CHAPTER 1

A LOWER BOUND FOR SNAKE-IN-THE-BOX CODES

§1.1. Introduction

Given a natural number d, let the *d-dimensional cube* $I[d]$ be the graph defined as follows. Let the vertex set of $I[d]$ be the set of all d-tuples of binary digits, and for vertices $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ of $I[d]$, let (x, y) be an edge of $I[d]$ if x and y differ in exactly one coordinate.

Let G_1 and G_2 be graphs. We say that G_1 is an *induced subgraph* of G_2 if G_1 is a subgraph of G_2, and if for every pair of vertices x and y of G_1 such that (x, y) is an edge of G_2, (x, y) is also an edge of G_1.

A *snake-in-the-box code*, or a *snake*, is an induced cycle in $I[d]$ for a certain integer d. For each $d \in \mathbb{N}$, let $S(d)$ be the length of the longest snake in $I[d]$.

Snakes were introduced by Kautz [37] as a type of error-checking code for a certain analogue-to-digital conversion system. Consider the following problem. We want to encode the position of a rotating wheel using ordered d-tuples of binary digits in such a way that a small error resulting in changing one digit does little harm. Thus, we would like to partition the circle which is the boundary of the wheel into many segments of equal length, assigning a d-tuple of binary digits to each of them so that the following conditions are satisfied:

(i) different d-tuples are assigned to different segments,

(ii) if a d-tuple x is assigned to a segment A, then any d-tuple differing in one coordinate from x is either assigned to one of the neighbours of A or is not assigned to any segment at all, thus allowing the error to be detected.

Ideally then, we would use a snake of length $S(d)$ for the encoding. Clearly, we then want to bound $S(d)$ from below (preferably by giving an explicit construction).

Let us first review the lower bounds for $S(d)$ that have already been obtained. Kautz [37] showed that

$$S(d) \geq \lambda\sqrt{2^d}.$$

This bound was later improved by Ramanujacharyulu and Menon [46], who proved that

$$S(d) \geq (3/2)^d,$$

whilst Brown (unpublished, quoted by Danzer and Klee [15]) and Singleton [51] got

$$S(d) \geq \lambda(\sqrt[4]{6})^d.$$

Abbott [1] obtained

$$S(d) \geq \lambda(\sqrt{5/2})^d,$$

and later Vasil'ev [58] showed that

$$S(d) \geq \frac{2^d}{d} \qquad \text{when } d \text{ is a power of 2,}$$

and further,

$$S(d) \geq (1-\varepsilon(d))\frac{2^{d-1}}{d} \qquad \text{with } \varepsilon(d) \to 0 \text{ as } d \to \infty.$$

Finally, Danzer and Klee [15] proved that

$$S(d) \geq \frac{2^{d+1}}{d} \qquad \text{when } d \text{ is a power of 2,}$$

and

$$S(d) \geq \frac{7}{4}\frac{2^d}{d-1} \qquad \text{for all } d \geq 5.$$

In this chapter we shall prove a linear lower bound for $S(d)$, namely

$$S(d) \geq \frac{9}{64}2^d;$$

see also [60].

§1.2. The Main Lemma

Our aim in this section is to state and prove Lemma 1.3, which will provide a construction of long snakes, leading to the proof of the lower bound stated in the introduction to this chapter.

Let us first introduce some notation. If F is a subgraph of $I[d]$, then let us denote by $F^{(0)}$ the subgraph of $I[d+1]$ obtained as the image of F under the embedding

$$\psi_0 : I[d] \to I[d+1]$$

such that

$$\psi_0[(v^1, \ldots, v^d)] = (v^1, \ldots, v^d, 0).$$

Analogously, let $F^{(1)}$ be the image of F under

$$\psi_1 : I[d] \to I[d+1]$$

such that

$$\psi_1[(v^1, \ldots, v^d)] = (v^1, \ldots, v^d, 1).$$

For each $d \geq 2$, let

$$R_d : [2^d + 1, 2^{d+1}] \to [2^d]$$

be the order reversing bijection, i.e. let

$$R_d(i) = 2^{d+1} + 1 - i.$$

Now, for each $d \geq 2$, we shall define a function

$$H_d : [2^d] \to V(I[d])$$

such that if

$$\overline{H_d} = (H_d(1), \ldots, H_d(2^d), H_d(1)),$$

then $\overline{H_d}$ is a Hamiltonian cycle in $I[d]$. Set

$$\overline{H_2} = ((0,0), (0,1), (1,1), (1,0), (0,0)),$$

and

$$H_{d+1}(i) = \begin{cases} (H_d(i))^{(0)} & \text{if } 1 \le i \le 2^d, \\ (H_d \circ R_d(i))^{(1)} & \text{if } 2^d + 1 \le i \le 2^{d+1}. \end{cases} \tag{1}$$

In other words, $\overline{H_{d+1}}$ is obtained by taking $\overline{H_d}^{(0)}$ and $\overline{H_d}^{(1)}$, removing the edges connecting their last vertices with their first vertices, joining the first vertex of $\overline{H_d}^{(0)}$ with the first vertex of $\overline{H_d}^{(1)}$, and analogously the last with the last.

The Hamiltonian cycle $\overline{H_d}$ will be used later to construct a snake in $I[d+i]$ for a certain integer i, so we are interested in describing when an edge of $I[d]$ is not an edge of $\overline{H_d}$. The following lemma gives such a description in an inductive way.

Lemma 1.1. *For each $d \ge 2$, if $1 \le i < j \le 2^{d+1}$ and*

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]) \setminus E(\overline{H_{d+1}}),$$

then exactly one of the following conditions holds:

(i) $1 \le i < j \le 2^d$, $(i,j) \ne (1, 2^d)$ *and* $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$,

(ii) $i = 1$ *and* $j = 2^d$,

(iii) $2^d + 1 \le i < j \le 2^{d+1}$, $(i,j) \ne (2^d+1, 2^{d+1})$ *and* $(H_d \circ R_d(i), H_d \circ R_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$,

(iv) $i = 2^d + 1$ *and* $j = 2^{d+1}$,

(v) $2 \le i \le 2^d - 1$ *and* $i = R_d(j)$.

Proof. It is obvious that the conditions (i)–(v) are mutually exclusive, so assuming that

$$(H_{d+1}(i), H_{d+1}(j)) \in E(i[d+1]) \setminus E(\overline{H_{d+1}}),$$

it is enough to show that one of them is satisfied. If the last coordinates of $H_{d+1}(i)$ and $H_{d+1}(j)$ are both equal to 0, then by (1) we have $1 \le i < j \le 2^d$. If (ii) does not hold, then it follows from (i) that $(H_d(i), H_d(j))$ is not an edge of $\overline{H_d}$ and thus (i) is satisfied.

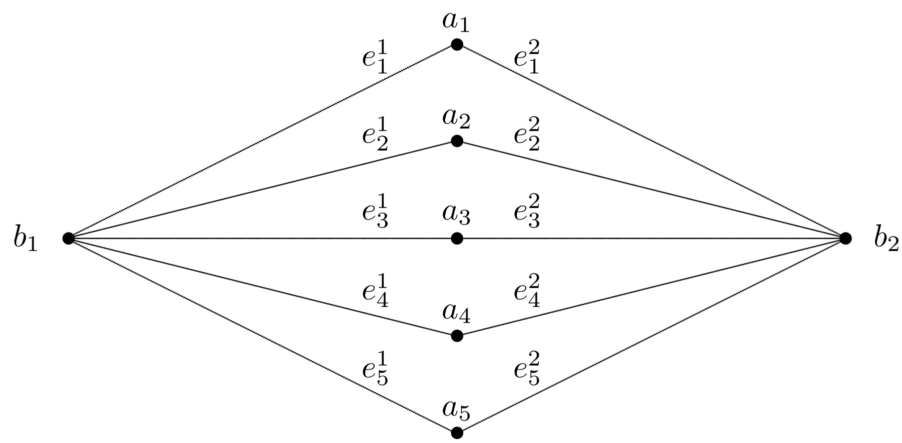

Fig. 1. The graph G

If the last coordinates of $H_{d+1}(i)$ and $H_{d+1}(j)$ are both equal to 1, then (1) above implies that $2^d + 1 \leq i < j \leq 2^{d+1}$; similarly as above we conclude that either (iii) or (iv) is satisfied.

The remaining case to consider is when the last coordinate of $H_{d+1}(i)$ is equal to 0 and the last coordinate of $H_{d+1}(j)$ is equal to 1. Then it follows from (1) that $1 \leq i \leq 2^d$ and $2^d + 1 \leq j \leq 2^{d+1}$. Since

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]),$$

$H_{d+1}(i)$ and $H_{d+1}(j)$ differ only at the last coordinate, and thus (1) implies that

$$H_d(i) = H_d \circ R_d(j).$$

Since H_d is a bijection, we have $i = R_d(j)$, and because

$$(H_{d+1}(i), H_{d+1}(j)) \notin E(\overline{H_{d+1}}),$$

we get $2 \leq i \leq 2^d - 1$. Thus (v) is satisfied, and the lemma is proved. □

Let G be the complete bipartite graph $K_{2,5}$, with the vertex set $A \cup B$ where $A = \{a_1, \ldots, a_5\}$ and $B = \{b_1, b_2\}$, and the edge set $E_1 \cup E_2$ where $E_1 = \{e_i^1 = (a_i, b_1) : 1 \leq i \leq 5\}$ and $E_2 = \{e_i^2 = (a_i, b_2) : 1 \leq i \leq 5\}$ (see Fig. 1).

We shall use the graph G in our construction of a snake. In order to present the construction, let us introduce some operations on the set of edges of G. Let S_5 be the set of permutations on the set $[1,5]$. Given a permutation $\sigma \in S_5$, let φ_σ and φ_σ^+ be permutations on the set of edges of G such that we have

$$\varphi_\sigma(e_i^j) = e_{\sigma(i)}^j,$$

and

$$\varphi_\sigma^+(e_i^j) = e_{\sigma(i)}^{3-j},$$

for $i = 1, \ldots, 5$ and $j = 1, 2$. Note that φ_σ permutes the edges in E_1 and E_2 by permuting their endpoints belonging to A according to σ, and φ_σ^+ also transposes their endpoints belonging to B. Therefore, for each $e, e' \in E(G)$ and $\sigma \in S_5$, the edges e and e' have the same number of vertices in common as the edges $\varphi_\sigma(e)$ and $\varphi_\sigma(e')$, and the same as the edges $\varphi_\sigma^+(e)$ and $\varphi_\sigma^+(e')$. If e and e' have one vertex in common, then it belongs to A if and only if the common vertex of $\varphi_\sigma(e)$ and $\varphi_\sigma(e')$ belongs to A, and if and only if the common vertex of $\varphi_\sigma^+(e)$ and $\varphi_\sigma^+(e')$ belongs to A.

Let us consider the following permutations of the set $[1,5]$.

$$\sigma_1 = (3\ 5),$$
$$\sigma_2 = (1\ 3)(2\ 4\ 5),$$
$$\sigma_3 = (1\ 2)(4\ 5).$$

We have the folowing lemma.

Lemma 1.2. *For each $e \in E(G)$ the edges $\varphi_{\sigma_1}(e)$ and $\varphi_{\sigma_2}^+(e)$ are vertex–disjoint, and the edges $\varphi_{\sigma_1}(e)$ and $\varphi_{\sigma_3}^+(e)$ are vertex–disjoint.*

Proof. By symmetry it is enough to prove the lemma for $e \in E_1$. The following table shows all possible cases:

e	$\varphi_{\sigma_1}(e)$	$\varphi^+_{\sigma_2}(e)$	$\varphi^+_{\sigma_3}(e)$
e^1_1	e^1_1	e^2_3	e^2_2
e^1_2	e^1_2	e^2_4	e^2_1
e^1_3	e^1_5	e^2_1	e^2_3
e^1_4	e^1_4	e^2_5	e^2_5
e^1_5	e^1_3	e^2_2	e^2_4

It is clear that in every row of the above table the edge in the second column is vertex disjoint from the edges in both the third and fourth columns. Thus the lemma is proved. □

Now we can state our key lemma. We claim in it the existence, for each $d \geq 2$, of a closed walk of length 2^d in G which will provide a construction of long snakes. We shall 'combine' our walk with the cycle $\overline{H_d}$ in an appropriate way. The walk will start from a vertex belonging to the set $\{a_1, \ldots, a_5\}$, will not use any edge twice in turn and will possess the following property with respect to the Hamiltonian cycle $\overline{H_d}$; if we regard this walk and the Hamiltonian cycle $\overline{H_d}$ as sequences of length 2^d, the walk as a sequence of edges, and $\overline{H_d}$ as a sequence of vertices, then any two edges corresponding to two nonconsecutive vertices of $\overline{H_d}$ which are neighbours in $I[d]$ will be vertex–disjoint.

Lemma 1.3. *For every $d \geq 2$ there is a function $\Phi_d : [2^d] \to E(G)$ such that*

(i) if $1 \leq i \leq 2^d - 1$, $j = i + 1$, or $i = 2^d$, $j = 1$, then $\Phi_d(i)$ and $\Phi_d(j)$ have exactly one vertex in common, and

(ii) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then $\Phi_d(i)$ and $\Phi_d(j)$ are vertex disjoint.

Proof. To prove the lemma we shall use induction on d, and we shall prove a statement which is stronger than the lemma itself. We shall show that there are functions

$$\Phi^{k,l}_d : [2^d] \to E(G)$$

for each $d \geq 2$, and

$$(k,l) \in I = \{(1,1),(1,3),(1,4),(2,3),(2,4)\}$$

such that each of the following conditions holds:

(2) $\Phi_d^{k,l}(1) = e_k^1$ and $\Phi_d^{k,l}(2^d) = e_l^2$,

(3) if $1 \leq i \leq 2^d - 1$, then $\Phi_d^{k,l}(i)$ and $\Phi_d^{k,l}(i+1)$ have exactly one vertex v_i in common such that $v_i \in A$ for i even and $v_i \in B$ for i odd,

(4) if $2 \leq i \leq 2^d - 1$ and $(k,l) \neq (1,1) \neq (k',l')$, then $\Phi_d^{k,l}(i) = \Phi_d^{k',l'}(i)$,

(5) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then $\Phi_d^{k,l}(i)$ and $\Phi_d^{k,l}(j)$ are vertex–disjoint.

In other words, the function $\Phi_d^{k,l}$ will describe a walk in G starting from the vertex a_k and the edge e_k^1, ending in the edge e_l^2 and the vertex a_l, and having all the properties we require for the walk described by the function Φ_d, i.e. it will not use any edge twice in turn, and any of its edges corresponding to two nonconsecutive vertices of $\overline{H_d}$ which are neighbours in $I[d]$ will be vertex–disjoint. Also, given $d \geq 2$, all the walks described by $\Phi_d^{k,l}$, for $(k,l) \in I \setminus \{(1,1)\}$, will differ only at the first and the last vertices.

The construction of such functions will complete the proof of Lemma 1.3 because if we set $\Phi_d = \Phi_d^{1,1}$, then (ii) will follow from (5), and (i) will follow from (2) and (3).

Set

$$(\Phi_2^{k,l}(1), \Phi_2^{k,l}(2), \Phi_2^{k,l}(3), \Phi_2^{k,l}(4)) = (e_k^1, e_5^1, e_5^2, e_l^2).$$

If $(k,l) \neq (1,1)$, then let

$$\Phi_{d+1}^{k,l}(i) = \begin{cases} \varphi_{\sigma_1} \circ \Phi_d^{k,3}(i) & \text{if } 1 \leq i \leq 2^d, \\ \varphi_{\sigma_2}^{+} \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(i) & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

and set

$$\Phi_{d+1}^{1,1}(i) = \begin{cases} \varphi_{\sigma_1} \circ \Phi_d^{1,3}(i) & \text{if } 1 \leq i \leq 2^d, \\ \varphi_{\sigma_3}^{+} \circ \Phi_d^{2,4} \circ R_d(i) & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

where R_d is the order reversing bijection.

In the inductive construction given above, the walk w corresponding to $\Phi_{d+1}^{k,l}$, $(k,l) \neq (1,1)$, is obtained from the walks w_1 and w_2 described by $\Phi_d^{k,3}$ and $\Phi_d^{\sigma_2^{-1}(l),4}$. To obtain w, we permute the edges of w_1 with φ_{σ_1}, and the edges of w_2 with $\varphi_{\sigma_2}^+$, getting w_1' and w_2'; then we reverse the order of edges of w_2', getting w_2'', and finally we identify the last vertex of w_1' with the first vertex of w_2''.

It can be checked directly that for $d = 2$ conditions (2)–(5) are satisfied. Given $d \geq 2$, let us assume that conditions (2)–(5) are satisfied for d. We shall prove that they are satisfied for $d + 1$.

Proof of Condition (2). If $(k,l) \in I \setminus \{(1,1)\}$, then

$$\Phi_{d+1}^{k,l}(1) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(1) = \varphi_{\sigma_1}(e_k^1) = e_k^1,$$

and

$$\Phi_{d+1}^{k,l}(2^{d+1}) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(2^{d+1}) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4}(1) = \varphi_{\sigma_2}^+(e^1_{\sigma_2^{-1}(l)}) = e_l^2.$$

For $(k,l) = (1,1)$ we have

$$\Phi_{d+1}^{1,1}(1) = \varphi_{\sigma_1} \circ \Phi_d^{1,3}(1) = \varphi_{\sigma_1}(e_1^1) = e_1^1,$$

and

$$\Phi_{d+1}^{1,1}(2^{d+1}) = \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4} \circ R_d(2^{d+1}) = \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4}(1) = \varphi_{\sigma_3}^+(e_2^1) = e_1^2.$$

Thus condition (2) is satisfied for $d + 1$.

Proof of Condition (3). We have to show that if $1 \leq i \leq 2^{d+1} - 1$, then $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(i+1)$ have exactly one vertex in common, which belongs to A for i even and to B for i odd. If $i \neq 2^d$, then condition (3) follows from condition (3) of the induction hypothesis and the definition of the permutations φ_σ and φ_σ^+, for $\sigma \in S_5$. If $i = 2^d$ and $(k,l) \neq (1,1)$, then the edges

$$\Phi_{d+1}^{k,l}(2^d) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(2^d) = \varphi_{\sigma_1}(e_3^2) = e_5^2, \tag{6}$$

and

$$\Phi_{d+1}^{k,l}(i+1) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(2^d+1) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4}(2^d) = \varphi_{\sigma_2}^+(e_4^2) = e_5^1, \quad (7)$$

have the vertex a_5 in common, so (3) holds. For $(k,l) = (1,1)$, the edges

$$\Phi_{d+1}^{1,1}(i) = \varphi_{\sigma_1} \circ \Phi_d^{1,3}(2^d) = \varphi_{\sigma_1}(e_3^2) = e_5^2, \quad (8)$$

and

$$\Phi_{d+1}^{1,1}(i+1) = \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4} \circ R_d(2^d+1) = \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4}(2^d) = \varphi_{\sigma_3}^+(e_4^2) = e_5^1, \quad (9)$$

have the vertex a_5 in common also. Thus condition (3) is satisfied for $d+1$.

Proof of Condition (4). We have to show that if $2 \leq i \leq 2^{d+1} - 1$ and $(k,l) \neq (1,1) \neq (k',l')$ then $\Phi_{d+1}^{k,l}(i) = \Phi_{d+1}^{k',l'}(i)$. If $2^d \neq i \neq 2^d + 1$, then this follows from condition (4) of the induction hypothesis; otherwise by (6)

$$\Phi_{d+1}^{k,l}(2^d) = e_5^2 = \Phi_{d+1}^{k',l'}(2^d),$$

and by (7)

$$\Phi_{d+1}^{k,l}(2^d+1) = e_5^1 = \Phi_{d+1}^{k',l'}(2^d+1).$$

Thus condition (4) is satisfied for $d+1$.

Proof of Condition (5). Let us fix $i < j$ such that

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]) \setminus E(\overline{H_{d+1}}).$$

We have to show that for each $(k,l) \in I$, $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

First let us assume that $(k,l) \neq (1,1)$. By our assumption about i and j, we can apply Lemma 1.1. If condition (i) of Lemma 1.1 is satisfied, then by condition (5) of the induction hypothesis, $\Phi_d^{k,3}(i)$ and $\Phi_d^{k,3}(j)$ are vertex–disjoint. By the definition of φ_σ and φ_σ^+, for $\sigma \in S_5$, we conclude that $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

If condition (iii) of Lemma 1.1 holds, then by condition (5) of the induction hypothesis, $\Phi_d^{\sigma_2^{-1}(l),4}(R_d(i))$ and $\Phi_d^{\sigma_2^{-1}(l),4}(R_d(j))$ are vertex disjoint. So analogously to the above we conclude that $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

If condition (ii) of Lemma 1.1 holds, then $i = 1$ and $j = 2^d$. By condition (2), we have

$$\Phi_{d+1}^{k,l}(1) = e_k^1,$$

and by (6),

$$\Phi_{d+1}^{k,l}(2^d) = e_5^2.$$

If condition (iv) of Lemma 1.1 holds, then $i = 2^d + 1$ and $j = 2^{d+1}$. By (7) we have

$$\Phi_{d+1}^{k,l}(i) = e_5^1,$$

and by condition (2),

$$\Phi_{d+1}^{k,l}(j) = e_l^2.$$

In both cases the required edges are vertex–disjoint.

Assume now that condition (v) of Lemma 1.1 holds. By condition (4) of the induction hypothesis, we have

$$\Phi_d^{k,3}(i) = \Phi_d^{\sigma_2^{-1}(l),4}(i) = e,$$

for some $e \in E(G)$. Hence

$$\Phi_{d+1}^{k,l}(i) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(i) = \varphi_{\sigma_1}(e),$$

and

$$\Phi_{d+1}^{k,l}(j) = \varphi_{\sigma_2}^{+} \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(j) = \varphi_{\sigma_2}^{+}(e).$$

By Lemma 1.2, the edges $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

If $(k, l) = (1, 1)$, then the argument is exactly the same as above. We use the condition (5) of the induction hypotesis for $\Phi_d^{1,3}(i)$ and $\Phi_d^{1,3}(j)$ when condition (i) of Lemma 1.1 is satisfied, and for $\Phi_d^{2,4}(R_d(i))$ and $\Phi_d^{2,4}(R_d(j))$ when condition (iii) of Lemma 1.1 is satisfied.

If condition (ii) of Lemma 1.1 is satisfied, then $i = 1$ and $j = 2^d$. By condition (2) we have

$$\Phi_{d+1}^{1,1}(i) = e_1^1,$$

and by (8),

$$\Phi_{d+1}^{1,1}(j) = e_5^2.$$

If condition (iv) of Lemma 1.1 is satisfied, then $i = 2^d + 1$ and $j = 2^{d+1}$. By (9) we have

$$\Phi_{d+1}^{1,1}(i) = e_1^2,$$

and by condition (2),

$$\Phi_{d+1}^{1,1}(j) = e_5^1.$$

In both cases we get vertex–disjoint edges.

Finally, if condition (v) of Lemma 1.1 is satisfied, then by the condition (4) of the induction hypothesis we have

$$\Phi_{d+1}^{1,1}(i) = \varphi_{\sigma_1}(e),$$

and

$$\Phi_{d+1}^{1,1}(j) = \varphi_{\sigma_3}^{+}(e),$$

for a certain edge $e \in E(G)$. By Lemma 1.2, the edges $\Phi_{d+1}^{1,1}(i)$ and $\Phi_{d+1}^{1,1}(j)$ are vertex–disjoint, and the proof of the lemma is finished. □

§1.3. The Lower Bound

In this section we are going to give a construction of long snakes C_d in $I[d]$, for $d \geq 7$, which will allow us to prove the main result of this chapter, Theorem 1.6. The construction of C_{d+5}, for $d \geq 2$, will combine the Hamiltonian cycle $\overline{H_d}$ defined in Section 1.2 and a sequence of induced paths $P_i^d = (v_i^1, v_i^2, \ldots, v_i^{r_i})$, $i = 1, \ldots, 2^d$, in $I[5]$, which will be defined in this section. In the sequel, given d, if either $1 \leq i \leq 2^d - 1$ and $j = i + 1$, or $i = 2^d$ and $j = 1$, then we shall say that j is the *successor* of i. The paths P_i^d, $1 \leq i \leq 2^d$ will satisfy the following two conditions:

(10) if j is the successor of i, then the paths P_i^d and P_j^d have exactly the vertex $v_i^{r_i} = v_j^1$ in common,

(11) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then the paths P_i^d and P_j^d are vertex–disjoint.

Let us first assume that the induced paths P_i^d, $i = 1, \ldots, 2^d$, satisfying conditions (10) and (11) are given. Let us regard $I[d+5]$ as $I[d] \times I[5]$. To construct the induced cycle C_{d+5} in $I[d+5]$, we consider $I[d+5]$ as the d-dimensional cube $I[d]$, with each vertex being a copy of $I[5]$. Let us take the path P_j^d in the copy of $I[5]$ corresponding to the vertex $H_d(j)$ in $I[d]$, $j = 1, \ldots, 2^d$; see Figure 2 for the case $d = 3$ where the edges of the Hamiltonian cycle $\overline{H_d}$ are denoted by bolder lines. Then, let us join the vertex $v_i^{r_i}$ from the i-th copy of $I[5]$ with the vertex v_j^1 from the jth copy of $I[5]$ for all $i, j \in \{1, \ldots, 2^d\}$, such that j is a successor of i. Hence, we have

$$
\begin{aligned}
C_{d+5} = \Big(&\big(H_d(1), v_1^1\big), \big(H_d(1), v_1^2\big), \ldots, \big(H_d(1), v_1^{r_1}\big), \\
&\big(H_d(2), v_2^1\big), \big(H_d(2), v_2^2\big), \ldots, \big(H_d(2), v_2^{r_2}\big), \\
&\vdots \\
&\big(H_d(2^d), v_{2^d}^1\big), \big(H_d(2^d), v_{2^d}^2\big), \ldots, \big(H_d(2^d), v_{2^d}^{r_{2^d}}\big), \big(H_d(1), v_1^1\big) \Big).
\end{aligned}
$$

Clearly C_{d+5} is a cycle. The following lemma states that it is an induced cycle in $I[d+5]$.

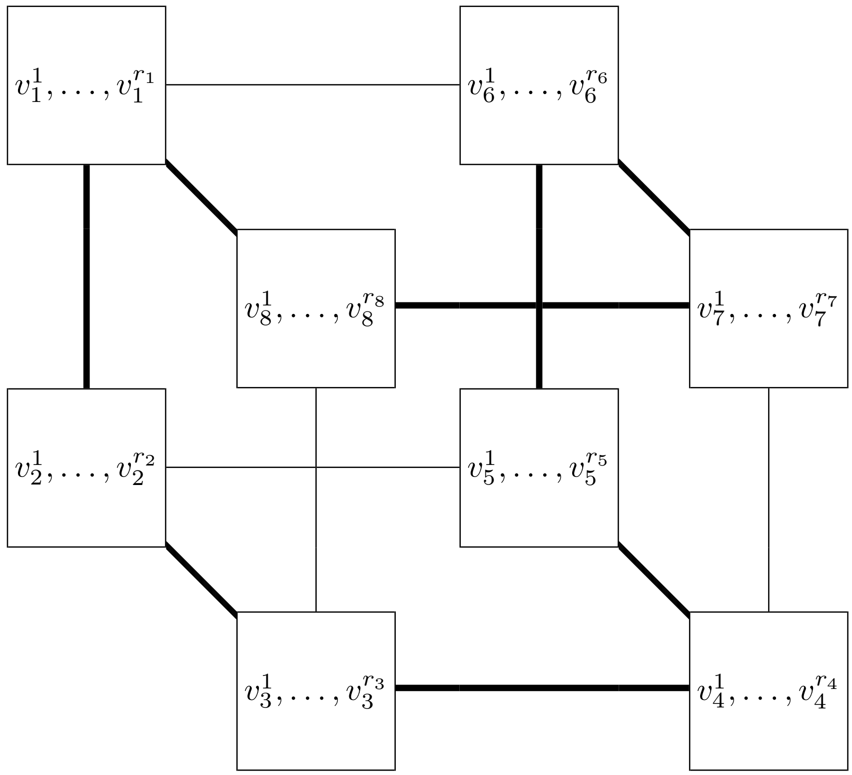

Fig. 2. The cycle C_8

Lemma 1.4. *For every $d \geq 2$, the cycle C_{d+5} is an induced subgraph of $I[d+5]$.*

Proof. Assume that

$$\big((H_d(i), v_i^k), (H_d(j), v_j^\ell)\big) \in E(I[d+5]). \tag{12}$$

To prove the lemma we shall show that

$$\big((H_d(i), v_i^k), (H_d(j), v_j^\ell)\big) \in E(C_{d+5}). \tag{13}$$

Indeed, by (12) we have either $H_d(i) = H_d(j)$ or $v_i^k = v_j^\ell$. If $H_d(i) = H_d(j)$, then since H_d is an injection, we have $i = j$. It also follows from (12) that v_i^k and v_j^ℓ are neighbours in $I[5]$. Since the path $P_i^d = P_j^d$ is an induced graph in $I[5]$, we have

$$(v_i^k, v_j^\ell) \in E(P_i^d).$$

Thus, (13) follows from the definition of C_{d+5}.

If $v_i^k = v_j^\ell$, then it follows from (12) that $H_d(i)$ and $H_d(j)$ are neighbours in $I[d]$. For supposing that

$$(H_d(i), H_d(j)) \notin E(\overline{H_d}),$$

we get

$$(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d}),$$

which contradicts (11). Thus

$$(H_d(i), H_d(j)) \in E(\overline{H_d}),$$

and we may assume that j is the successor of i. By (10), we have $k = r_i$ and $\ell = 1$, so (13) follows from the definition of C_{d+5}. Thus the lemma is proved. $\square$

Now we shall define the induced paths P_i^d, $i = 1, \dots, 2^d$, satisfying conditions (10) and (11). We shall use the graph G defined in Section 1.2 and the function Φ_d from Lemma 1.3. First let us define the following subdivision G' of G. Let G' be obtained by subdividing the edge e_k^1 of G with two new vertices c_k^1 and c_k^2 in such a way that we get the path (b_1, c_k^1, c_k^2, a_k), and subdividing the edge e_k^2 of G with three new vertices c_k^4, c_k^5 and c_k^6, giving rise to the path $(a_k, c_k^4, c_k^5, c_k^6, b_2)$, for each $k \leq 5$. To have a uniform notation, set $c_k^3 = a_k$. Let

$$\xi : V[G'] \to V(I[5])$$

be the injection defined as follows. Set

$$\xi(b_1) = (0,0,0,0,0),$$

$$\xi(c_1^1) = (1,0,0,0,0),$$

$$\xi(c_1^2) = (1,1,0,0,0),$$

$$\xi(c_1^3) = (1,1,0,1,0),$$

$$\xi(c_1^4) = (0,1,0,1,0),$$

$$\xi(c_1^5) = (0,1,1,1,0),$$

$$\xi(c_1^6) = (0,1,1,1,1),$$

$$\xi(b_2) = (1,1,1,1,1).$$

To obtain the image of ξ on c_k^j, $2 \leq k \leq 5$, $1 \leq j \leq 6$, we shall apply a cyclic permutation on the coordinates of the image of c_1^j, namely if $\xi(c_1^i) = (\alpha_1, \ldots, \alpha_5)$, then let

$$\xi(c_k^i) = (\alpha_k, \ldots, \alpha_5, \alpha_1, \ldots, \alpha_{k-1}).$$

Let the path P_i^d, $i = 1, \ldots, 2^d$, be the image under ξ of the path of G' which is the subdivision of the edge $\Phi_d(i)$, where Φ_d is the function whose existence is claimed in Lemma 1.3. Thus, if $\Phi_d(i) = (a_k, b_1)$, $1 \leq k \leq 5$, then we have

$$P_i^d = (\xi(b_1), \xi(c_k^1), \xi(c_k^2), \xi(c_k^3)),$$

and if $\Phi_d(i) = (a_k, b_2)$, $1 \leq k \leq 5$, then we have

$$P_i^d = (\xi(c_k^3), \xi(c_k^4), \xi(c_k^5), \xi(c_k^6), \xi(b_2)).$$

We can now prove the following lemma.

Lemma 1.5. *Given $d \geq 2$, the paths P_i^d, $1 \leq i \leq 2^d$, are induced graphs in $I[d]$ and they satisfy conditions (10) and (11) above.*

Proof. Assume that $\Phi_d(i) = (a_1, b_1)$. Then P_i^d is equal to the following path:

$$\big((0,0,0,0,0), (1,0,0,0,0), (1,1,0,0,0), (1,1,0,1,0)\big) \tag{14}$$

which is clearly an induced path in $I[5]$. Similarly if $\Phi_d(i) = (a_1, b_2)$, then P_i^d is equal to the folowing path:

$$\big((1,1,0,1,0), (0,1,0,1,0), (0,1,1,1,0), (0,1,1,1,1), (1,1,1,1,1)\big) \tag{15}$$

which is also an induced path in $I[5]$.

If $\Phi_d(i) = (a_k, b_\ell)$, $2 \leq k \leq 5$ and $1 \leq \ell \leq 2$, then it follows from the definition of ξ that P_i^d is obtained either from the path (14) or from (15) after applying a certain cyclic permutation on the coordinates of each vertex. Since both the paths (14) and (15) are induced in $I[5]$, then P_i^d is also an induced path in $I[5]$.

Note that to prove conditions (10) and (11), it is enough to show that for every $i, j = 1, \ldots, 2^d$ if the edge $\Phi_d(i)$ has one common vertex with the edge $\Phi_d(j)$, then the path P_i^d has one common vertex with the path P_j^d, and if the edges $\Phi_d(i)$ and $\Phi_d(j)$ are vertex–disjoint then so are the paths P_i^d and P_j^d. Indeed having established the above facts, condition (10) follows from Lemma 1.3(i) and condition (11) follows from Lemma 1.3(ii).

Now we shall prove the above observation. Assume that $1 \leq i, j \leq 2^d$ and that $\Phi_d(i)$ has one common vertex with $\Phi_d(j)$. If this common vertex is a_1 then we can assume that $\Phi_d(i) = (a_1, b_1)$ and $\Phi_d(j) = (a_1, b_2)$. Then P_i^d is equal to the path (14) and P_j^d is equal to the path (15). It is thus clear that P_i^d and P_j^d have exactly the vertex $(1,1,0,1,0)$ in common. If the common vertex of $\Phi_d(i)$ and $\Phi_d(j)$ is a_k, $2 \leq k \leq 5$, then it follows from the definition of ξ that P_i^d and P_j^d have only one vertex in common, which is obtained after a cyclic permutation of the coordinates of the vertex $(1,1,0,1,0)$.

If the common vertex of $\Phi_d(i)$ and $\Phi_d(j)$ is b_1, then it follows from the definition of ξ that the coordinates of each vertex of P_i^d are obtained by a cyclic permutation of the coordinates of the corresponding vertex of the path (14). The same is true for the vertex P_j^d, but the cyclic permutation that is applied is different. However, it is clear for every vertex of the path (14) except the first $(0,0,0,0,0)$, that two different cyclic permutations give different results. Thus P_i^d and P_j^d have only the vertex $(0,0,0,0,0)$ in common. Analogously, if the common vertex of $\Phi_d(i)$ and $\Phi_d(j)$ is b_2, then P_i^d and P_j^d have only the vertex $(1,1,1,1,1)$ in common.

Thus the lemma is proved. □

Now we are ready to prove our main result of this chapter.

Theorem 1.6. *For each $d_0 \geq 2$, the length $S(d_0)$ of the longest induced cycle in*

$I[d_0]$ *satisfies*

$$S(d_0) \geq \frac{9}{64} 2^{d_0}.$$

Proof. It is known (see [16], [27], [37]) that $S(2) = 4$, $S(3) = 6$, $S(4) = 8$, $S(5) = 14$, $S(6) = 26$, so we can assume that $d_0 \geq 7$.

Let us fix $d_0 = d + 5$, $d \geq 2$. It is clear from the definition of the paths P_i^d, $1 \leq i \leq 2^d$, that the length of P_i^d is 4 for $i \equiv 1$ or 2 mod 4, and the length of P_i^d is 5 for $i \equiv 3$ or 0 mod 4. Thus it follows from the definition of C_{d+5} that its length is equal to

$$4 \cdot 2^{d-1} + 5 \cdot 2^{d-1} = \frac{9}{64} 2^{d+5}.$$

It follows from Lemma 1.4 that C_{d+5} is an induced cycle in $I[d+5]$, so the proof of the theorem is complete. □

§1.4. Further Remarks

There is still a gap between the lower bound for the length of snake-in-the-box codes proved in this chapter and the best known upper bound. This best upper bound is due to Solov'jeva [54], who improved the following earlier bound of Deimer [17]:

$$S(d) \leq 2^{d-1} - \frac{2^{d-1}}{d(d-5)+7}, \qquad \text{for} \qquad d \geq 7.$$

by proving that

$$S(d) \leq 2^{d-1}\left(1 - \frac{2}{d^2 - d + 2}\right) \qquad \text{for} \qquad d \geq 7.$$

We believe that the upper bound can be further improved to the form of $c2^d$, where c is a constant smaller than $1/2$.

I was informed by the referees of [60] that the lower bound of the form $\lambda 2^d$ was first obtained by Evdokimov [25]. He showed that

$$S(d) \geq 2^{-11} 2^d,$$

but he stated that after certain changes in his construction, the constant λ can be increased to $2^{-9}S(8)$.

The idea of his proof is similar to that used in the present proof, in the sense that the snake in $I[d] = I[d-d_0] \times I[d_0]$ (where d_0 is a constant) is constructed in such a way that its projection on $I[d-d_0]$ is a Hamiltonian cycle. However the rest of the construction is quite different.

Glagolev and Evdokimov [33] proved a theorem about the chromatic number of a certain infinite graph, and stated that it can be used to further increase the constant λ so that $\lambda \in (\frac{3}{16}, \frac{1}{4})$. In addition, one of the referees of [60] informed me also that in his dissertation, Evdokimov [26] proved that $S(d) \geq 0.26 \cdot 2^d$.

Recently Abbot and Katchalski [2] found a completely different way of proving a lower bound of the form $\lambda 2^d$. They use induction in a way resembling the proof of Danzer and Klee [15], but they construct the so-called *accessible* snake, which is a snake with some additional paths between its vertices, which allows them to keep the induction going without decreasing the ratio of vertices used by the snake (which could not be avoided in the proof of Danzer and Klee).