^0, \ldots, r_i^{p-1}$ and hence at least one of $s_i^0, \ldots, s_i^{p-1}$ must be equal to 0.

Let α be the restriction of f to $|Y_{n,p}|$. We shall show that α is equivariant with respect to ω. Let $x \in |Y_{n,p}|$ and assume that $\alpha(x) \in |Y_{n,p}|$ has the following barycentric representation:

$$\alpha(x) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^j x_{n,i}^j.$$

By the definition of ω, we have that

$$\omega(\alpha(x)) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^{(j+1) \bmod p} x_{n,i}^j. \tag{7}$$

Assume that $\alpha(\omega(x)) \in |Y_{n,p}|$ has the following barycentric representation:

$$\alpha(\omega(x)) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \overline{s}_i^j x_{n,i}^j. \tag{8}$$

From the definition of f it follows that

$$\overline{s}_i^j = s_i^{(j+1) \bmod p},$$

for $i = 1, \ldots, n$ and $j = 0, \ldots, p-1$. Therefore (7) and (8) imply that $\omega(\alpha(x)) = \alpha(\omega(x))$, and α is equivariant with respect to ω.

To finish the proof of Lemma 4.11, it is enough to show that α is null homotopic. We shall define a homotopy from α to a constant map using the extension f of α. Let $H : |Y_{n,p}| \times [0,1] \to |Y_{n,p}|$ be defined as follows. For $y \in |Y_{n,p}|$ with the following barycentric representation

$$y = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_i^j x_{n,i}^j,$$

and $t \in [0,1]$, set

$$H(y,t) = f\left[\sum_{i=1}^{n} \sum_{j=0}^{p-1} (1-t)\mu_i^j x_{n+1,n+1}^j + t x_{n+1,i}^0\right].$$

Thus

$$H(y,0) = f(y) = \alpha(y),$$

and

$$H(y,1) = f(x_{n+1,n+1}^0)$$

for all $y \in |Y_{n,p}|$. So H is a homotopy from α to a constant map proving that α is null homotopic. □

§4.4. Proof of the Generalization of the Borsuk-Ulam Theorem

In this section we are going to prove Theorem 4.2. We shall define an equivariant map

$$\zeta : |X_{m,p}| \to \mathbb{S}_p^{m(p-1)}$$

and apply Lemma 4.10.

Given a positive integer m, let $q = (p-1)m+1$ and let $Z_{m,p}$ be the subcomplex of the simplicial complex $Y_{q,p}$ such that T is the skeleton of a simplex of $Z_{m,p}$ if and only if

$$|T \cap T_{q,i}| \leq 1$$

for every $i = 1, \ldots, q$. It is clear that if we restrict the action ω on $Y_{q,p}$ to $Z_{m,p}$, we get a free Z_p-action on $Z_{m,p}$. We shall denote it also by ω.

We shall define the function ζ as the composition of two equivariant maps

$$\gamma : |X_{m,p}| \to |Z_{m,p}|,$$

and

$$g : |Z_{m,p}| \to \mathbb{S}_p^{m(p-1)}.$$

The map g is easy to define because there is a straightforward equivariant map from $|Z_{m,p}|$ to $\mathbb{S}_p^{m(p-1)}$ which happens to be a homeomorphism. The hard part is to define the function γ.

Lemma 4.12. *There exists a homeomorphism*

$$g : |Z_{m,p}| \to \mathbb{S}_p^{m(p-1)}$$

which is equivariant with respect to the action ω on $|Z_{m,p}|$ and ω on $\mathbb{S}_p^{m(p-1)}$.

Proof. The map g we are to define has to satisfy $g \circ \omega = \omega \circ g$ where ω acts on $|Z_{m,p}|$ on the left-hand side and on $\mathbb{S}_p^{m(p-1)}$ on the right-hand side. Let $x \in |Z_{m,p}|$ have the following barycentric representation with respect to $Z_{m,p}$:

$$x = \sum_{i=1}^{q} \sum_{j=0}^{p-1} \mu_{i,j} x_{q,i}^{j}.$$

It follows from the definition of $Z_{m,p}$ that for every i, $1 \le i \le q$, there is at most one j, $0 \le j \le p-1$, such that $\mu_{i,j} > 0$. Set

$$g(x) = \Big((\mu_{1,j_1}, j_1), (\mu_{2,j_2}, j_2), \dots, (\mu_{q,j_q}, j_q) \Big) \in \mathbb{R}^q_{+,p}$$

where j_i, $1 \le i \le q$, is such that $\mu_{i,j} = 0$ for all $j \ne j_i$, $0 \le j \le p-1$.

Since

$$\sum_{i=1}^{q} \mu_{i,j_i} = 1,$$

we have $g(x) \in \mathbb{S}_p^{m(p-1)}$. It is straightforward to verify that g is a homeomorphism and that $g \circ \omega = \omega \circ g$. Thus the lemma is proved. □

Before we define the function γ, we need some more preliminary lemmas. Given a prime p, let

$$P = 2^{[0,p-1]} \setminus \{\emptyset, [0, p-1]\}$$

be the set of all subsets of $[0, p-1] \subset \mathbb{N}$ which are nonempty and different from $[0, p-1]$.

Let $\eta : [0, p-1] \to [0, p-1]$ be the function defined in Section 4.1; $\eta(i) = (i+1)$ mod p and let $\Theta : P \to P$ be defined by

$$\Theta(A) = \{\eta(a) : a \in A\}.$$

We are going to define a function $\varphi : P \to [0, p-1]$ satisfying

$$\varphi(\Theta(A)) = \eta(\varphi(A)).$$

If $A \in P$, then set

$$\xi(A) = \sum_{i \in A} 2^i,$$

and let

$$B_A = \left\{ \xi(A), \xi(\Theta(A)), \xi(\Theta^2(A)), \dots, \xi(\Theta^{p-1}(A)) \right\}.$$

The following lemma holds.

Lemma 4.13. *B_A contains p different numbers.*

Proof. Suppose that $\xi(\Theta^j(A)) = \xi(\Theta^{j+k}(A))$ and $1 \le k \le p-1$. Since p is a prime, k is relatively prime to p, and hence $\xi(A) = \xi(\Theta(A))$. But this is possible only when $A = \emptyset$ or $A = [0, p-1]$. Since $1 \le |A| \le p-1$, the resulting contradiction finishes the proof of the lemma. □

We can now define φ. Let

$$\varphi(A) = \eta^{-j}(\max(\Theta^j(A)))$$

where j is such that

$$\xi(\Theta^j(A)) = \max(B_A).$$

By Lemma 4.13, φ is well defined; also we have the folowing lemma.

Lemma 4.14. *The function φ is such that for all $A \in P$ we have*

$$\varphi(\Theta(A)) = \eta(\varphi(A)).$$

Proof. We have

$$\varphi(A) = \eta^{-j}(\max(\Theta^j(A)))$$

where j satisfies

$$\xi(\Theta^j(A)) = \max(B_A).$$

We also have

$$\varphi(\Theta(A)) = \eta^{-j'}(\max(\Theta^{j'}(\Theta(A))))$$

where j' satisfies

$$\xi(\Theta^{j'}(\Theta(A))) = \max(B_{\Theta(A)}).$$

Since $B_{\Theta(A)} = B_A$, we have $j' = (j-1) \bmod p$ and hence

$$\varphi(\Theta(A)) = \eta^{-j+1}(\max(\Theta^j(A))) = \eta(\varphi(A)).$$

Thus the proof of the lemma is complete. □

If K is a simplicial complex, then the *barycentric subdivision* K' of K is the simplicial decomposition of $|K|$ obtained as follows. For a simplex $\Delta = (x_0, \ldots, x_k) \in K$, let

$$c_\Delta = \frac{1}{k+1} \sum_{i=0}^{k} x_i$$

be the *barycentre* of Δ. Let K' consist of all simplices $(c_{\Delta_0}, \ldots, c_{\Delta_k})$ such that $\Delta_i \in K$, $i = 0, 1, \ldots, k$, and Δ_i is a proper face of Δ_{i+1}, $i = 0, 1, \ldots, k-1$.

We are now going to define the *quasi barycentric subdivision* $X'_{m,p}$ of $X_{m,p}$. Let $\mathcal{A}_i$, $1 \leq i \leq m+1$, be the set of simplices Δ of $X_{m,p}$ such that the vertices of Δ are contained in $T_{m+1,i}$. Let

$$\left\{ c_\Delta : \Delta \in \bigcup_{i=1}^{m+1} \mathcal{A}_i \right\}$$

be the set of vertices of $X'_{m,p}$ where c_Δ is the barycentre of Δ. Let T be the skeleton of a simplex of $X'_{m,p}$ if and only if for every i, $1 \leq i \leq m+1$, we have

$$\{c_\Delta \in T : \Delta \in \mathcal{A}_i\} = \{c_{\Delta_0}, \ldots, c_{\Delta_k}\}$$

where Δ_i is a proper face of Δ_{i+1}, $i = 0, 1, \ldots, k-1$. It is straightforward to verify the following lemma.

Lemma 4.15. *$X'_{m,p}$ is a simplicial decomposition of $|X_{m,p}|$.* □

We shall define $\gamma : |X_{m,p}| \to |Z_{m,p}|$ on the vertices of $X'_{m,p}$ first. The map γ restricted to the vertices of $X'_{m,p}$ will take its values in the set of vertices of $Z_{m,p}$. Given a vertex c_Δ of $X'_{m,p}$, let i, $1 \leq i \leq m+1$, be such that $\Delta \in \mathcal{A}_i$. Let T be the skeleton of Δ and

$$A = \left\{ j : 0 \leq j \leq p-1,\ x^j_{m+1,i} \in T \right\}.$$

By the definition of $X_{m,p}$, we have

$$1 \leq |A| \leq p-1$$

if $1 \le i \le m$, and

$$|A| = 1$$

if $i = m + 1$. Set

$$\gamma(c_\Delta) = x_{q,(p-1)(i-1)+|A|}^{\varphi(A)} .$$

We shall now show that γ maps skeletons of simplices of $X'_{m,p}$ to skeletons of simplices of $Z_{m,p}$.

Lemma 4.16. *If $(c_{\Delta_0}, \ldots, c_{\Delta_k})$ is a simplex of $X'_{m,p}$, then $(\gamma(c_{\Delta_0}), \ldots, \gamma(c_{\Delta_k}))$ is a simplex of $Z_{m,p}$.*

Proof. Assume that $(c_{\Delta_0}, \ldots, c_{\Delta_k})$ is a simplex of $X'_{m,p}$ and

$$\gamma(c_{\Delta_i}) = x_{q,r_i}^{a_i}$$

for $i = 0, \ldots, k$. By the definition of $Z_{m,p}$, to prove that $(x_{q,r_0}^{a_0}, \ldots, x_{q,r_k}^{a_k})$ is a simplex of $Z_{m,p}$ we have to show that all $r_0, \ldots, r_k$ are distinct. Suppose $r_j = r_\ell$ and $0 \le j \le \ell \le k$. There is exactly one i and one s, $1 \le i \le m+1$, $1 \le s \le p-1$, such that

$$r_j = r_\ell = (p-1)(i-1) + s.$$

Hence, by the definition of γ, we have

$$c_{\Delta_j}, c_{\Delta_\ell} \in \mathcal{A}_i$$

and

$$|T_j| = |T_\ell| = s$$

where T_j and T_ℓ are skeletons of Δ_j and Δ_ℓ respectively. This contradicts the definition of $X'_{m,p}$ since, according to this definition, Δ_j is a proper face of Δ_ℓ. Thus the lemma is proved. □

We now extend γ linearly to $|X'_{m,p}|$. If $x \in (c_{\Delta_0}, \ldots, c_{\Delta_k}) \in X'_{m,k}$ has the following barycentric representation

$$x = \sum_{i=0}^{k} \mu_i c_{\Delta_i},$$

then let

$$\gamma(x) = \sum_{i=0}^{k} \mu_i \gamma(c_{\Delta_i}).$$

Lemma 4.17. *The map γ is equivariant with respect to ω.*

Proof. We have to show that for every $x \in |X_{m,p}|$ we have

$$\gamma(\omega(x)) = \omega(\gamma(x)).$$

It is enough to prove this equality for x being a vertex of $X'_{m,p}$. Let $x = c_\Delta$ be the barycentre of $\Delta \in \mathcal{A}_i$, let T be the skeleton of Δ and set

$$A = \left\{ j : 0 \leq j \leq p-1,\ x^j_{m+1,i} \in T \right\}.$$

If $\omega(c_\Delta) = c_{\Delta'}$, then by the definition of ω, we have $\Delta' \in \mathcal{A}_i$. If T' is the skeleton of Δ', then

$$\left\{ j : 0 \leq j \leq p-1,\ x^j_{m+1,i} \in T' \right\} = \Theta(A).$$

Therefore we have

$$\gamma(\omega(x)) = x^{\varphi(\Theta(A))}_{q,(p-1)(i-1)+|\Theta(A)|}.$$

Since $|\Theta(A)| = |A|$ and $\varphi(\Theta(A)) = \eta(\varphi(A))$, we have

$$\gamma(\omega(x)) = x^{\eta(\varphi(A))}_{q,(p-1)(i-1)+|A|}.$$

We also have

$$\omega(\gamma(x)) = x^{\eta(\varphi(A))}_{q,(p-1)(i-1)+|A|},$$

so

$$\gamma(\omega(x)) = \omega(\gamma(x)),$$

as required. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let $h : \mathbb{S}_p^{m(p-1)} \to \mathbb{R}^m$ be a continuous function. Let $g : |Z_{m,p}| \to \mathbb{S}_p^{m(p-1)}$ be a homeomorphism satisfying $g \circ \omega = \omega \circ g$ (see Lemma 4.12). Let us consider the function

$$h \circ g \circ \gamma : |X_{m,p}| \to \mathbb{R}^m.$$

By Lemma 4.11 there exists $y \in |X_{m,p}|$ satisfying

$$h \circ g \circ \gamma(y) = h \circ g \circ \gamma(\omega(y)) = \ldots = h \circ g \circ \gamma(\omega^{p-1}(y)).$$

Let $x = g \circ \gamma(y)$. Since $g \circ \gamma(\omega(y)) = \omega(x)$, we have

$$h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x)),$$

and the proof of Theorem 4.2 is complete. □

§4.5. Concluding remark

Although our proof of Theorem 4.1 is much more combinatorial than the original one given by Alon [4], it is still based upon a result from algebraic topology. It would be desirable to find a purely combinatorial proof. Probably the way to give such a proof would be to find a purely combinatorial proof of our generalization of the Borsuk-Ulam antipodal theorem (Theorem 4.2). Recall that the Borsuk-Ulam theorem has a purely combinatorial proof which perhaps could be generalized.