# LONG INDUCED CYCLES IN THE HYPERCUBE AND COLOURINGS OF GRAPHS

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## Declaration

No part of this dissertation is derived from any other source, except where explicitly stated otherwise.

Chapter 2 presents joint work with Y. Kohayakawa. The rest of this dissertation is my own unaided work.


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# CONTENTS

# INTRODUCTION

Cycles are very simple combinatorial structures, yet there are many interesting problems concerning them. The major part of this dissertation is concerned with problems about cycles.

One very natural question about cycles goes as follows: given a graph $G$, what is the length of the longest induced cycle in $G$? In Chapter 1 we deal with this question when $G$ is the $d$-dimensional hypercube (also called the cube). Since the vertices of the $d$-dimensional cube can be considered as $d$-tuples of binary digits, a long induced cycle in the $d$-dimensional cube can be applied as a type of error-checking code as explained in the introduction to Chapter 1 (see also Kautz [37]). The main result of Chapter 1 is a very natural explicit construction of an induced cycle of length $(9/64)2^d$ in the $d$-dimensional cube.

The first lower bound for the maximal possible length of an induced cycle in the $d$-dimensional cube was given by Kautz [37]. He proved that such cycles can have length greater than $\lambda\sqrt{2^d}$, where $\lambda$ is a constant. This bound was later improved many times leading eventually to the bound given by Danzer and Klee [15], who proved the lower bound $2^{d+1}/d$ when $d$ is a power of 2, and $(7/4)2^d/(d-1)$ for all $d \geq 5$. The best upper bound at present is $2^{d-1}(1 - 2/(d^2 - d + 2))$ for $d \geq 7$, given by Solov'jeva [54].

We improve the bound of Danzer and Klee by giving a construction that can be outlined as follows. We regard the $d$-dimensional cube $I[d]$ as the $(d-5)$-dimensional cube with vertices being copies of the 5-dimensional cube. Our induced cycle visits every vertex of the $(d-5)$-dimensional cube exactly once, thus it is an 'expansion' of a Hamiltonian cycle in $I[d-5]$. The Hamiltonian cycle in $I[d-5]$ is constructed by induction in a way that allows us to give an exact and simple characterization of the pairs of vertices of $I[d-5]$ which are connected by an edge but are not consecutive in the cycle. This is crucial since for each such pair $(x, y)$ the 'expansion' of the Hamiltonian cycle in $I[d-5]$ to an induced cycle in $I[d]$ cannot use the same vertex of $I[5]$ in both of its copies corresponding to the vertices $x$ and $y$.

Having constructed the Hamiltonian cycle in $I[d-5]$ and having characterized the 'bad' edges of $I[d-5]$, we embed a certain subdivision of the complete bipartite graph $K_{2,5}$ into $I[5]$. To construct our 'expansion' of the Hamiltonian cycle, we use at each copy of $I[5]$ one of the paths obtained as the images of the edges of $K_{2,5}$.

In Chapter 2, which presents joint work with Yoshiharu Kohayakawa, we consider the following colouring problem. Let an integer $s \geq 1$ and a graph $G$ be given. Let us denote by $\chi_s(G)$ the smallest integer $\chi$ for which there exists a vertex-colouring of $G$ with $\chi$ colours such that any two distinct vertices of the same colour are at distance greater than $s$. Note that $\chi_1(G)$ is the usual chromatic number of $G$, and hence $\chi_s(G)$ is a very natural generalization of $\chi_1(G)$. Let us denote by $\omega_s(G)$ the maximal cardinality of a subset of the vertices of $G$ with diameter at most $s$. Clearly $\chi_s(G) \geq \omega_s(G)$. For $s \geq 1$ and $h \geq 0$ set $\gamma_s(G) = \chi_s(G) - \omega_s(G)$ and

$$\nu_s(h) = \max\left\{n \in \mathbb{N} : \text{for any graph } G,\ |G| < n \text{ implies } \gamma_s(G) < h\right\}.$$

Gionfriddo [30] has given estimates for $\nu_s(h)$. We improve the recent bound $\nu_2(h) \leq 6h$ $(h \geq 3)$ of Gionfriddo and Milici [31] to $\nu_2(h) \leq 5h$ $(h \geq 3)$. More generally, we give the following tight bounds for arbitrary $s \geq 1$ and large enough $h$:

$$2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2} \leq \nu_s(h) \leq 2h + h^{1-\epsilon_s},$$

where $\epsilon_s > 0$ depends only on $s$. The upper bound is proved entirely by constructive methods.

In Chapter 3 we consider a problem concerning colourings of cycles. Before we state the problem let us present some background. In 1963 Ringel [47] conjectured that for any natural number $n$ and any tree $T$ with $n$ edges, the complete graph $K_{2n+1}$ could be decomposed into $2n+1$ subgraphs isomorphic to the tree $T$. Later Kotzig (reported by Rosa [48]) strengthened Ringel's conjecture by adding a condition of cyclic symmetry on the decomposition. This Ringel-Kotzig conjecture remains open, and so does its weaker version due to Ringel.

In connection with the Ringel-Kotzig conjecture, Rosa [48] studied four classes of labellings of graphs, i.e. assignments of natural numbers to their vertices and edges satisfying the condition that the label of an edge is the absolute value of the difference of the labels of its end-points. Showing that one of Rosa's classes of labellings could be used to label all trees would prove the Ringel-Kotzig conjecture. The smallest class of Rosa's labellings for which it is still unknown whether they can be used to label all trees is the class of $\beta$-labellings, also called graceful labellings. The condition for a labelling of a graph with $n$ edges to be graceful is that the labels of its vertices should be distinct elements of the set $[0, n] \subset \mathbb{N}$ and that the labels of its edges should be distinct elements of $[1, n] \subset \mathbb{N}$.

Since the conjecture whether all trees can be labelled gracefully has proved to be very difficult, Bloom [10] defined an analogous notion, namely that of minimally $k$-equitable labellings of graphs, where $k$ is a natural number. A labelling of a graph on $n$ vertices is *minimally $k$-equitable* if the labels of vertices are distinct elements of $[1, n] \subset \mathbb{N}$ and every edge label occurs either $k$-times or does not occur at all. Thus for trees graceful labellings are essentially equivalent to minimally 1-equitable labellings. Bloom was mainly interested in minimally $k$-equitable labellings of cycles. The obvious necessary condition for the cycle $C_n$ to have a minimally $k$-equitable labelling is that $k$ should be a proper divisor of $n$ (i.e. different from 1 and $n$).

Bloom [10] has asked whether this simple necessary divisibility condition is in fact sufficient. In Chapter 3 we answer Bloom's question in the positive. The proof we give is constructive. We consider three cases; $k$ odd, $k \equiv 2 \bmod 4$, and $k \equiv 0 \bmod 4$. In each case the proof goes by induction on $m = n/k$. When performing our construction we look into the problem from a different point of view. Instead of trying to label the vertices of the cycle $C_n$, we try to build a cycle on the vertex-set $\{1, 2, \ldots, n\}$ using edges of $k$ different lengths, where the *length* of an edge is the absolute value of its end-points. This approach proves to be very useful.

We start from a simple observation that for $k$ odd and $n = 2k$, it is enough to connect $i$ with $i+k$, $i = 1, 2, \ldots, k$ and $j$ with $j+1$, $j = 1, 3, 5, \ldots, 2k-1$. We give a similar but a bit more complicated construction for $n = mk$, $m = 3$, 4, 5. Then we apply induction. In the inductive step we subdivide edges of a certain length in such a way that we get edges of two different lengths ($k$ edges of each) and also different from the lengths of other edges. In each of the two remaining cases the proof is analogous; they differ only in the first step of the construction.

In Chapter 4 we consider a problem concerned with an 'opened' coloured cycle, i.e. a coloured path. Assume that the vertices of a path $N$ are coloured with the integers $1, 2, \ldots, t$. We shall call such a path $N$ an *opened $t$-coloured necklace*. Suppose we want to cut only a small number of edges of our necklace and use the obtained segments to partition the set of vertices of $N$ into $k$ classes such that, for each colour $i$, the vertices of colour $i$ are partitioned evenly between them. Let us call such a partition a *$k$-splitting* and let its *size* be the minimal number of cuts required to obtain it. The problem of calculating the size of a $k$-splitting has some applications to VLSI circuit design, as noted by Bhatt and Leiserson [9] and Bhatt and Leighton [8].

If the vertices of each colour are consecutive in $N$, then for any $k$-splitting of $N$, each segment of vertices of one colour must be cut at $k-1$ points at least. Thus any $k$-splitting of $N$ has size at least $t(k-1)$. Goldberg and West [34] proved that this trivial lower bound is also an upper bound for 2-splittings, and they posed a question about the general case of arbitrary $k$. Alon and West [5] conjectured that $t(k-1)$ is an upper bound on the size of $k$-splittings for any $k$ and $t$. Alon [4] proved this conjecture. His proof uses many techniques from algebraic topology. In Chapter 4 we present a different, more combinatorial, proof of Alon's result using a theorem from algebraic topology only as a starting point. In our proof, the main tool is a new very natural generalization of the Borsuk-Ulam antipodal theorem which says that for any continuous map $h : \mathbb{S}^m \to \mathbb{R}^m$, there is a point $x \in \mathbb{S}^m$ such that $h(x) = h(-x)$.

To formulate our generalization we first define a generalization $\mathbb{S}_p^{m(p-1)}$ of the $m$-dimensional $\ell_1$-sphere $\mathbb{S}^m = \mathbb{S}_2^m$, for any prime number $p$. If we think of the set $\mathbb{R}$ of reals as two half-lines with a common end-point 0, then its natural generalization is the set $\mathbb{R}_{+,p}$ of $p$ half-lines having a common end-point (we denote it also by 0). In analogy to $\mathbb{S}_2^m$ being the set of points of $\mathbb{R}^{m+1}$ which are at distance 1 from the point $(0, 0, \ldots, 0) \in \mathbb{R}^{m+1}$, we define $\mathbb{S}_p^{m(p-1)}$ as the set of points of $\mathbb{R}_{+,p}^{m(p-1)+1} = (\mathbb{R}_{+,p})^{m(p-1)+1}$ which are at distance 1 from $(0, 0, \ldots, 0) \in \mathbb{R}_{+,p}^{m(p-1)+1}$.

As an analogue of the antipodal map on $\mathbb{S}_2^m$, we have a very natural free $\mathbb{Z}_p$-action $\omega$ on $\mathbb{S}_p^{m(p-1)}$. Note that the antipodal map swaps the two half-lines in every coordinate of $x \in \mathbb{S}_2^m$; in the general case we define $\omega$ to permute the half-lines cyclicly in every coordinate. Our generalization of the Borsuk-Ulam antipodal theorem says that for any continuous map

$$h : \mathbb{S}_p^{m(p-1)} \to \mathbb{R}^m$$

there is a point $x \in \mathbb{S}_p^{m(p-1)}$ such that

$$h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x)).$$

This theorem easily implies Alon's result.

In Chapter 5 we again consider a problem connected with the Borsuk-Ulam antipodal theorem. Bajmóczy and Bárány [6] proved that if $\Delta$ is the closure of an $(n+1)$-dimensional simplex and $f : \Delta \to \mathbb{R}^n$ is a continuous map, then there are two disjoint faces of $\Delta$ whose images intersect. Since the Borsuk-Ulam theorem says that for any continuous map $h : \mathbb{S}^n \to \mathbb{R}^n$ there exists $x \in \mathbb{S}^n$ with $h(x) = h(-x)$, to prove the Bajmóczy-Bárány theorem it is enough to show that there is a continuous map $g : \mathbb{S}^n \to \Delta$ such that for every $x \in \mathbb{S}^n$ the supports of $g(x)$ and $g(-x)$ are disjoint. In Chapter 5 we give a very natural construction of such a function $g$.

Finally, in Chapter 6 we present a simple observation allowing us to give a positive answer to a question posed by Sen, Das, Roy and West [50]. They asked whether each digraph can be represented as an intersection digraph of convex sets in two dimensional Euclidean space. Sen, Das, Roy and West defined intersection digraphs as digraphs with ordered pairs of sets assigned to vertices, where $\overrightarrow{uv}$ is a directed edge when the 'source set' of $u$ intersects the 'terminal set' of $v$.

# CHAPTER 1

# A LOWER BOUND FOR SNAKE-IN-THE-BOX CODES

## §1.1. Introduction

Given a natural number $d$, let the *$d$-dimensional cube* $I[d]$ be the graph defined as follows. Let the vertex set of $I[d]$ be the set of all $d$-tuples of binary digits, and for vertices $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ of $I[d]$, let $(x, y)$ be an edge of $I[d]$ if $x$ and $y$ differ in exactly one coordinate.

Let $G_1$ and $G_2$ be graphs. We say that $G_1$ is an *induced subgraph* of $G_2$ if $G_1$ is a subgraph of $G_2$, and if for every pair of vertices $x$ and $y$ of $G_1$ such that $(x, y)$ is an edge of $G_2$, $(x, y)$ is also an edge of $G_1$.

A *snake-in-the-box code*, or a *snake*, is an induced cycle in $I[d]$ for a certain integer $d$. For each $d \in \mathbb{N}$, let $S(d)$ be the length of the longest snake in $I[d]$.

Snakes were introduced by Kautz [37] as a type of error-checking code for a certain analogue-to-digital conversion system. Consider the following problem. We want to encode the position of a rotating wheel using ordered $d$-tuples of binary digits in such a way that a small error resulting in changing one digit does little harm. Thus, we would like to partition the circle which is the boundary of the wheel into many segments of equal length, assigning a $d$-tuple of binary digits to each of them so that the following conditions are satisfied:

(i) different $d$-tuples are assigned to different segments,

(ii) if a $d$-tuple $x$ is assigned to a segment $A$, then any $d$-tuple differing in one coordinate from $x$ is either assigned to one of the neighbours of $A$ or is not assigned to any segment at all, thus allowing the error to be detected.

Ideally then, we would use a snake of length $S(d)$ for the encoding. Clearly, we then want to bound $S(d)$ from below (preferably by giving an explicit construction).

Let us first review the lower bounds for $S(d)$ that have already been obtained. Kautz [37] showed that

$$S(d) \geq \lambda\sqrt{2^d}.$$

This bound was later improved by Ramanujacharyulu and Menon [46], who proved that

$$S(d) \geq (3/2)^d,$$

whilst Brown (unpublished, quoted by Danzer and Klee [15]) and Singleton [51] got

$$S(d) \geq \lambda(\sqrt[4]{6})^d.$$

Abbott [1] obtained

$$S(d) \geq \lambda(\sqrt{5/2})^d,$$

and later Vasil'ev [58] showed that

$$S(d) \geq \frac{2^d}{d} \qquad \text{when } d \text{ is a power of 2,}$$

and further,

$$S(d) \geq (1 - \varepsilon(d))\frac{2^{d-1}}{d} \qquad \text{with } \varepsilon(d) \to 0 \text{ as } d \to \infty.$$

Finally, Danzer and Klee [15] proved that

$$S(d) \geq \frac{2^{d+1}}{d} \qquad \text{when } d \text{ is a power of 2,}$$

and

$$S(d) \geq \frac{7}{4}\frac{2^d}{d-1} \qquad \text{for all } d \geq 5.$$

In this chapter we shall prove a linear lower bound for $S(d)$, namely

$$S(d) \geq \frac{9}{64}2^d;$$

see also [60].

## §1.2. The Main Lemma

Our aim in this section is to state and prove Lemma 1.3, which will provide a construction of long snakes, leading to the proof of the lower bound stated in the introduction to this chapter.

Let us first introduce some notation. If $F$ is a subgraph of $I[d]$, then let us denote by $F^{(0)}$ the subgraph of $I[d+1]$ obtained as the image of $F$ under the embedding

$$\psi_0 : I[d] \to I[d+1]$$

such that

$$\psi_0[(v^1, \ldots, v^d)] = (v^1, \ldots, v^d, 0).$$

Analogously, let $F^{(1)}$ be the image of $F$ under

$$\psi_1 : I[d] \to I[d+1]$$

such that

$$\psi_1[(v^1, \ldots, v^d)] = (v^1, \ldots, v^d, 1).$$

For each $d \geq 2$, let

$$R_d : [2^d + 1, 2^{d+1}] \to [2^d]$$

be the order reversing bijection, i.e. let

$$R_d(i) = 2^{d+1} + 1 - i.$$

Now, for each $d \geq 2$, we shall define a function

$$H_d : [2^d] \to V(I[d])$$

such that if

$$\overline{H_d} = (H_d(1), \ldots, H_d(2^d), H_d(1)),$$

then $\overline{H_d}$ is a Hamiltonian cycle in $I[d]$. Set

$$\overline{H_2} = ((0,0), (0,1), (1,1), (1,0), (0,0)),$$

and

$$H_{d+1}(i) = \begin{cases} (H_d(i))^{(0)} & \text{if } 1 \le i \le 2^d, \\ (H_d \circ R_d(i))^{(1)} & \text{if } 2^d+1 \le i \le 2^{d+1}. \end{cases} \tag{1}$$

In other words, $\overline{H_{d+1}}$ is obtained by taking $\overline{H_d}^{(0)}$ and $\overline{H_d}^{(1)}$, removing the edges connecting their last vertices with their first vertices, joining the first vertex of $\overline{H_d}^{(0)}$ with the first vertex of $\overline{H_d}^{(1)}$, and analogously the last with the last.

The Hamiltonian cycle $\overline{H_d}$ will be used later to construct a snake in $I[d+i]$ for a certain integer $i$, so we are interested in describing when an edge of $I[d]$ is not an edge of $\overline{H_d}$. The following lemma gives such a description in an inductive way.

**Lemma 1.1.** *For each $d \ge 2$, if $1 \le i < j \le 2^{d+1}$ and*

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]) \setminus E(\overline{H_{d+1}}),$$

*then exactly one of the following conditions holds:*

*(i)* $1 \le i < j \le 2^d$, $(i,j) \ne (1, 2^d)$ *and* $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$,

*(ii)* $i = 1$ *and* $j = 2^d$,

*(iii)* $2^d+1 \le i < j \le 2^{d+1}$, $(i,j) \ne (2^d+1, 2^{d+1})$ *and* $(H_d \circ R_d(i), H_d \circ R_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$,

*(iv)* $i = 2^d+1$ *and* $j = 2^{d+1}$,

*(v)* $2 \le i \le 2^d - 1$ *and* $i = R_d(j)$.

*Proof.* It is obvious that the conditions (i)–(v) are mutually exclusive, so assuming that

$$(H_{d+1}(i), H_{d+1}(j)) \in E(i[d+1]) \setminus E(\overline{H_{d+1}}),$$

it is enough to show that one of them is satisfied. If the last coordinates of $H_{d+1}(i)$ and $H_{d+1}(j)$ are both equal to 0, then by (1) we have $1 \le i < j \le 2^d$. If (ii) does not hold, then it follows from (i) that $(H_d(i), H_d(j))$ is not an edge of $\overline{H_d}$ and thus (i) is satisfied.

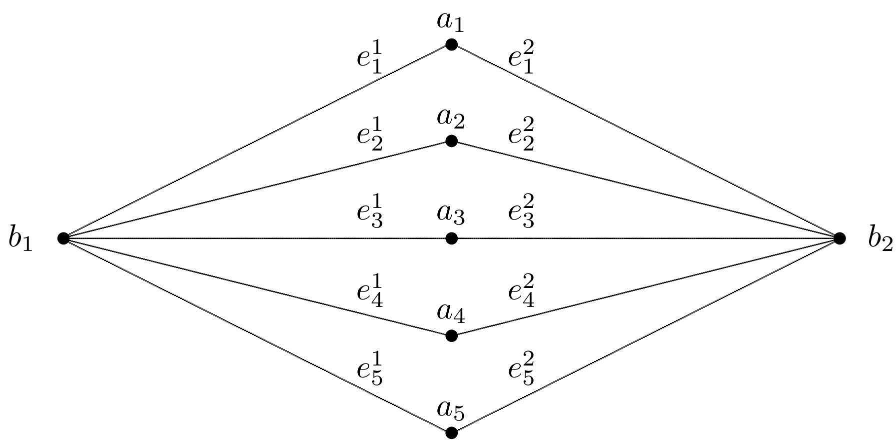


Fig. 1. The graph $G$

If the last coordinates of $H_{d+1}(i)$ and $H_{d+1}(j)$ are both equal to 1, then (1) above implies that $2^d + 1 \leq i < j \leq 2^{d+1}$; similarly as above we conclude that either (iii) or (iv) is satisfied.

The remaining case to consider is when the last coordinate of $H_{d+1}(i)$ is equal to 0 and the last coordinate of $H_{d+1}(j)$ is equal to 1. Then it follows from (1) that $1 \leq i \leq 2^d$ and $2^d + 1 \leq j \leq 2^{d+1}$. Since

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]),$$

$H_{d+1}(i)$ and $H_{d+1}(j)$ differ only at the last coordinate, and thus (1) implies that

$$H_d(i) = H_d \circ R_d(j).$$

Since $H_d$ is a bijection, we have $i = R_d(j)$, and because

$$(H_{d+1}(i), H_{d+1}(j)) \notin E(\overline{H_{d+1}}),$$

we get $2 \leq i \leq 2^d - 1$. Thus (v) is satisfied, and the lemma is proved. $\square$

Let $G$ be the complete bipartite graph $K_{2,5}$, with the vertex set $A \cup B$ where $A = \{a_1, \ldots, a_5\}$ and $B = \{b_1, b_2\}$, and the edge set $E_1 \cup E_2$ where $E_1 = \{e_i^1 = (a_i, b_1) : 1 \leq i \leq 5\}$ and $E_2 = \{e_i^2 = (a_i, b_2) : 1 \leq i \leq 5\}$ (see Fig. 1).

We shall use the graph $G$ in our construction of a snake. In order to present the construction, let us introduce some operations on the set of edges of $G$. Let $S_5$ be the set of permutations on the set $[1,5]$. Given a permutation $\sigma \in S_5$, let $\varphi_\sigma$ and $\varphi_\sigma^+$ be permutations on the set of edges of $G$ such that we have

$$\varphi_\sigma(e_i^j) = e_{\sigma(i)}^j,$$

and

$$\varphi_\sigma^+(e_i^j) = e_{\sigma(i)}^{3-j},$$

for $i = 1, \ldots, 5$ and $j = 1, 2$. Note that $\varphi_\sigma$ permutes the edges in $E_1$ and $E_2$ by permuting their endpoints belonging to $A$ according to $\sigma$, and $\varphi_\sigma^+$ also transposes their endpoints belonging to $B$. Therefore, for each $e, e' \in E(G)$ and $\sigma \in S_5$, the edges $e$ and $e'$ have the same number of vertices in common as the edges $\varphi_\sigma(e)$ and $\varphi_\sigma(e')$, and the same as the edges $\varphi_\sigma^+(e)$ and $\varphi_\sigma^+(e')$. If $e$ and $e'$ have one vertex in common, then it belongs to $A$ if and only if the common vertex of $\varphi_\sigma(e)$ and $\varphi_\sigma(e')$ belongs to $A$, and if and only if the common vertex of $\varphi_\sigma^+(e)$ and $\varphi_\sigma^+(e')$ belongs to $A$.

Let us consider the following permutations of the set $[1,5]$.

$$\sigma_1 = (3\ 5),$$
$$\sigma_2 = (1\ 3)(2\ 4\ 5),$$
$$\sigma_3 = (1\ 2)(4\ 5).$$

We have the folowing lemma.

**Lemma 1.2.** *For each $e \in E(G)$ the edges $\varphi_{\sigma_1}(e)$ and $\varphi_{\sigma_2}^+(e)$ are vertex–disjoint, and the edges $\varphi_{\sigma_1}(e)$ and $\varphi_{\sigma_3}^+(e)$ are vertex–disjoint.*

*Proof.* By symmetry it is enough to prove the lemma for $e \in E_1$. The following table shows all possible cases:

| $e$ | $\varphi_{\sigma_1}(e)$ | $\varphi^+_{\sigma_2}(e)$ | $\varphi^+_{\sigma_3}(e)$ |
|---|---|---|---|
| $e^1_1$ | $e^1_1$ | $e^2_3$ | $e^2_2$ |
| $e^1_2$ | $e^1_2$ | $e^2_4$ | $e^2_1$ |
| $e^1_3$ | $e^1_5$ | $e^2_1$ | $e^2_3$ |
| $e^1_4$ | $e^1_4$ | $e^2_5$ | $e^2_5$ |
| $e^1_5$ | $e^1_3$ | $e^2_2$ | $e^2_4$ |

It is clear that in every row of the above table the edge in the second column is vertex disjoint from the edges in both the third and fourth columns. Thus the lemma is proved. $\square$

Now we can state our key lemma. We claim in it the existence, for each $d \geq 2$, of a closed walk of length $2^d$ in $G$ which will provide a construction of long snakes. We shall 'combine' our walk with the cycle $\overline{H_d}$ in an appropriate way. The walk will start from a vertex belonging to the set $\{a_1, \ldots, a_5\}$, will not use any edge twice in turn and will possess the following property with respect to the Hamiltonian cycle $\overline{H_d}$; if we regard this walk and the Hamiltonian cycle $\overline{H_d}$ as sequences of length $2^d$, the walk as a sequence of edges, and $\overline{H_d}$ as a sequence of vertices, then any two edges corresponding to two nonconsecutive vertices of $\overline{H_d}$ which are neighbours in $I[d]$ will be vertex–disjoint.

**Lemma 1.3.** *For every $d \geq 2$ there is a function $\Phi_d : [2^d] \to E(G)$ such that*

*(i) if $1 \leq i \leq 2^d - 1$, $j = i + 1$, or $i = 2^d$, $j = 1$, then $\Phi_d(i)$ and $\Phi_d(j)$ have exactly one vertex in common, and*

*(ii) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then $\Phi_d(i)$ and $\Phi_d(j)$ are vertex disjoint.*

*Proof.* To prove the lemma we shall use induction on $d$, and we shall prove a statement which is stronger than the lemma itself. We shall show that there are functions

$$\Phi^{k,l}_d : [2^d] \to E(G)$$

for each $d \geq 2$, and

$$(k,l) \in I = \{(1,1),(1,3),(1,4),(2,3),(2,4)\}$$

such that each of the following conditions holds:

(2) $\Phi_d^{k,l}(1) = e_k^1$ and $\Phi_d^{k,l}(2^d) = e_l^2$,

(3) if $1 \leq i \leq 2^d - 1$, then $\Phi_d^{k,l}(i)$ and $\Phi_d^{k,l}(i+1)$ have exactly one vertex $v_i$ in common such that $v_i \in A$ for $i$ even and $v_i \in B$ for $i$ odd,

(4) if $2 \leq i \leq 2^d - 1$ and $(k,l) \neq (1,1) \neq (k',l')$, then $\Phi_d^{k,l}(i) = \Phi_d^{k',l'}(i)$,

(5) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then $\Phi_d^{k,l}(i)$ and $\Phi_d^{k,l}(j)$ are vertex–disjoint.

In other words, the function $\Phi_d^{k,l}$ will describe a walk in $G$ starting from the vertex $a_k$ and the edge $e_k^1$, ending in the edge $e_l^2$ and the vertex $a_l$, and having all the properties we require for the walk described by the function $\Phi_d$, i.e. it will not use any edge twice in turn, and any of its edges corresponding to two nonconsecutive vertices of $\overline{H_d}$ which are neighbours in $I[d]$ will be vertex–disjoint. Also, given $d \geq 2$, all the walks described by $\Phi_d^{k,l}$, for $(k,l) \in I \setminus \{(1,1)\}$, will differ only at the first and the last vertices.

The construction of such functions will complete the proof of Lemma 1.3 because if we set $\Phi_d = \Phi_d^{1,1}$, then (ii) will follow from (5), and (i) will follow from (2) and (3).

Set

$$(\Phi_2^{k,l}(1), \Phi_2^{k,l}(2), \Phi_2^{k,l}(3), \Phi_2^{k,l}(4)) = (e_k^1, e_5^1, e_5^2, e_l^2).$$

If $(k,l) \neq (1,1)$, then let

$$\Phi_{d+1}^{k,l}(i) = \begin{cases} \varphi_{\sigma_1} \circ \Phi_d^{k,3}(i) & \text{if } 1 \leq i \leq 2^d, \\ \varphi_{\sigma_2}^{+} \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(i) & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

and set

$$\Phi_{d+1}^{1,1}(i) = \begin{cases} \varphi_{\sigma_1} \circ \Phi_d^{1,3}(i) & \text{if } 1 \leq i \leq 2^d, \\ \varphi_{\sigma_3}^{+} \circ \Phi_d^{2,4} \circ R_d(i) & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

where $R_d$ is the order reversing bijection.

In the inductive construction given above, the walk $w$ corresponding to $\Phi_{d+1}^{k,l}$, $(k,l) \neq (1,1)$, is obtained from the walks $w_1$ and $w_2$ described by $\Phi_d^{k,3}$ and $\Phi_d^{\sigma_2^{-1}(l),4}$. To obtain $w$, we permute the edges of $w_1$ with $\varphi_{\sigma_1}$, and the edges of $w_2$ with $\varphi_{\sigma_2}^+$, getting $w_1'$ and $w_2'$; then we reverse the order of edges of $w_2'$, getting $w_2''$, and finally we identify the last vertex of $w_1'$ with the first vertex of $w_2''$.

It can be checked directly that for $d = 2$ conditions (2)–(5) are satisfied. Given $d \geq 2$, let us assume that conditions (2)–(5) are satisfied for $d$. We shall prove that they are satisfied for $d+1$.

*Proof of Condition* (2). If $(k,l) \in I \setminus \{(1,1)\}$, then

$$\Phi_{d+1}^{k,l}(1) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(1) = \varphi_{\sigma_1}(e_k^1) = e_k^1,$$

and

$$\Phi_{d+1}^{k,l}(2^{d+1}) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(2^{d+1}) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4}(1) = \varphi_{\sigma_2}^+(e^1_{\sigma_2^{-1}(l)}) = e_l^2.$$

For $(k,l) = (1,1)$ we have

$$\Phi_{d+1}^{1,1}(1) = \varphi_{\sigma_1} \circ \Phi_d^{1,3}(1) = \varphi_{\sigma_1}(e_1^1) = e_1^1,$$

and

$$\Phi_{d+1}^{1,1}(2^{d+1}) = \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4} \circ R_d(2^{d+1}) = \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4}(1) = \varphi_{\sigma_3}^+(e_2^1) = e_1^2.$$

Thus condition (2) is satisfied for $d+1$.

*Proof of Condition* (3). We have to show that if $1 \leq i \leq 2^{d+1} - 1$, then $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(i+1)$ have exactly one vertex in common, which belongs to $A$ for $i$ even and to $B$ for $i$ odd. If $i \neq 2^d$, then condition (3) follows from condition (3) of the induction hypothesis and the definition of the permutations $\varphi_\sigma$ and $\varphi_\sigma^+$, for $\sigma \in S_5$. If $i = 2^d$ and $(k,l) \neq (1,1)$, then the edges

$$\Phi_{d+1}^{k,l}(2^d) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(2^d) = \varphi_{\sigma_1}(e_3^2) = e_5^2, \tag{6}$$

and

$$\Phi_{d+1}^{k,l}(i+1) = \varphi_{\sigma_2}^{+} \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(2^d+1) = \varphi_{\sigma_2}^{+} \circ \Phi_d^{\sigma_2^{-1}(l),4}(2^d) = \varphi_{\sigma_2}^{+}(e_4^2) = e_5^1, \quad (7)$$

have the vertex $a_5$ in common, so (3) holds. For $(k,l) = (1,1)$, the edges

$$\Phi_{d+1}^{1,1}(i) = \varphi_{\sigma_1} \circ \Phi_d^{1,3}(2^d) = \varphi_{\sigma_1}(e_3^2) = e_5^2, \quad (8)$$

and

$$\Phi_{d+1}^{1,1}(i+1) = \varphi_{\sigma_3}^{+} \circ \Phi_d^{2,4} \circ R_d(2^d+1) = \varphi_{\sigma_3}^{+} \circ \Phi_d^{2,4}(2^d) = \varphi_{\sigma_3}^{+}(e_4^2) = e_5^1, \quad (9)$$

have the vertex $a_5$ in common also. Thus condition (3) is satisfied for $d+1$.

*Proof of Condition* (4). We have to show that if $2 \le i \le 2^{d+1}-1$ and $(k,l) \neq (1,1) \neq (k',l')$ then $\Phi_{d+1}^{k,l}(i) = \Phi_{d+1}^{k',l'}(i)$. If $2^d \neq i \neq 2^d+1$, then this follows from condition (4) of the induction hypothesis; otherwise by (6)

$$\Phi_{d+1}^{k,l}(2^d) = e_5^2 = \Phi_{d+1}^{k',l'}(2^d),$$

and by (7)

$$\Phi_{d+1}^{k,l}(2^d+1) = e_5^1 = \Phi_{d+1}^{k',l'}(2^d+1).$$

Thus condition (4) is satisfied for $d+1$.

*Proof of Condition* (5). Let us fix $i < j$ such that

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]) \setminus E(\overline{H_{d+1}}).$$

We have to show that for each $(k,l) \in I$, $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

First let us assume that $(k,l) \neq (1,1)$. By our assumption about $i$ and $j$, we can apply Lemma 1.1. If condition (i) of Lemma 1.1 is satisfied, then by condition (5) of the induction hypothesis, $\Phi_d^{k,3}(i)$ and $\Phi_d^{k,3}(j)$ are vertex–disjoint. By the definition of $\varphi_\sigma$ and $\varphi_\sigma^{+}$, for $\sigma \in S_5$, we conclude that $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

If condition (iii) of Lemma 1.1 holds, then by condition (5) of the induction hypothesis, $\Phi_d^{\sigma_2^{-1}(l),4}(R_d(i))$ and $\Phi_d^{\sigma_2^{-1}(l),4}(R_d(j))$ are vertex disjoint. So analogously to the above we conclude that $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

If condition (ii) of Lemma 1.1 holds, then $i = 1$ and $j = 2^d$. By condition (2), we have

$$\Phi_{d+1}^{k,l}(1) = e_k^1,$$

and by (6),

$$\Phi_{d+1}^{k,l}(2^d) = e_5^2.$$

If condition (iv) of Lemma 1.1 holds, then $i = 2^d + 1$ and $j = 2^{d+1}$. By (7) we have

$$\Phi_{d+1}^{k,l}(i) = e_5^1,$$

and by condition (2),

$$\Phi_{d+1}^{k,l}(j) = e_l^2.$$

In both cases the required edges are vertex–disjoint.

Assume now that condition (v) of Lemma 1.1 holds. By condition (4) of the induction hypothesis, we have

$$\Phi_d^{k,3}(i) = \Phi_d^{\sigma_2^{-1}(l),4}(i) = e,$$

for some $e \in E(G)$. Hence

$$\Phi_{d+1}^{k,l}(i) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(i) = \varphi_{\sigma_1}(e),$$

and

$$\Phi_{d+1}^{k,l}(j) = \varphi_{\sigma_2}^{+} \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(j) = \varphi_{\sigma_2}^{+}(e).$$

By Lemma 1.2, the edges $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex–disjoint.

If $(k, l) = (1, 1)$, then the argument is exactly the same as above. We use the condition (5) of the induction hypotesis for $\Phi_d^{1,3}(i)$ and $\Phi_d^{1,3}(j)$ when condition (i) of Lemma 1.1 is satisfied, and for $\Phi_d^{2,4}(R_d(i))$ and $\Phi_d^{2,4}(R_d(j))$ when condition (iii) of Lemma 1.1 is satisfied.

If condition (ii) of Lemma 1.1 is satisfied, then $i = 1$ and $j = 2^d$. By condition (2) we have

$$\Phi_{d+1}^{1,1}(i) = e_1^1,$$

and by (8),

$$\Phi_{d+1}^{1,1}(j) = e_5^2.$$

If condition (iv) of Lemma 1.1 is satisfied, then $i = 2^d + 1$ and $j = 2^{d+1}$. By (9) we have

$$\Phi_{d+1}^{1,1}(i) = e_1^2,$$

and by condition (2),

$$\Phi_{d+1}^{1,1}(j) = e_5^1.$$

In both cases we get vertex–disjoint edges.

Finally, if condition (v) of Lemma 1.1 is satisfied, then by the condition (4) of the induction hypothesis we have

$$\Phi_{d+1}^{1,1}(i) = \varphi_{\sigma_1}(e),$$

and

$$\Phi_{d+1}^{1,1}(j) = \varphi_{\sigma_3}^{+}(e),$$

for a certain edge $e \in E(G)$. By Lemma 1.2, the edges $\Phi_{d+1}^{1,1}(i)$ and $\Phi_{d+1}^{1,1}(j)$ are vertex–disjoint, and the proof of the lemma is finished. $\square$

## §1.3. The Lower Bound

In this section we are going to give a construction of long snakes $C_d$ in $I[d]$, for $d \geq 7$, which will allow us to prove the main result of this chapter, Theorem 1.6. The construction of $C_{d+5}$, for $d \geq 2$, will combine the Hamiltonian cycle $\overline{H_d}$ defined in Section 1.2 and a sequence of induced paths $P_i^d = (v_i^1, v_i^2, \ldots, v_i^{r_i})$, $i = 1, \ldots, 2^d$, in $I[5]$, which will be defined in this section. In the sequel, given $d$, if either $1 \leq i \leq 2^d - 1$ and $j = i + 1$, or $i = 2^d$ and $j = 1$, then we shall say that $j$ is the *successor* of $i$. The paths $P_i^d$, $1 \leq i \leq 2^d$ will satisfy the following two conditions:

(10) if $j$ is the successor of $i$, then the paths $P_i^d$ and $P_j^d$ have exactly the vertex $v_i^{r_i} = v_j^1$ in common,

(11) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then the paths $P_i^d$ and $P_j^d$ are vertex–disjoint.

Let us first assume that the induced paths $P_i^d$, $i = 1, \ldots, 2^d$, satisfying conditions (10) and (11) are given. Let us regard $I[d+5]$ as $I[d] \times I[5]$. To construct the induced cycle $C_{d+5}$ in $I[d+5]$, we consider $I[d+5]$ as the $d$-dimensional cube $I[d]$, with each vertex being a copy of $I[5]$. Let us take the path $P_j^d$ in the copy of $I[5]$ corresponding to the vertex $H_d(j)$ in $I[d]$, $j = 1, \ldots, 2^d$; see Figure 2 for the case $d = 3$ where the edges of the Hamiltonian cycle $\overline{H_d}$ are denoted by bolder lines. Then, let us join the vertex $v_i^{r_i}$ from the $i$-th copy of $I[5]$ with the vertex $v_j^1$ from the $j$th copy of $I[5]$ for all $i, j \in \{1, \ldots, 2^d\}$, such that $j$ is a successor of $i$. Hence, we have

$$
\begin{aligned}
C_{d+5} = \Big( &\big(H_d(1), v_1^1\big), \big(H_d(1), v_1^2\big), \ldots, \big(H_d(1), v_1^{r_1}\big), \\
&\big(H_d(2), v_2^1\big), \big(H_d(2), v_2^2\big), \ldots, \big(H_d(2), v_2^{r_2}\big), \\
&\vdots \\
&\big(H_d(2^d), v_{2^d}^1\big), \big(H_d(2^d), v_{2^d}^2\big), \ldots, \big(H_d(2^d), v_{2^d}^{r_{2^d}}\big), \big(H_d(1), v_1^1\big) \Big).
\end{aligned}
$$

Clearly $C_{d+5}$ is a cycle. The following lemma states that it is an induced cycle in $I[d+5]$.

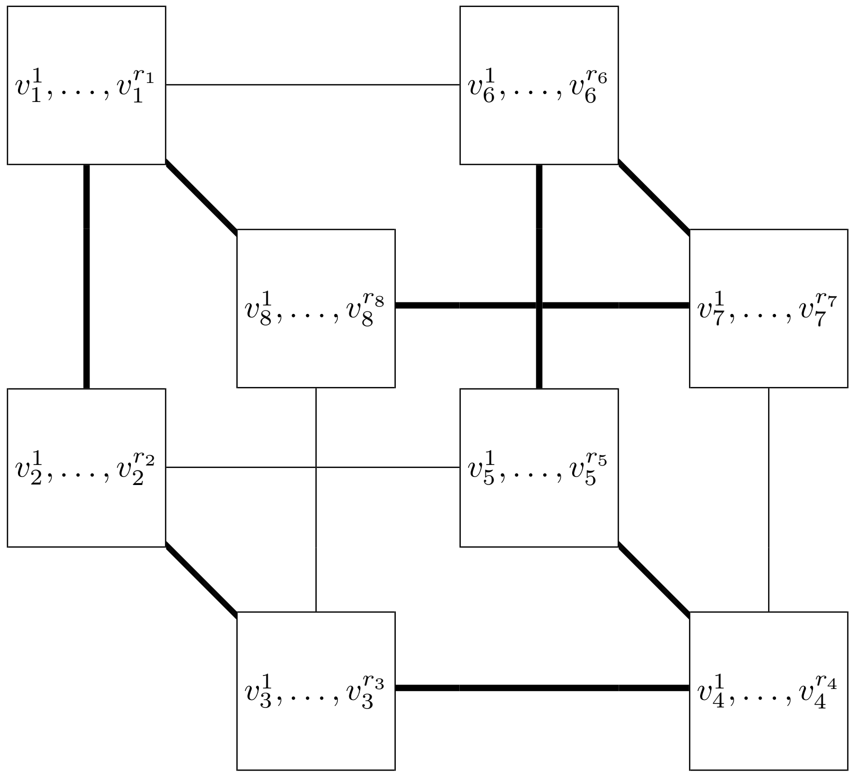


Fig. 2. The cycle $C_8$

**Lemma 1.4.** *For every $d \geq 2$, the cycle $C_{d+5}$ is an induced subgraph of $I[d+5]$.*

*Proof.* Assume that

$$\big((H_d(i), v_i^k), (H_d(j), v_j^\ell)\big) \in E(I[d+5]). \tag{12}$$

To prove the lemma we shall show that

$$\big((H_d(i), v_i^k), (H_d(j), v_j^\ell)\big) \in E(C_{d+5}). \tag{13}$$

Indeed, by (12) we have either $H_d(i) = H_d(j)$ or $v_i^k = v_j^\ell$. If $H_d(i) = H_d(j)$, then since $H_d$ is an injection, we have $i = j$. It also follows from (12) that $v_i^k$ and $v_j^\ell$ are neighbours in $I[5]$. Since the path $P_i^d = P_j^d$ is an induced graph in $I[5]$, we have

$$(v_i^k, v_j^\ell) \in E(P_i^d).$$

Thus, (13) follows from the definition of $C_{d+5}$.

If $v_i^k = v_j^\ell$, then it follows from (12) that $H_d(i)$ and $H_d(j)$ are neighbours in $I[d]$. For supposing that

$$(H_d(i), H_d(j)) \notin E(\overline{H_d}),$$

we get

$$(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d}),$$

which contradicts (11). Thus

$$(H_d(i), H_d(j)) \in E(\overline{H_d}),$$

and we may assume that $j$ is the successor of $i$. By (10), we have $k = r_i$ and $\ell = 1$, so (13) follows from the definition of $C_{d+5}$. Thus the lemma is proved. $\square$

Now we shall define the induced paths $P_i^d$, $i = 1, \ldots, 2^d$, satisfying conditions (10) and (11). We shall use the graph $G$ defined in Section 1.2 and the function $\Phi_d$ from Lemma 1.3. First let us define the following subdivision $G'$ of $G$. Let $G'$ be obtained by subdividing the edge $e_k^1$ of $G$ with two new vertices $c_k^1$ and $c_k^2$ in such a way that we get the path $(b_1, c_k^1, c_k^2, a_k)$, and subdividing the edge $e_k^2$ of $G$ with three new vertices $c_k^4$, $c_k^5$ and $c_k^6$, giving rise to the path $(a_k, c_k^4, c_k^5, c_k^6, b_2)$, for each $k \leq 5$. To have a uniform notation, set $c_k^3 = a_k$. Let

$$\xi : V[G'] \to V(I[5])$$

be the injection defined as follows. Set

$$\xi(b_1) = (0,0,0,0,0),$$

$$\xi(c_1^1) = (1,0,0,0,0),$$

$$\xi(c_1^2) = (1,1,0,0,0),$$

$$\xi(c_1^3) = (1,1,0,1,0),$$

$$\xi(c_1^4) = (0,1,0,1,0),$$

$$\xi(c_1^5) = (0,1,1,1,0),$$

$$\xi(c_1^6) = (0,1,1,1,1),$$

$$\xi(b_2) = (1,1,1,1,1).$$

To obtain the image of $\xi$ on $c_k^j$, $2 \le k \le 5$, $1 \le j \le 6$, we shall apply a cyclic permutation on the coordinates of the image of $c_1^j$, namely if $\xi(c_1^i) = (\alpha_1, \ldots, \alpha_5)$, then let

$$\xi(c_k^i) = (\alpha_k, \ldots, \alpha_5, \alpha_1, \ldots, \alpha_{k-1}).$$

Let the path $P_i^d$, $i = 1, \ldots, 2^d$, be the image under $\xi$ of the path of $G'$ which is the subdivision of the edge $\Phi_d(i)$, where $\Phi_d$ is the function whose existence is claimed in Lemma 1.3. Thus, if $\Phi_d(i) = (a_k, b_1)$, $1 \le k \le 5$, then we have

$$P_i^d = (\xi(b_1), \xi(c_k^1), \xi(c_k^2), \xi(c_k^3)),$$

and if $\Phi_d(i) = (a_k, b_2)$, $1 \le k \le 5$, then we have

$$P_i^d = (\xi(c_k^3), \xi(c_k^4), \xi(c_k^5), \xi(c_k^6), \xi(b_2)).$$

We can now prove the following lemma.

**Lemma 1.5.** *Given $d \ge 2$, the paths $P_i^d$, $1 \le i \le 2^d$, are induced graphs in $I[d]$ and they satisfy conditions (10) and (11) above.*

*Proof.* Assume that $\Phi_d(i) = (a_1, b_1)$. Then $P_i^d$ is equal to the following path:

$$\big((0,0,0,0,0), (1,0,0,0,0), (1,1,0,0,0), (1,1,0,1,0)\big) \tag{14}$$

which is clearly an induced path in $I[5]$. Similarly if $\Phi_d(i) = (a_1, b_2)$, then $P_i^d$ is equal to the folowing path:

$$\big((1,1,0,1,0), (0,1,0,1,0), (0,1,1,1,0), (0,1,1,1,1), (1,1,1,1,1)\big) \tag{15}$$

which is also an induced path in $I[5]$.

If $\Phi_d(i) = (a_k, b_\ell)$, $2 \le k \le 5$ and $1 \le \ell \le 2$, then it follows from the definition of $\xi$ that $P_i^d$ is obtained either from the path (14) or from (15) after applying a certain cyclic permutation on the coordinates of each vertex. Since both the paths (14) and (15) are induced in $I[5]$, then $P_i^d$ is also an induced path in $I[5]$.

Note that to prove conditions (10) and (11), it is enough to show that for every $i, j = 1, \ldots, 2^d$ if the edge $\Phi_d(i)$ has one common vertex with the edge $\Phi_d(j)$, then the path $P_i^d$ has one common vertex with the path $P_j^d$, and if the edges $\Phi_d(i)$ and $\Phi_d(j)$ are vertex–disjoint then so are the paths $P_i^d$ and $P_j^d$. Indeed having established the above facts, condition (10) follows from Lemma 1.3(i) and condition (11) follows from Lemma 1.3(ii).

Now we shall prove the above observation. Assume that $1 \leq i, j \leq 2^d$ and that $\Phi_d(i)$ has one common vertex with $\Phi_d(j)$. If this common vertex is $a_1$ then we can assume that $\Phi_d(i) = (a_1, b_1)$ and $\Phi_d(j) = (a_1, b_2)$. Then $P_i^d$ is equal to the path (14) and $P_j^d$ is equal to the path (15). It is thus clear that $P_i^d$ and $P_j^d$ have exactly the vertex $(1, 1, 0, 1, 0)$ in common. If the common vertex of $\Phi_d(i)$ and $\Phi_d(j)$ is $a_k$, $2 \leq k \leq 5$, then it follows from the definition of $\xi$ that $P_i^d$ and $P_j^d$ have only one vertex in common, which is obtained after a cyclic permutation of the coordinates of the vertex $(1, 1, 0, 1, 0)$.

If the common vertex of $\Phi_d(i)$ and $\Phi_d(j)$ is $b_1$, then it follows from the definition of $\xi$ that the coordinates of each vertex of $P_i^d$ are obtained by a cyclic permutation of the coordinates of the corresponding vertex of the path (14). The same is true for the vertex $P_j^d$, but the cyclic permutation that is applied is different. However, it is clear for every vertex of the path (14) except the first $(0, 0, 0, 0, 0)$, that two different cyclic permutations give different results. Thus $P_i^d$ and $P_j^d$ have only the vertex $(0, 0, 0, 0, 0)$ in common. Analogously, if the common vertex of $\Phi_d(i)$ and $\Phi_d(j)$ is $b_2$, then $P_i^d$ and $P_j^d$ have only the vertex $(1, 1, 1, 1, 1)$ in common.

Thus the lemma is proved. $\square$

Now we are ready to prove our main result of this chapter.

**Theorem 1.6.** *For each $d_0 \geq 2$, the length $S(d_0)$ of the longest induced cycle in*

*$I[d_0]$ satisfies*

$$S(d_0) \geq \frac{9}{64} 2^{d_0}.$$

*Proof.* It is known (see [16], [27], [37]) that $S(2) = 4$, $S(3) = 6$, $S(4) = 8$, $S(5) = 14$, $S(6) = 26$, so we can assume that $d_0 \geq 7$.

Let us fix $d_0 = d + 5$, $d \geq 2$. It is clear from the definition of the paths $P_i^d$, $1 \leq i \leq 2^d$, that the length of $P_i^d$ is 4 for $i \equiv 1$ or 2 mod 4, and the length of $P_i^d$ is 5 for $i \equiv 3$ or 0 mod 4. Thus it follows from the definition of $C_{d+5}$ that its length is equal to

$$4 \cdot 2^{d-1} + 5 \cdot 2^{d-1} = \frac{9}{64} 2^{d+5}.$$

It follows from Lemma 1.4 that $C_{d+5}$ is an induced cycle in $I[d+5]$, so the proof of the theorem is complete. □

## §1.4. Further Remarks

There is still a gap between the lower bound for the length of snake-in-the-box codes proved in this chapter and the best known upper bound. This best upper bound is due to Solov'jeva [54], who improved the following earlier bound of Deimer [17]:

$$S(d) \leq 2^{d-1} - \frac{2^{d-1}}{d(d-5)+7}, \qquad \text{for} \qquad d \geq 7.$$

by proving that

$$S(d) \leq 2^{d-1}\left(1 - \frac{2}{d^2 - d + 2}\right) \qquad \text{for} \qquad d \geq 7.$$

We believe that the upper bound can be further improved to the form of $c2^d$, where $c$ is a constant smaller than $1/2$.

I was informed by the referees of [60] that the lower bound of the form $\lambda 2^d$ was first obtained by Evdokimov [25]. He showed that

$$S(d) \geq 2^{-11} 2^d,$$

but he stated that after certain changes in his construction, the constant $\lambda$ can be increased to $2^{-9}S(8)$.

The idea of his proof is similar to that used in the present proof, in the sense that the snake in $I[d] = I[d-d_0] \times I[d_0]$ (where $d_0$ is a constant) is constructed in such a way that its projection on $I[d-d_0]$ is a Hamiltonian cycle. However the rest of the construction is quite different.

Glagolev and Evdokimov [33] proved a theorem about the chromatic number of a certain infinite graph, and stated that it can be used to further increase the constant $\lambda$ so that $\lambda \in (\frac{3}{16}, \frac{1}{4})$. In addition, one of the referees of [60] informed me also that in his dissertation, Evdokimov [26] proved that $S(d) \geq 0.26 \cdot 2^d$.

Recently Abbot and Katchalski [2] found a completely different way of proving a lower bound of the form $\lambda 2^d$. They use induction in a way resembling the proof of Danzer and Klee [15], but they construct the so-called *accessible* snake, which is a snake with some additional paths between its vertices, which allows them to keep the induction going without decreasing the ratio of vertices used by the snake (which could not be avoided in the proof of Danzer and Klee).

# CHAPTER 2

# ON SMALL GRAPHS WITH HIGHLY IMPERFECT POWERS

## §2.1. Introduction

A celebrated result first proved by Blanche Descartes (cf. [18] and [19]) is the existence of triangle-free graphs of arbitrarily high chromatic number. In particular, there are graphs for which the difference $h$ between their chromatic and clique numbers is arbitrarily large. Supposing we want the order of these graphs to be small with respect to $h$, we cannot hope to have graphs with $O(h^2)$ vertices if we keep the requirement that the graphs should be triangle-free (see [11]). A rather natural problem suggests itself: can we do any better *without* any assumption on the clique number? In fact, this problem has been posed in the literature in the following more general setting.

Let an integer $s \geq 1$ and a graph $G$ be given. A vertex-colouring of $G$ is said to be a *proper $L_s$-colouring* if any two distinct vertices of the same colour are at a distance greater than $s$ (see Chartrand, Geller and Hedetniemi [14], Kramer and Kramer [41], [42] and Speranza [56]). Let us denote by $\chi_s(G)$ the smallest integer $\chi$ such that there exists a proper $L_s$-colouring of $G$ with $\chi$ colours. Let us denote by $\omega_s(G)$ the maximal cardinality of a subset of $V(G)$ of diameter at most $s$. Thus $\chi_1(G)$ is the ordinary chromatic number of $G$ and $\omega_1(G)$ its ordinary clique number. Clearly $\chi_s(G) \geq \omega_s(G)$. The main question we shall study in this chapter is the following problem raised by Gionfriddo and others ([30], [32]). If a graph $G$ satisfies $\chi_s(G) - \omega_s(G) \geq h$, then how small can the order $|G|$ of $G$ be?

For $s \geq 1$ and $h \geq 0$, let us set $\gamma_s(G) = \chi_s(G) - \omega_s(G)$ and

$$\nu_s(h) = \max\{n \in \mathbb{N} : \text{for any graph } G,\ |G| < n \text{ implies } \gamma_s(G) < h\}.$$

We are then interested in estimating the function $\nu_s$. As one would expect, the exact value of $\nu_s(h)$ is known only for very few $s$ and $h$. For instance, $\nu_2(1) = 7$ and $\nu_2(2) = 11$ are the only exact results for $s = 2$. Moreover, having established that $15 \leq \nu_2(3) \leq 18$, Gionfriddo [30] asks what the value of $\nu_2(3)$ is. More generally, Gionfriddo and Milici [31] have proved that $\nu_2(h) \leq 6h$ for $h \geq 3$. As to $\nu_s(h)$ for $s \geq 3$, the following estimates are given in [30]. For $h \geq 3$,

$$\nu_s(h) \leq \frac{1}{2}(3s+1)(h+1)$$

if $s$ is odd and

$$\nu_s(h) \leq \frac{1}{2}(3s+4)(h+1) - 2$$

if $s$ is even. Our main concern in this chapter is to improve the bounds above. We first show that $\nu_2(h) \leq 5h$ for $h \geq 3$, proving that $\nu_2(3) = 15$. We then study the growth of $\nu_s(h)$ as a function of $h$; we shall prove that for fixed $s \geq 1$ and sufficiently large $h$

$$2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2} \leq \nu_s(h) \leq 2h + h^{1-\epsilon_s}, \tag{1}$$

where $\epsilon_s > 0$ is a constant which depends only on $s$; in particular $\nu_s(h) = (2 + o(1))h$ for any fixed $s \geq 1$ and $h \to \infty$.

Let us also mention that the upper bound in (1) improves previous bounds for certain related functions [30]. Let us denote by $m_s(h)$ the smallest number of *edges* in a graph $G$ with $\gamma_s(G) \geq h$. Let us define $\delta_s(h)$ to be the smallest integer $n$ such that there is a graph $G$ of diameter $s$ that can be extended to a graph $G'$ with (*i*) $\gamma_s(G') \geq h$ and (*ii*) $|G'| - |G| \leq n$. Upper bounds for $m_s(h)$ and $\delta_s(h)$ trivially follow from (1); it turns out that they are better than those in [30].

Let us introduce some of the definitions we shall need. We generally follow [11] for graph-theoretical terms. In particular, given a graph $G$, a *walk* in $G$ is a sequence $v_0$, $v_1, \ldots, v_\ell$ of vertices of $G$ such that $v_{i-1}v_i \in E(G)$ for all $1 \leq i \leq \ell$. The *length* of the walk above is defined to be $\ell$; it is said *to connect* $v_0$ *to* $v_\ell$ and thus it is referred to as a $v_0$–$v_\ell$ *walk*. For convenience's sake, we write $\chi(G) = \chi_1(G)$, $\omega(G) = \omega_1(G)$ and $\gamma(G) = \gamma_1(G)$. The complement of $G$ is denoted by $G^{\mathrm{c}}$. The independence number of $G$ is denoted by $\alpha(G)$, hence $\alpha(G) = \omega(G^{\mathrm{c}})$. Given a graph $G$ and $s \geq 1$, we define its $s$th power $G^s$ to be the graph on $V(G)$ with two distinct vertices joined to each other iff their distance is at most $s$. Note that then $\chi_s(G)$ and $\omega_s(G)$ are simply the ordinary chromatic and clique numbers of $G^s$ and thus $\gamma_s(G) = \gamma(G^s)$.

Finally we outline the organisation of this chapter. We shall prove the new upper bound for $\nu_2(h)$, $h \geq 3$, in Section 2.2. In the following section we consider some generalities about the case of arbitrary $s \geq 1$, and draw some easy corollaries concerning $\nu_s(h)$ from estimates on certain Ramsey numbers. In particular, we give the proof of the lower bound in (1). In Section 2.4 we describe the key result, Theorem 2.9, and then we draw the upper bound in (1) as a corollary. (A more precise statement of this bound is given in Corollary 2.10.) The proof of Theorem 2.9 is given in Section 2.5.

The results of this chapter will appear in [39].

## §2.2. A new simple upper bound for $\nu_2(h)$

Let us start by showing a simple construction allowing us to improve the upper bound on $\nu_2(h)$ of Gionfriddo and Milici [31]. This construction is in fact a special case of a much more general construction considered in Sections 2.4 and 2.5.

**Theorem 2.1.** *For every $h \geq 3$ we have $\nu_2(h) \leq 5h$.*

*Proof.* Let us fix $h \geq 3$. It is enough to construct a graph $G$ with $|G| = 5h$ and $\gamma_2(G) = h$. Let $C^5$ be a cycle of order 5 and $K^h$ a complete graph of order $h$. Let us define the graph $G$ on $V(C^5) \times V(K^h)$ by joining the vertices $(c, k)$ to $(c', k')$ iff $cc' \in E(C^5)$ and $kk' \in E(K^h)$

Obviously $|G| = 5h$. As pointed out in the introduction, $\gamma_2(G) = \gamma(G^2)$ and so we proceed to compute $G^2$.

We claim that $G^2$ is the complement of the disjoint union of $h$ pentagons, i.e. cycles of order 5, say $C_1, C_2, \ldots, C_h$. The claim implies that $\gamma(G^2) = h$. Indeed, a maximal clique in $G^2$ has cardinality $2h$ (two nonconsecutive vertices in each $C_i$), and the chromatic number of $G^2$ is $3h$ (three colours for each $C_i$). Therefore, it only remains to check the claim.

Let $v_1 = (c_1, k_1)$, $v_2 = (c_2, k_2)$ be a pair of distinct vertices of $G$. We shall show that their distance $d(v_1, v_2)$ in $G$ is greater than 2 if and only if $c_1$ is adjacent to $c_2$ in $C^5$ and $k_1 = k_2$. Note that this proves the claim. Let us consider the following three cases.

*Case* 1. $c_1c_2 \in E(C^5)$ and $k_1 = k_2$.

By definition $(c_1, k_1)$ is not adjacent to $(c_2, k_2)$ in $G$ since $k_1 = k_2$. No vertex of $G$ is adjacent to both $(c_1, k_1)$ and $(c_2, k_2)$ since no vertex of $C^5$ is adjacent to both $c_1$ and $c_2$. Therefore $d((c_1, k_1), (c_2, k_2)) \geq 3$ in $G$.

*Case* 2. $c_1c_2 \in E(C^5)$ and $k_1 \neq k_2$.

By definition the vertices $(c_1, k_1)$ and $(c_2, k_2)$ are adjacent in $G$.

*Case* 3. $c_1c_2 \notin E(C^5)$ (including the case $c_1 = c_2$).

There is a vertex adjacent to both $c_1$ and $c_2$ in $C^5$, and there is a vertex adjacent to both $k_1$ and $k_2$ in $K^h$ (since $h \geq 3$). So $d((c_1, k_1), (c_2, k_2)) \leq 2$ in $G$ concluding the proof of Theorem 2.1.

$\square$

The theorem above solves the question about the determination of $\nu_2(3)$, posed by Gionfriddo in [30]. He has proved that $\nu_2(3) \geq 15$ and from Theorem 2.1 it follows that $\nu_2(3) \leq 15$, so we obtain the result that 15 is the exact value of $\nu_2(3)$.

### §2.3. Bounds arising from estimates on Ramsey numbers

In this section we start a more systematic study of $\nu_s(h)$ for arbitrary $s \geq 1$. Let us first consider the case $s = 1$ and remark that certain bounds for Ramsey numbers give us rather good information about $\nu_1(h)$. As usual, let us denote by $R(s, t)$ the smallest positive integer $n$ such that any graph of order at least $n$ has either a clique of order at least $s$ or an independent set of order at least $t$. Erdős [22], with an ingenious probabilistic proof, established that

$$R(s, 3) \geq c(s/\log s)^2 \tag{2}$$

for some $c > 0$. In fact (2) holds for any $0 < c < 1/27$ and large enough $s$, cf. [12], Chapter XII, §2. The following result is an immediate corollary of (2).

**Theorem 2.2.** *For sufficiently large $h$,*

$$\nu_1(h) < 2h + 20h^{1/2} \log h.$$

*Proof.* By taking $s = \lfloor (n/c)^{1/2} \log n \rfloor$, it can be easily checked that Erdős's lower bound for $R(s, 3)$ tells us the following: for any $0 < c < 1/27$ there is an integer $n_0 = n_0(c)$ such that, for any $n \geq n_0$, there is a graph of order $n$ with clique number less than $(n/c)^{1/2}\log n$ and independence number at most 2. Let us fix $c = 1/28$ and a large enough $h$ (it will be clear that our inequalities hold if $h \geq h_0$, where $h_0$ is an absolute constant). Let $n$ satisfy

$$\frac{n}{2} - \left(\frac{n}{c}\right)^{1/2} \log n \geq h > \frac{n-1}{2} - \left(\frac{n-1}{c}\right)^{1/2} \log(n-1) \geq \frac{n}{3}$$

and $n \geq n_0(c)$. Let $G$ be a graph of order $n$ with $\omega(G) < (n/c)^{1/2}\log n$ and $\alpha(G) \leq 2$. Clearly $\chi(G) \geq n/2$ and so

$$\gamma(G) \geq \frac{n}{2} - \left(\frac{n}{c}\right)^{1/2} \log n \geq h.$$

Moreover, by the choice of $n$,

$$\begin{aligned} |G| &= n \\ &\leq 2h + 2\left(\frac{n-1}{c}\right)^{1/2} \log(n-1) + 1 \\ &< 2h + 20h^{1/2}\log h. \end{aligned}$$

Hence this $G$ proves the bound in the theorem. $\square$

We now turn our attention to arbitrary $s \geq 1$. An obvious way of generalising Theorem 2.2 is to prove the existence of graphs with large order and small clique and independence numbers which are, furthermore, powers. Neither the probabilistic approach of Erdős in [22] nor a more recent one by Spencer [55] based on the Erdős–Lovász sieve seems to be directly applicable; we shall use instead an explicit construction of Erdős [23] which proves that $R(s, 3)$ grows at least as fast as a power of $s$.

In order to describe Erdős's construction, let us recall the definition of the $n$-dimensional cube $I[n]$. It is the graph whose vertices are the 0–1 sequences of length $n$, two of them being adjacent iff they differ in exactly one coordinate. The graph $I[n]$ induces a natural metric on its set of vertices; let us denote this metric by $d$. Hence $d(x, y)$, which is usually called the *Hamming distance* between $x$ and $y$, is simply the number of coordinates in which $x$ and $y$ differ. Erdős's graph $J_r$, $r \geq 1$, has as its set of vertices the 0–1 sequences of length $3r+1$, two distinct vertices being adjacent iff their distance is at most $2r$. Thus $J_r$ is the $2r$th power of $I[3r+1]$.

It is easy to check that in $J_r$ any three distinct vertices span at least one edge. The fact that it has only small cliques is a consequence of the following theorem conjectured by Erdős and proved by Kleitman [38].

**Theorem 2.3.** *Let $n$ and $r \geq 1$ be integers with $n \geq 2r$. Let $S \subset I[n]$ be a set vertices of the $n$-dimensional cube $I[n]$ with diameter at most $2r$. Then*

$$|S| \leq \sum_{i=0}^{r} \binom{n}{i}. \qquad \square$$

We thus have the following.

**Theorem 2.4.**

*(i) The independence number of $J_r$ is 2 for all $r \geq 1$.*

*(ii) Set $c = (5\log 2 - 3\log 3)/(3\log 2) = \cdot 0817\ldots$ and let $0 < \epsilon < c$. Then, for $r \geq r_0(\epsilon)$,*

$$\omega(J_r) < |J_r|^{1-\epsilon}/2. \qquad \square$$

In particular, we conclude that

$$\gamma(J_r) > \frac{1}{2}|J_r|\left(1 - |J_r|^{-c+o(1)}\right) \tag{3}$$

as $r \to \infty$. Since $J_r$ has an $s$th root when $s$ divides $r$, we immediately notice the following.

**Corollary 2.5.** *For all $s \geq 1$, we have that $\liminf_h \nu_s(h)/h \leq 2$.*

*Proof.* For all $s$ and $t \geq 1$, let us define the graph $J(s,t)$ on 0–1 sequences of length $3st+1$ by joining two distinct sequences iff their distance is at most $2t$. Clearly $J(s,t)^s = J_{st}$ for all $s$ and $t$. This remark coupled with inequality (3) completes the proof. $\square$

A moment's thought reveals that the drawback of using the $J_r$ only is that the set $\{|J_r| : r \geq 1\}$ is much too sparse. Indeed, with such an approach we can merely conclude that $\limsup_h \nu_1(h)/h \leq 16$. In the next section, we introduce a technique to generate more graphs $F$ with large $\gamma(F^s)$ and thus improve Corollary 2.5.

Let us now turn to the problem of bounding $\nu_1(h)$ from below. Trivially, $\nu_s(h) \geq \nu_t(h)$ for all $h$ if $t$ divides $s$; hence the lower bound we shall prove for $\nu_1(h)$ bounds $\nu_s(h)$ for arbitrary $s \geq 1$ as well. We shall need the following simple lemma.

**Lemma 2.6.** *Let $G$ be a graph. Then for any induced subgraph $H$ of $G$*

$$|G| \geq 2\gamma(G) + \omega(G) + |H| - 2\chi(H).$$

*Proof.* Let us set $G' = G - V(H)$. Note that

$$\chi(G') + \chi(H) \geq \chi(G) = \gamma(G) + \omega(G),$$

and so

$$\chi(G') \geq \gamma(G) + \omega(G) - \chi(H). \tag{4}$$

Clearly, in a proper minimal colouring of a graph the union of any two colour classes must span an edge. Hence, in such a colouring, the set of vertices which are assigned colours which occur only once must span a complete graph. Thus

$$|G'| \geq 2\chi(G') - \omega(G').$$

By (4) we conclude that

$$\begin{aligned} |G'| &\geq 2(\gamma(G) + \omega(G) - \chi(H)) - \omega(G') \\ &\geq 2\gamma(G) + \omega(G) - 2\chi(H). \end{aligned}$$

As $|G| = |G'| + |H|$, the proof is complete. □

A way of applying the lemma above is to take $V(H)$ to be an independent set of order $\alpha(G)$. Doing so, we conclude that

$$\begin{aligned} |G| &\geq 2\gamma(G) + \omega(G) + \alpha(G) - 2 \\ &> 2\gamma(G) + (\log|G|)/\log 4, \end{aligned} \tag{5}$$

where the second inequality follows from the well-known bound of Erdős and Szekeres

$$R(s,s) \leq \binom{2s-2}{s-1} < \frac{1}{6} 4^s s^{-1/2},$$

for $s \geq 4$. We can in fact improve the log term in (5) by choosing a better subgraph $H$; we shall make use of an upper bound for off-diagonal Ramsey numbers to find a suitable $H$.

Ajtai, Komlós and Szemerédi [3] were the first to prove that

$$R(s,3) = O(s^2/\log s),$$

and Shearer [52] a little later gave a simple and elegant proof of a slightly stronger result (see also [12], Chapter XII, §3). The following bound is sufficient for our purposes:

$$R(s,3) \leq \frac{(s-1)(s-2)^2}{(s-1)\log(s-1) - s + 2} + 1 \leq \frac{2s^2}{\log s} \tag{6}$$

for $s$ large enough. It follows immediately from this bound that any graph of order $n$ has either three independent vertices or a clique of order at least $(n \log n)^{1/2}/3$, provided $n$ is sufficiently large.

**Theorem 2.7.** *For all graphs $G$ of sufficiently large order,*

$$|G| > 2\gamma(G) + \frac{1}{6}(|G| \log |G|)^{1/2}. \tag{7}$$

*In particular, for all $s \geq 1$ and large enough $h$,*

$$\nu_s(h) > 2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2}. \tag{8}$$

*Proof.* Throughout the proof of (7) we assume that $n$ is a large enough integer. Let $G$ be a graph of order $n$, which we may trivially assume is not complete. We may furthermore assume that

$$\omega(G) < \frac{1}{6}(n \log n)^{1/2},$$

since otherwise Lemma 2.6 completes the proof: we simply choose $H$ to be two independent vertices. By the remark following (6), we can find an independent 3-set $W_0 \subset V(G)$ in $G$. Define $G_1 = G - W_0$ and $n_1 = |G_1| = n-3$. We have that

$$\omega(G_1) \le \omega(G) < \frac{1}{6}(n \log n)^{1/2} < \frac{1}{3}(n_1 \log n_1)^{1/2}.$$

Hence we can find an independent 3-set $W_1$ in $G_1$. Define $G_2 = G_1 - W_1$ and $n_2 = |G_2| = n - 6$. In this fashion we obtain $G = G_0 \supset G_1 \supset \cdots \supset G_t$ with $W_i = V(G_i) \setminus V(G_{i+1})$ an independent 3-set in $G_i$, $0 \le i \le t-1$, and $n_i = |G_i| = n - 3i$ for all $i$. We claim that if $t < n/5$, and hence $n_t > 2n/5$, we can still continue the process. Indeed

$$\omega(G_t) \le \omega(G) < \frac{1}{6}(n \log n)^{1/2} < \frac{1}{6}((5n_t/2) \log n)^{1/2} < \frac{1}{3}(n_t \log n_t)^{1/2},$$

and again we know that there is an independent 3-set in $G_t$. Thus we find $s = \lceil n/5 \rceil$ pairwise disjoint independent 3-sets $W_0, \dots, W_{s-1}$ in $G$. Set $H$ to be the subgraph of $G$ induced by the union of these $W_i$. Then $|H| = 3s$ and $\chi(H) \le s$ and hence $|H| - 2\chi(H) \ge s = \lceil n/5 \rceil$. Therefore an application of Lemma 2.6 with this $H$ completes the proof of (7).

Finally, given a large enough $h$, if $G$ is a graph with $\gamma(G) \ge h$ then (7) tells us that

$$\begin{aligned} |G| &\ge 2h + \frac{1}{6}(|G| \log |G|)^{1/2} \\ &> 2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2}, \end{aligned}$$

which completes the proof of (8), since trivially $\nu_s(h) \ge \nu_1(h)$ for all $s$ and $h$. □

We conclude this section by remarking the following. In Theorem 2.2, our approach in the search for graphs $G$ with large $\gamma(G)$ is rather crude in the sense that we guarantee a large $\chi(G)$ simply by taking a $G$ with $\alpha(G) = 2$. Indeed, by (6), we must have a large clique in such a $G$ and this forces $\gamma(G)$ down. However, Theorem 2.7 tells us that this simple approach gives us in fact a reasonable bound.

## §2.4. The main construction and the asymptotic upper bound

Our aim in this section is to introduce a new class of graphs in order to prove our upper bound (1) for $\nu_s(h)$. We shall make use of the following two operations. Given two graphs $G$ and $H$, let us define their (*categorical*) *product* $G \times H$ as the graph on $V(G) \times V(H)$ whose edges are

$$E(G \times H) = \{(g_1, h_1)(g_2, h_2) : g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)\}.$$

Also, we define their $*$-*product* $G * H$ as the graph on $V(G) \times V(H)$ whose edges are

$$\begin{aligned} E(G * H) = \{(g_1, h_1)(g_2, h_2) : \text{either } g_1 = g_2 \text{ and } h_1 h_2 \in E(H) \\ \text{or } g_1 g_2 \in E(G) \text{ and } h_1 = h_2\}. \end{aligned}$$

In the last section we considered the graphs $J_r$, as their chromatic numbers are large and their clique numbers small. The reason $\chi(J_r)$ is large is that $\alpha(J_r) = 2$ or, in other words, their complement $G_r = J_r^{\mathrm{c}}$ is triangle-free. The point of considering the $*$-product is that $G * H$ is triangle-free if both $G$ and $H$ are. Moreover, the independence number of $G * H$ is trivially at most $|H|\alpha(G)$. Thus, if $H$ is triangle-free,

$$\begin{aligned} \gamma\left[(G_r * H)^{\mathrm{c}}\right] &= \chi\left[(G_r * H)^{\mathrm{c}}\right] - \omega\left[(G_r * H)^{\mathrm{c}}\right] \\ &\geq |H||G_r|/2 - |H|\alpha(G_r) \\ &= (1/2 - o(1))|G_r * H|, \end{aligned}$$

as $r \to \infty$, by (3). Thus, if we can find a triangle-free $H$ for which $G_r * H$ is the complement of a square, say of $F^2$, then we shall have a good upper bound for $\nu_2(\gamma(F^2))$, namely, $|F^2| = (2 + o(1))\gamma(F^2)$.

Let us define two families of graphs. First, for each $q$ and $r \geq 1$, we denote by $G_{r,q}$ the graph whose vertices are the 0–1 sequences of length $(2q+1)r + 1$, two of them being adjacent iff they differ in at least $2qr + 1$ coordinates. Thus, for instance, we have $G_{r,1} = G_r = J_r^{\mathrm{c}}$. Secondly, for each $k \geq 1$ and $\ell \geq 0$, set $m = (2\ell + 1)k + 2$ and denote the cycle of order $m$ by $C^m$; we define $H_{k,\ell}$ as the graph whose vertices are the vertices of $C^m$, two of them being adjacent in our $H_{k,\ell}$ iff their distance in $C^m$ is at least $\ell k+1$. Note that in $H_{k,\ell}$ the neighbours of a vertex $h$ are the farthest $k+1$ points from $h$ in $C^m$.

It is easy to check that $G_{r,q}$ is triangle-free for all $q$ and $r \geq 1$. Moreover, Theorem 2.3 gives us the following upper bound for $\alpha(G_{r,q}) = \omega(G^{\mathrm{c}}_{r,q})$.

**Lemma 2.8.** *Let $q \geq 1$ be fixed and set*

$$\eta_q = \left( \frac{(2q+1)^{2q+1}}{q^q(q+1)^{q+1}} \right)^{1/(2q+1)}.$$

*Then, for sufficiently large $r$,*

$$\omega(G^{\mathrm{c}}_{r,q}) < \frac{1}{2}\eta_q^{(2q+1)r+1}. \qquad \square$$

For all $r$ and $k \geq 1$ and $s \geq 2$, let us set

$$F_{r,k,s} = G_{r,\lfloor s/2 \rfloor} \times H_{k,\lfloor (s-1)/2 \rfloor}.$$

As usual, a graph with no edges is said to be *empty*; we denote the empty graph of order $m$ by $E^m$. For any graph $G$, we note that $G * E^m$ is simply the disjoint union of $m$ copies of $G$. We are now ready to state our key result.

**Theorem 2.9.** *Let $r$, $k \geq 1$ and $s \geq 2$. Set $q = \lfloor s/2 \rfloor$, $\ell = \lfloor (s-1)/2 \rfloor$ and $m = |H_{k,\ell}| = (2\ell+1)k+2$. Then*

$$(F_{r,k,s})^s = \begin{cases} (G_{r,q} * E^m)^{\mathrm{c}} & \text{if } s \text{ is even} \\ (G_{r,q} * H_{k,\ell})^{\mathrm{c}} & \text{if } s \text{ is odd.} \end{cases}$$

Theorem 2.9, whose proof is given in the next section, implies the promised upper bound for $\nu_s(h)$.

**Corollary 2.10.** *Let $s \geq 2$ be fixed, $q = \lfloor s/2 \rfloor$ and $\eta_q$ as defined in Lemma 2.8. Moreover, set $\epsilon_0 = \epsilon_0(s) = 1 - (\log \eta_q)/\log 2 > 0$ and $C_s = 4 + s2^{s+1}$. Then for sufficiently large $h$*

$$\nu_s(h) < 2h + C_s h^{1/(1+\epsilon_0)}. \tag{9}$$

*Proof.* Fix an $h$ and $s \geq 2$. We shall assume throughout the proof that $h$ is large enough; it will be clear that our inequalities hold if $h \geq h_0$ for some absolute constant $h_0$. We shall choose suitable parameters $r$ and $k$ for which $F = F_{r,k,s}$ shows that (9) holds.

First, let $r \geq 1$ be the minimal integer such that setting $n = (2q+1)r+1$ we have

$$2^n \geq h^{1/(1+\epsilon_0)}. \tag{10}$$

Now put $\ell = \lfloor (s-1)/2 \rfloor$ and let $k \geq 1$ be the minimal integer such that setting $m = (2\ell+1)k+2$ we have

$$m \geq 2^{1-n} h \left(1 + 2(2h)^{-\epsilon_0/(1+\epsilon_0)}\right). \tag{11}$$

*Claim.* We have

$$\gamma\left[(F_{r,k,s})^s\right] \geq \left|(F_{r,k,s})^s\right|/2 - m\omega(G^{\mathrm{c}}_{r,q}) > h \tag{12}$$

and

$$\left|(F_{r,k,s})^s\right| = 2^n m < 2h + C_s h^{1/(1+\epsilon_0)}. \tag{13}$$

Note that the claim above proves (9); it now remains to check (12) and (13). Let us start with (12). We first note that $G_{r,q}$ ($r$, $q \geq 1$) and $H_{k,\ell}$ ($k$, $\ell \geq 1$) are triangle-free (see Lemmas 2.12($i$) and 2.13($i$)), and hence so are $G_{r,q} * E^m$ and $G_{r,q} * H_{k,\ell}$. Therefore, by Theorem 2.9, we have that $\alpha\left[(F_{r,k,s})^s\right] = 2$ and so $\chi\left[(F_{r,k,s})^s\right] \geq \left|(F_{r,k,s})^s\right|/2$. Secondly, since $G_{r,q} * E^m$ is a spanning subgraph of $G_{r,q} * H_{k,\ell}$, we trivially have that $\omega\left[(G_{r,q} * E^m)^{\mathrm{c}}\right] = \alpha\left(G_{r,q} * E^m\right) \geq \alpha\left(G_{r,q} * H_{k,\ell}\right) = \omega\left[(G_{r,q} * H_{k,\ell})^{\mathrm{c}}\right]$. Theorem 2.9 then tells us that

$$\begin{aligned}\omega\left[(F_{r,k,s})^s\right] &\leq \omega\left[(G_{r,q} * E^m)^{\mathrm{c}}\right] \\ &\leq m\omega\left(G^{\mathrm{c}}_{r,q}\right).\end{aligned}$$

Furthermore, by the definition of $\epsilon_0$ and Lemma 2.8, we know that

$$\omega\left(G^{\mathrm{c}}_{r,q}\right) < \eta_q^n/2 = 2^{n-1}(2^n)^{-\epsilon_0}.$$

Hence, by (10) and (11),

$$\gamma\left[(F_{r,k,s})^s\right] = \chi\left[(F_{r,k,s})^s\right] - \omega\left[(F_{r,k,s})^s\right]$$

$$
\begin{aligned}
&\geq \left|(F_{r,k,s})^s\right|/2 - m\omega(G^{c}_{r,q}) \\
&> m2^{n-1}\left(1-(2^n)^{-\epsilon_0}\right) \\
&\geq h\left(1+2(2h)^{-\epsilon_0/(1+\epsilon_0)}\right)\left(1-(2^n)^{-\epsilon_0}\right) \\
&\geq h\left(1+\frac{2^{1/(1+\epsilon_0)}}{h^{\epsilon_0/(1+\epsilon_0)}}\right)\left(1-\frac{1}{h^{\epsilon_0/(1+\epsilon_0)}}\right) \\
&> h,
\end{aligned}
$$

proving (12).

Inequality (13) follows from the choices of $r$ and $k$. Indeed, we first note that by the minimality of $r$

$$2^n < 2^{2q+1}h^{1/(1+\epsilon_0)}.$$

By the minimality of $k$, we have that

$$(m-(2\ell+1))2^n < 2h\left(1+2(2h)^{-\epsilon_0/(1+\epsilon_0)}\right).$$

Thus

$$
\begin{aligned}
m2^n &< 2h + 2(2h)^{1/(1+\epsilon_0)} + (2\ell+1)2^{2q+1}h^{1/(1+\epsilon_0)} \\
&< 2h + 4h^{1/(1+\epsilon_0)} + s2^{s+1}h^{1/(1+\epsilon_0)} \\
&= 2h + C_s h^{1/(1+\epsilon_0)},
\end{aligned}
$$

completing the proof of the claim and hence establishing our result. □

We now remark that (9) trivially improves some upper bounds for certain functions mentioned in [30]. Let us recall the following two definitions given in the introduction. Given $s \geq 1$ and $h \geq 0$, set

$$m_s(h) = \max\{m \in \mathbb{N} : \text{for any graph } G,\ |E(G)| < m \text{ implies } \gamma_s(G) < h\}$$

and

$$\delta_s(h) = \max\{n \in \mathbb{N} : \text{for any graph } G,\ |G| < n + \omega_s(G) \text{ implies } \gamma_s(G) < h\}.$$

It has been known [30] that for $h \geq 3$ one has $m_2(h) \leq 13h^2$ and $\delta_2(h) \leq 3h$. Moreover, for $s \geq 3$,

$$m_s(h) \leq \begin{cases} (3s+1)h^2 - 2 & \text{if } s \text{ is odd} \\ (3s+4)h^2 & \text{if } s \text{ is even,} \end{cases}$$

and

$$\delta_s(h) \leq \begin{cases} s(h+1) & \text{if } s \text{ is odd} \\ (s+1)(h+1)-1 & \text{if } s \text{ is even.} \end{cases}$$

Corollary 2.10 immediately gives us the following bounds.

**Corollary 2.11.** *Let $s$ be fixed and $C_s$ and $\epsilon_0 = \epsilon_0(s)$ as in Corollary 2.10. Then for sufficiently large $h$*

$$\begin{aligned} m_s(h) &\leq \binom{\nu_s(h)}{2} \\ &< 2h^2 + 3C_s h^{(2+\epsilon_0)/(1+\epsilon_0)} \end{aligned}$$

*and*

$$\delta_s(h) \leq \nu_s(h) < 2h + C_s h^{1/(1+\epsilon_0)} \qquad \square$$

## §2.5. Proof of the key result

In this section we prove Theorem 2.9 . We shall need the following two lemmas about walks in the graphs $G_{r,q}$ and $H_{k,\ell}$.

**Lemma 2.12.** *For any $r \geq 1$ and $q \geq 1$ the following conditions hold.*

*(i) Any odd closed walk in $G_{r,q}$ has length at least $2q+3$.*

*(ii) Let $g$ and $g'$ be nonadjacent vertices in $G_{r,q}$. Then they are connected by a walk of length $2q$. If they are furthermore distinct then they are also connected by a walk of length $2q+1$.*

*Proof.* $(i)$ Assume $g_1, g_2, \ldots, g_{2j+1}$ is a walk in $G_{r,q}$ with $j \leq q$. We claim that $g_1$ is not adjacent to $g_{2j+1}$. Indeed, for $i = 1, 2, \ldots, 2j$ we have $d(g_i, g_{i+1}) \geq 2qr+1$, so $g_i$ and $g_{i+1}$ agree at no more than $r$ coordinates. Therefore, for $i = 1, 2, \ldots, 2j-1$,

$$d(g_i, g_{i+2}) \leq 2r,$$

since if $g_i$ and $g_{i+2}$ disagree at a coordinate $j$, say, then $g_{i+1}$ agrees at $j$ either with $g_i$ or else with $g_{i+2}$. Hence

$$d\left(g_1, g_{2j+1}\right) \leq 2jr \leq 2qr,$$

and $g_1$ is not adjacent to $g_{2j+1}$ concluding the proof of $(i)$.

$(ii)$ If $g$ and $g'$ are two nonadjacent vertices in $G_{r,q}$ then $d(g,g') \leq 2qr$ by definition. Let us construct a walk of length $2q$ from $g$ to $g'$. Let $C$ be the set of coordinates on which $g$ and $g'$ disagree. Since the cardinality of $C$ is at most $2qr$ we can write

$$C = C_1 \cup C_2 \cup \cdots \cup C_{2q},$$

where the $C_i$ are pairwise disjoint and satisfy

$$0 \leq |C_i| \leq r$$

for all $i$. Let us consider the walk $g = g_0, g_1, \ldots, g_{2q}$ in $G_{r,q}$ defined by the condition that $C_i$ is the set of coordinates on which $g_{i-1}$ and $g_i$ agree. It is easy to check that $g_{2q} = g'$, and so we have found the required $g$–$g'$ walk.

We now assume that $g \neq g'$. To find a walk of length $2q+1$ joining $g$ to $g'$ it is enough to find $g''$ adjacent to $g'$ but not adjacent to $g$. In order to construct such a sequence $g''$ put $D$ to be a set of coordinates of cardinality $r+1$ containing at least one coordinate at which $g$ and $g'$ disagree. Now let $g''$ be equal to $g$ at each coordinate in $D$ and different from $g'$ at each coordinate outside $D$.

The sequence $g''$ is not adjacent to $g$ since they can only differ on coordinates not in $D$, and so $d(g,g'') \leq 2qr$. On the other hand $g''$ differs from $g'$ on each coordinate outside $D$ and on at least one coordinate in $D$, hence $d(g',g'') \geq 2qr+1$ and so $g''$ is adjacent to $g'$. □

**Lemma 2.13.** *For any $k \geq 1$ and $\ell \geq 0$ the following conditions hold.*

*(i) Any odd closed walk in $H_{k,\ell}$ has length at least $2\ell+3$.*

*(ii) Let $h$ and $h'$ be two distinct vertices in $H_{k,\ell}$. Then they are connected both by a walk of length $2\ell+1$ and by a walk of length $2\ell+2$. If they are furthermore nonadjacent, then they are also connected by a walk of length $2\ell$*

*Proof.* Let $h_0$ be a fixed vertex of $H_{k,\ell}$. Let $U_i$ be the set of the $2ik+1$ nearest vertices to $h_0$ in $C^m$, $m = (2\ell+1)k+2$, $i = 1, \ldots, \ell$ (see Fig. 1).

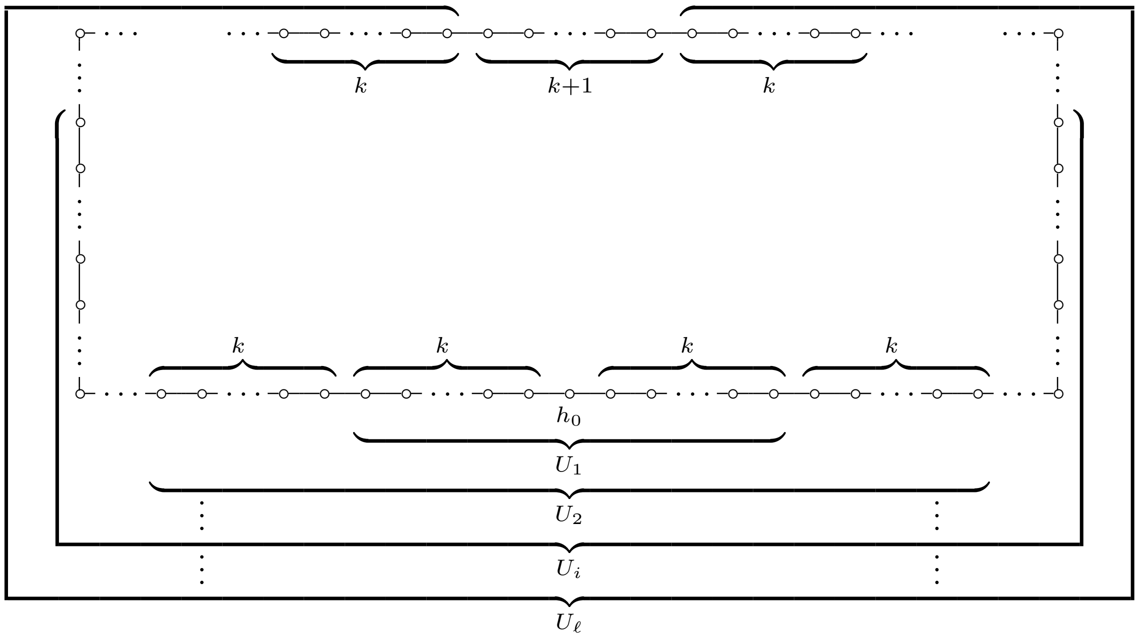


Fig. 1. The sets $U_i$ in $C^m$

Note that the complement of $U_\ell$ is the set of vertices adjacent to $h_0$.

It is easy to check that $U_1$ is the set of vertices $h$ of $H_{k,\ell}$ such that there is a walk of length 2 from $h_0$ to $h$. By induction, $U_i$ is the set of vertices connected to $h_0$ by a walk of length $2i$. So $U_\ell$ is the set of vertices $h$ for which there is a walk of length $2\ell$ from $h_0$ to $h$.

Since $h_0$ is not adjacent to any vertex of $U_\ell$ there are no walks of length $2\ell+1$ from $h_0$ to itself. This concludes the proof of $(i)$.

It can be easily seen that $h_0$ is the only vertex of $H_{k,\ell}$ not adjacent to any vertex in $U_\ell$. Hence there is a walk of length $2\ell + 1$ from $h_0$ to any other vertex of $H_{k,\ell}$. To show that there is a walk of length $2\ell+2$ from $h_0$ to any other vertex $h_1$ of $H_{k,\ell}$ let us consider any vertex $h_2$ adjacent to $h_1$ and different from $h_0$ (clearly $h_2$ exists since the degree of each vertex in $H_{k,\ell}$ is at least 2). We know that there is a walk of length $2\ell + 1$ from $h_0$ to $h_2$ and, since $h_2$ is adjacent to $h_1$, there is a walk of length $2\ell + 2$ from $h_0$ to $h_1$.

To finish our proof, it is enough to show that if $h_1$ is not adjacent to $h_0$, then there is a walk of length $2\ell$ between them. But this follows from the fact that the set of vertices nonadjacent to $h_0$ is $U_\ell$. Indeed, as remarked above, $U_\ell$ is precisely the set of vertices connected to $h_0$ by walks of length $2\ell$. $\square$

We are now ready to prove Theorem 2.9. Let us once and for all fix $r$ and $k \geq 1$. We shall analyse the cases $s$ even and $s$ odd separately. For $s \geq 2$ even, we have to prove that

$$(F_{r,k,s})^s = (G_{r,q} * E^m)^{\mathrm{c}}, \tag{14}$$

where $s = 2q = 2\ell + 2$ and $m = (2\ell + 1)k + 2$. On the other hand, for $s \geq 3$ odd we have to prove that

$$(F_{r,k,s})^s = (G_{r,q} * H_{k,\ell})^{\mathrm{c}}, \tag{15}$$

where $s = 2q + 1 = 2\ell + 1$.

*Proof of* (14). Let us fix an even $s \geq 2$ and let $q$ and $\ell$ satisfy $s = 2q = 2\ell + 2$. By definition we have

$$F_{r,k,s} = G_{r,q} \times H_{k,\ell}.$$

Let $(g_1, h_1)$, $(g_2, h_2)$ be any pair of distinct vertices of $F_{r,k,s}$. To prove (14), we have to show that if $g_1g_2 \in E(G_{r,q})$ and $h_1 = h_2$ then there are no $(g_1, h_1)$–$(g_2, h_2)$ walks of length at most $s$ in $F_{r,k,s}$. Furthermore, we have to show that there is such a walk otherwise.

Let us consider the following three cases. We want to show the nonexistence of our short $(g_1, h_1)$–$(g_2, h_2)$ walk in the first case, and its existence in the last two cases.

*Case* 1. $g_1g_2 \in E(G_{r,q})$ and $h_1 = h_2$.

Let us assume that there is a $(g_1, h_1)$–$(g_2, h_2)$ walk $W$ of length $t \leq s$ in $F_{r,k,s}$. If $t$ is odd then, by projecting $W$ onto the second coordinate, we get an odd closed walk of length $t \leq 2\ell + 1$ in $H_{k,\ell}$, contradicting Lemma 2.13$(i)$. On the other hand, if $t$ is even then, by projecting $W$ onto the first coordinate, we get an even $g_1$–$g_2$ walk of length $t \leq 2q$ in $G_{r,q}$. Since $g_1g_2 \in E(G_{r,q})$ we obtain an odd closed walk of length $t + 1 \leq 2q + 1$ in $G_{r,q}$, contradicting Lemma 2.12$(i)$.

*Case* 2. $g_1g_2 \in E(G_{r,q})$ and $h_1 \neq h_2$.

By Lemma 2.13($ii$), there is a $h_1$–$h_2$ walk of length $2\ell + 1 \leq s$ in $H_{k,\ell}$. Since $g_1g_2 \in E(G_{r,q})$ there clearly is a $g_1$–$g_2$ walk of length $2\ell + 1$ in $G_{r,q}$ (in fact of any odd length). Let $W$ be the sequence of vertices of $F_{r,k,s}$ whose projection onto the first and the second coordinates are the above walks in $G_{r,q}$ and in $H_{k,\ell}$. Clearly $W$ is a $(g_1, h_1)$–$(g_2, h_2)$ walk in $F_{r,k,s}$ and, since its length is $2\ell + 1 \leq s$, the proof of this case is finished.

*Case* 3. $g_1g_2 \notin E(G_{r,q})$.

As we have seen above, it is enough to show the existence of two suitable walks of the same length $t \leq s$, say, one connecting $g_1$ to $g_2$ in $G_{r,q}$ and the other $h_1$ to $h_2$ in $H_{k,\ell}$. Here we can take $t = s = 2q = 2\ell + 2$. Indeed, the existence of the required walk in $G_{r,q}$ follows from Lemma 2.12($ii$). To get a suitable walk in $H_{k,\ell}$ we apply Lemma 2.13($ii$) if $h_1 \neq h_2$ and if, on the other hand, $h_1 = h_2$ then we simply note that $s$ is even and that $H_{k,\ell}$ has no isolated vertices.

□

*Proof of* (15). Let us fix an odd $s \geq 3$ and let $q$ and $\ell$ satisfy $s = 2q + 1 = 2\ell + 1$. Let $(g_1, h_1)$ and $(g_2, h_2)$ be any pair of distinct vertices of $F_{r,k,s}$. To prove (15) we have to show that if either $h_1 = h_2$ and $g_1g_2 \in E(G_{r,q})$ or else $g_1 = g_2$ and $h_1h_2 \in E(H_{k,\ell})$, then there are no $(g_1, h_1)$–$(g_2, h_2)$ walks in $F_{r,k,s}$ of length at most $s$. Moreover we also need to show that otherwise there is such a walk.

Let us consider four cases. We shall prove the nonexistence of the appropriate walks in the first two cases and their existence in the last two.

*Case* 1. $g_1g_2 \in E(G_{r,q})$ and $h_1 = h_2$.

This is similar to the Case 1 of the proof of (14). The existence of a $(g_1, h_1)$–$(g_2, h_2)$ walk of length at most $s$ in $F_{r,k,s}$ requires either that there should be an odd closed walk of length at most $s = 2\ell + 1$ in $H_{k,\ell}$ or else that there should be an odd closed walk of length at most $s + 1 = 2q + 2$ in $G_{r,q}$. By Lemmas 2.12($i$)

and 2.13($i$), neither of the above walks can exist.

*Case* 2. $h_1h_2 \in E(H_{k,\ell})$ and $g_1 = g_2$.

If there is a $(g_1, h_1)$–$(g_2, h_2)$ walk of length $t \leq s$ in $F_{r,k,s}$, then either there is an odd closed walk of length at most $s = 2q+1$ in $G_{r,q}$ or else there is an odd closed walk of length at most $s+1 = 2\ell+2$ in $H_{k,\ell}$, contradicting either Lemma 2.12($i$) or 2.13($i$).

*Case* 3. $g_1 \neq g_2$ and $h_1 \neq h_2$.

We can get a $(g_1, h_1)$–$(g_2, h_2)$ walk of length $s = 2q + 1 = 2\ell + 1$ in $F_{r,k,s}$ by combining apropriate walks in $G_{r,q}$ and in $H_{k,\ell}$. If $g_1$ is not adjacent to $g_2$ then the required walk in $G_{r,q}$ exists by Lemma 2.12($ii$), otherwise its existence is obvious (since $s$ is odd). The existence of a suitable walk in $H_{k,\ell}$ follows from Lemma 2.13($ii$).

*Case* 4. Either $g_1 = g_2$ and $h_1h_2 \notin E(H_{k,\ell})$ or else $h_1 = h_2$ and $g_1g_2 \notin E(G_{r,q})$.

Now we combine appropriate walks of length $s - 1 = 2q = 2\ell$ from $G_{r,q}$ and from $H_{k,\ell}$. Their existence is either obvious (in the case their endpoints are equal) or follows from Lemmas 2.12($ii$) and 2.13($ii$).

$\square$

## §2.6. Concluding remarks

Although we have managed to estimate $\nu_s(h)$ quite accurately, some interesting questions concerning the function $e_s(h) = \nu_s(h) - 2h$ remain. Our results show that for large enough $h$

$$\frac{1}{3\sqrt{2}}(h \log h)^{1/2} < e_s(h) < h^{1-\epsilon_s} \tag{16}$$

where $\epsilon_s > 0$ depends only on $s$. What is clearly unsatisfactory is that the lower bound does not depend on $s$. Also, the exponent of $h$ in the upper bound is rather close to one, and in fact by our methods $\epsilon_s \to 0$ as $s \to \infty$. It is natural to ask whether $e_s(h) = O(h^{1-\epsilon})$ for some $\epsilon > 0$ independent of $s$.

Our proof of the upper bound in (16) is entirely constructive, and the question whether one can do better by probabilistic techniques naturally arises. Let us make the following remark, where for the sake of simplicity we restrict our attention to the case $s = 2$. It turns out that $\epsilon_2$ in (16) can be taken close to 1/2, provided there exists a triangle-free graph $G$ of order $n$, diameter 2, and with $\alpha(G) = O(n^c)$ for some $c$ close to 1/2. Indeed, the proof of (14) (or of the claim in the proof of Theorem 2.1) implies that $G * E^{k+2}$, $k \geq 1$, is the complement of a square. By straightforward computations as in the proof of Corollary 2.10, one then gets an improvement of the upper bound in (16), if $c$ is not much larger than 1/2.

## CHAPTER 3

## EQUITABLE LABELLINGS OF CYCLES

### §3.1. Introduction

A *labelling* of the vertices of a graph $G$ is an assignment of distinct natural numbers to the vertices of $G$. Every labelling induces a natural labelling of the edges: the label of an edge $(x, y)$ is the absolute value of the difference of the labels of $x$ and $y$. There are many natural questions one can ask about labellings. In particular, Bloom [10] defined a labelling of the vertices of a graph to be $k$-*equitable* if in the induced labelling of its edges, every label occurs exactly $k$ times, if at all. Furthermore, a $k$-equitable labelling of a graph of order $n$ is said to be *minimal* if the vertices are labelled with $1, 2, \ldots, n$. A graph is *minimally* $k$-*equitable* if it has a minimal $k$-equitable labelling.

Let us now restrict our attention to cycles. Let $C_n$ be the cycle on $n$ vertices. Given natural numbers $n$ and $k$, $n \geq 3$, if the cycle $C_n$ is $k$-equitable then obviously $k$ must be a divisor of $n$. It is also obvious that $k \neq n$. If the stronger assumption that $C_n$ is *minimally* $k$-equitable holds, then in the appropriate edge-labelling the largest label is at most $n-1$. Since there are $n$ edges in the cycle $C_n$, we conclude that $k \neq 1$. Thus a necessary condition for $C_n$ to be minimally $k$-equitable is that $k$ should be a proper divisor of $n$, i.e. different from 1 and $n$. Bloom [10] posed the question of whether this necessary condition is also sufficient. In this chapter we settle this problem by giving a positive answer to the above question.

The problem of minimally $k$-equitable labellings for cycles is connected with the very difficult conjecture of Ringel and Kotzig concerning decompositions of complete graphs with odd number of vertices into subgraphs isomorphic to trees. This is the main reason why Bloom raised the problem on $k$-equitable labellings of cycles. Let us briefly describe the Ringel-Kotzig conjecture and its connection with labellings.

In 1963 Ringel [47] conjectured that for any natural number $n$ and any $(n+1)$-vertex tree $T$, the complete graph $K_{2n+1}$ could be decomposed into $2n+1$ subgraphs isomorphic to $T$. As reported by Rosa [48], Kotzig strengthened Ringel's conjecture as follows. Let $S(K_{2n+1})$ be the set of all subgraphs of $K_{2n+1}$. Assume that the vertices of $K_{2n+1}$ are the numbers $0, 1, \ldots, 2n$, and let the *unit rotation* $R : S(K_{2n+1}) \to S(K_{2n+1})$ be defined by

$$R\big[\big(V(G), E(G)\big)\big] = \Big(\big\{s(v) : v \in V(G)\big\}, \big\{\big(s(u), s(v)\big) : (u, v) \in E(G)\big\}\Big),$$

where $s(v) = v + 1 \bmod 2n + 1$, $0 \le v \le 2n$. Assume that we are given a graph $G$ with $n$ edges. Let us say that $K_{2n+1}$ can be *cyclically $G$-decomposed* if there is a subgraph $G'$ of $K_{2n+1}$ isomorphic to $G$ such that the set $\{G', R(G'), \ldots, R^{2n}(G')\}$ is a decomposition of $K_{2n+1}$, i.e. $E(G') \cup E(R(G')) \cup \ldots \cup E(R^{2n}(G'))$ is a partition of $E(K_{2n+1})$. The Ringel-Kotzig conjecture asserts that $K_{2n+1}$ can be cyclically $T$-decomposed for any tree $T$ with $n$ edges.

For an edge $(u, v)$ of $K_{2n+1}$, let the reduced label $L(u, v)$ of $(u, v)$ be defined by

$$L(u, v) = \begin{cases} |u - v|, & \text{if } |u - v| \le n, \\ 2n + 1 - |u - v|, & \text{if } |u - v| > n. \end{cases}$$

Let $G'$ be a subgraph of $K_{2n+1}$ isomorphic to $G$ generating a cyclic $G$-decomposition of $K_{2n+1}$. Observe that the reduced labels of the edges of $G'$ are all distinct elements of the set $[1, n] \subset \mathbb{N}$. Rosa [48] defined a labelling of a graph $G$ with $n$ edges to be a $\rho$-labelling if the vertices of $G$ are assigned $n$ distinct integers from the set $\{0, 1, \ldots, 2n\}$ in such a way that for any pair of distinct edges $(u, v)$ and

$(u', v')$ of $G$ we have

$$\{|u-v|,\, 2n+1-|u-v|\} \cap \{|u'-v'|,\, 2n+1-|u'-v'|\} = \emptyset.$$

With the above definitions the Ringel-Kotzig conjecture is equivalent to saying that every tree has a $\rho$-labelling.

Rosa [48] has also defined another class of labellings; these are the $\beta$-labellings, more often refered to as graceful labellings. The requirement for a labelling of a graph $G$ with $n$ edges to be graceful is that the vertices of $G$ should be labelled with integers from the set $\{0, 1, \ldots, n\}$ in such a way that in the induced edge-labelling the edges of $G$ are labelled with distinct integers. The conjecture that any tree can be labelled gracefully is thus clearly stronger then the Ringel-Kotzig conjecture, and even this is still open. For problems connected with graceful labellings see also [21], [35], [40], [49], [53], [57].

Note that in the case of trees the graceful labellings are essentially the minimal 1-equitable labellings. In the case of cycles these two notions are different. We can easily see that no cycle has a minimal 1-equitable labelling and that the cycle $C_n$ has a graceful labelling if and only if the sum $1+2+\ldots+n$ is even, i.e. if and only if $n \equiv 3$ or $0 \bmod 4$.

We shall now turn to the main problem we are concerned with in this chapter. Let us first introduce the terminology we shall use. We shall call a graph $G$ an *integer graph* if its vertex set is a finite subset of $\mathbb{N}$, and we shall call $G$ a $[p, q]$-*graph* if $p$ is the smallest vertex of $G$ and $q$ is the largest vertex of $G$. If such a graph is a cycle, we shall call it an *integer cycle*. If $e = (v_1, v_2)$ is an edge of $G$, we will say that $e$ has *length* $|v_1 - v_2|$. Let $M = (a_{i,j})$ be an $s \times 2$ matrix with integer entries for which there is a partition $E(G) = E_1 \cup \ldots \cup E_s$ such that, for $i = 1, \ldots, s$, $a_{i,1}$ is the cardinality of the set $E_i$ and all the edges in $E_i$ have length $a_{i,2}$. Then, we will call $M$ a *distribution of edges* of $G$.

Let $G$ be a graph, and $k$ a positive integer. Observe that $G$ has a $k$-equitable labelling if $G$ is isomorphic to an integer graph $G'$ with either 0 or $k$ edges of

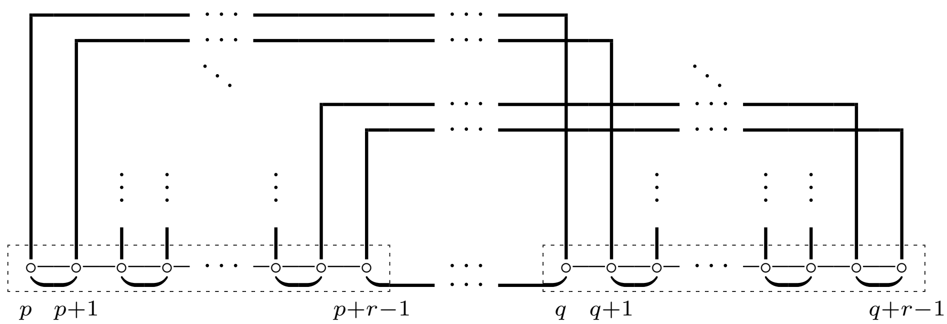


Fig. 1. The graph $C(p,q;r)$.

any length. We will call such $G'$ a *$k$-equitable representation* of $G$. Note that $G'$ is a $k$-equitable representation of a graph if and only if $G'$ has a distribution of edges with the first column having all entries equal to $k$ and the second column having all entries different. Note also that $G$ is minimally $k$-equitable if there is a $k$-equitable representation of $G$ which is a $[j, |V(G)|+j-1]$-graph for a certain integer $j$. Then, we shall call $G'$ a *minimal $k$-equitable representation* of $G$.

We are going to prove the following theorem.

**Theorem 3.1.** *If $k$ and $m$ are integers greater than 1, then the cycle $C_{mk}$ is minimally $k$-equitable.*

The proof of Theorem 3.1 will be broken down into several lemmas. Their proofs will contain several constructions of integer cycles. First we shall define the notions needed for these constructions. Let $p$, $q$ and $r$ be integers such that $r$ is greater than 2 and odd, and $p+r \le q$. Let $C(p,q;r)$ (see Fig. 1.) be a graph with the vertex set $[p, p+r-1] \cup [q, q+r-1]$, and the edge set

$$\{(p+i, q+i) : i = 0, 1, \ldots, r-1\} \cup \{(p+i, p+i+1) : i = 0, 2, 4, \ldots, r-3\}$$
$$\cup\{(p+r-1, q)\} \cup \{(q+i, q+i+1) : i = 1, 3, 5, \ldots, r-2\}.$$

It follows immediately from the definition that $C(p,q;r)$ is a cycle with the following distribution of edges:

$$\begin{pmatrix} r & q-p \\ r-1 & 1 \\ 1 & q-p-r+1 \end{pmatrix}. \tag{1}$$

Let $C$ be an integer cycle. We will say that $C$ is $(k_1, k_2; t)$-*outer* if it satisfies the following three conditions.

(i) $k_1 \geq 0$, $k_2 \geq 0$ and $k_1 + k_2 > 0$,

(ii) the set $V_1$ of the $k_1$ smallest and the set $V_2$ of the $k_2$ largest vertices of $C$ are disjoint segments in $\mathbb{N}$,

(iii) every edge of length $t$ has exactly one endvertex in $V_1 \cup V_2$, and every vertex in $V_1 \cup V_2$ is an endvertex of exactly one edge of length $t$.

Now we shall define a certain operation on outer cycles. Let $C$ be a $(k_1, k_2; t)$-outer cycle, $V_1 = [p, p + k_1 - 1]$ be the set of the $k_1$ smallest vertices of $C$, and $V_2 = [q - k_2 + 1, q]$ be the set of the $k_2$ largest vertices of $C$. Given a positive integer $d$, let the $(k_1, k_2; t; d)$-*extension* $C'$ of $C$ be an integer graph with the vertex set $V(C') = V(C) \cup [p - d - k_2 + 1, p - d] \cup [q + d, q + d + k_1 - 1]$ and the edge set defined as follows:

$$\begin{aligned} E(C') = E(C) &\setminus \{(p+i, p+t+i) : i = 0, \ldots, k_1 - 1\} \\ &\setminus \{(q-i, q-t-i) : i = 0, \ldots, k_2 - 1\} \\ &\cup \{(p+i, q+d+i) : i = 0, \ldots, k_1 - 1\} \\ &\cup \{(p+t+i, q+d+i) : i = 0, \ldots, k_1 - 1\} \\ &\cup \{(q-i, p-d-i) : i = 0, \ldots, k_2 - 1\} \\ &\cup \{(q-t-i, p-d-i) : i = 0, \ldots, k_2 - 1\}. \end{aligned}$$

What does this apparently complicated construction do? It subdivides every edge of length $t$ with one of the new vertices to get two edges of lengths $q - p + d$ and $q - p - t + d$. The set of new vertices is the union of two segments $V_1'$ and $V_2'$ of cardinalities $k_1$ and $k_2$ accordingly. The segment $V_1'$ is placed above the segment $[p, q]$ in the distance $d$ from $q$, and $V_2'$ is placed below $[p, q]$ in the distance $d$ from

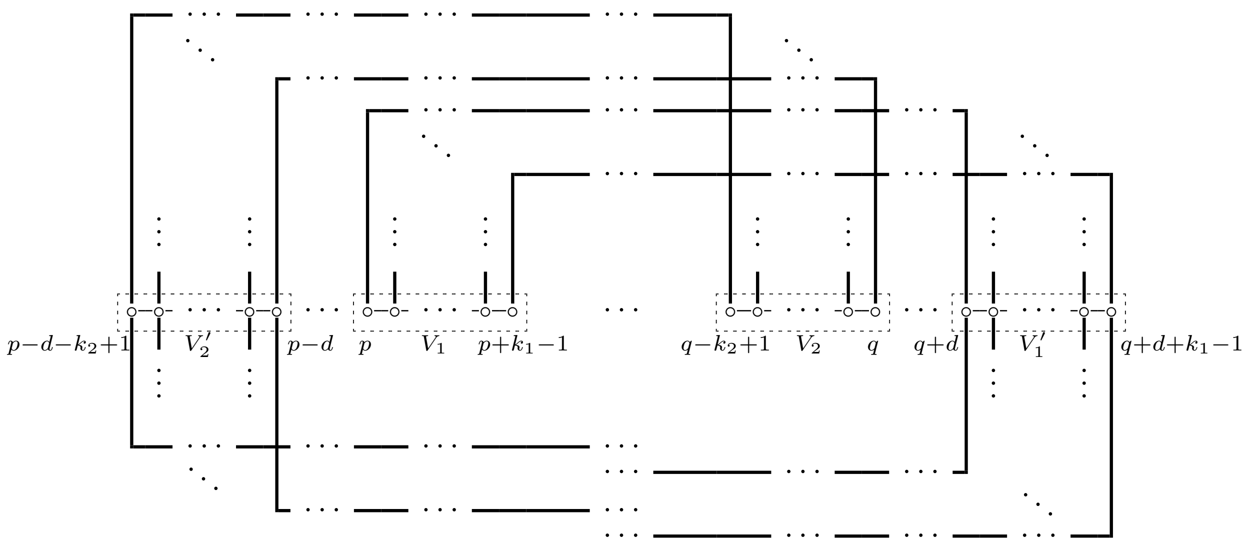


Fig. 2. The operation of taking the $(k_1, k_2; t; d)$-extension.

$p$. The vertices from the set $V_1'$ are used to subdivide edges with an endpoint in $V_1$ and the vertices from the set $V_2'$ are used to subdivide the edges with an endpoint in $V_2$ (see Fig. 2).

Therefore, assuming that the following matrix is a distribution of edges of $C$:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_i & b_i \\ k_1+k_2 & t \\ a_{i+1} & b_{i+1} \\ a_{i+2} & b_{i+2} \\ \vdots & \vdots \\ a_s & b_s \end{pmatrix},$$

$C'$ has clearly the following distribution of edges:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_i & b_i \\ k_1+k_2 & q-p+d \\ k_1+k_2 & q-p-t+d \\ a_{i+1} & b_{i+1} \\ a_{i+2} & b_{i+2} \\ \vdots & \vdots \\ a_s & b_s \end{pmatrix}. \tag{2}$$

Note that since $C$ is a $[p,q]$-graph, it does not have edges of length $q-p+d$. Thus, immediately from the definition, we get that $C'$ is a $(k_2,k_1;q-p+d)$-outer cycle. Also, if $C$ has no edges of length $q-p-t+d$ or if $q-p-t+d=t$, then $C'$ is a $(k_2,k_1;q-p-t+d)$-outer cycle as well.

To prove Theorem 3.1, we shall consider the following two cases.

(i) $k$ is odd,

(ii) $k$ is even.

Case (i) will be proved in Lemmas 3.2 and 3.3 in section 3.2, and case (ii) will be proved in Lemmas 3.4 and 3.5 in section 3.3.

## §3.2. The case $k$ odd

The following lemma takes care of the subcase $m \in [2,4]$ of case (i).

**Lemma 3.2.** *If $k$ is an odd integer greater than 2, and $m$ is 2, 3 or 4, then the cycle $C_{mk}$ is minimally $k$-equitable.*

*Proof.* To get a minimal $k$-equitable representation of $C_{2k}$, it is enough to take the cycle $C(1, k+1; k)$ which is a $[1, 2k]$-graph and, by (1), has the following distribution of edges:

$$\begin{pmatrix} k & k \\ k & 1 \end{pmatrix}.$$

Now, let us consider the case $m = 3$. Let $C$ be the $(k, 0; k; 1)$-extension of $C(1, k+1; k)$. As a result of this operation each edge of length $k$ got subdivided into two edges of lengths $2k$ and $k$. Indeed, by (2), $C$ is a cycle with the following distribution of edges:

$$\begin{pmatrix} k & 2k \\ k & k \\ k & 1 \end{pmatrix}.$$

$C$ is also clearly a $[1, 3k]$-graph, and thus a minimal $k$-equitable representation of $C_{3k}$. Therefore, to finish the proof of our lemma it remains to settle the case $m = 4$.

Unfortunately, we cannot continue the above construction. We have to start from the begining with a $[2k+1, 4k]$-graph $G_1 = C(2k+1, 3k+3; k-2)$ having, by (1), the following distribution of edges:

$$\begin{pmatrix} k-2 & k+2 \\ k-3 & 1 \\ 1 & 5 \end{pmatrix}.$$

Clearly, $G_1$ is a $(0, k-2; k+2)$-outer cycle. Let $G_2$ be the $(0, k-2; k+2; 2)$-extension of it. This way, each edge of length $k+2$ of $G_1$ got subdivided into two edges of lengths $2k+1$ and $k-1$ (see (2)). Thus, $G_2$ is a cycle with the following distribution of edges:

$$\begin{pmatrix} k-2 & 2k+1 \\ k-2 & k-1 \\ k-3 & 1 \\ 1 & 5 \end{pmatrix}.$$

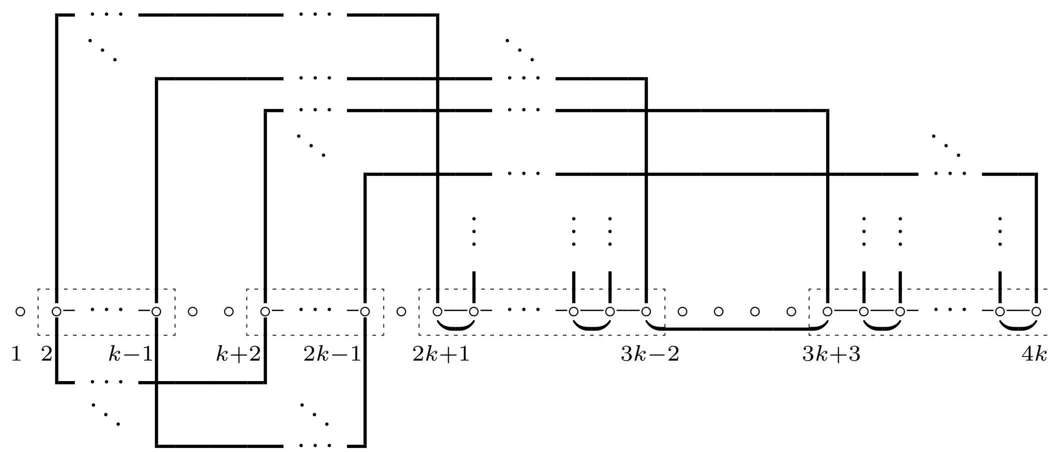


Fig. 3. The graph $G_3$.

Let $G_3$ be obtained from $G_2$ by adding the segment $S = [2, k-1]$ to the set of vertices and subdividing each edge of $G_2$ of length $k-1$ with an appropriate vertex from $S$ such that the resulting edges have lengths $k$ and $2k-1$, see Fig. 3.

Thus, $G_3$ is a cycle and has the following distribution of edges:

$$\begin{pmatrix} k-2 & 2k+1 \\ k-2 & 2k-1 \\ k-2 & k \\ k-3 & 1 \\ 1 & 5 \end{pmatrix}$$

To get a minimal $k$-equitable representation of $C_{4k}$ we must get rid of the edge of length 5 and create new edges of lengths $2k+1$, $2k-1$, $k$ and 1. We have also to remove the gaps from our graph. The graph $G_3$ is arranged in such a way that both this aims can be easily achieved. Let $C$ be obtained from $G_3$ by adding the set $\{1, k, k+1, 2k, 3k-1, 3k, 3k+1, 3k+2\}$ to the set of vertices and subdividing the edge $(3k-2, 3k+3)$ with the new vertices in such a way that we get the

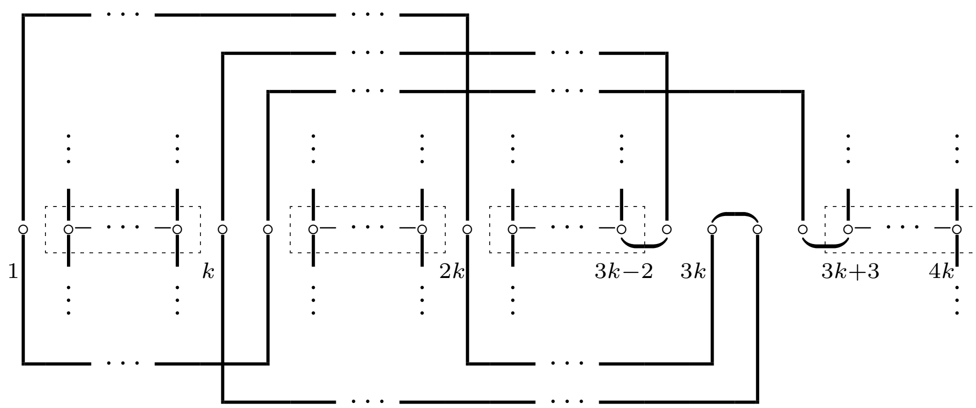


Fig. 4. The subdivision of $G_3$.

following new edges, see Fig. 4.

$$e_1 = (3k-2, 3k-1),$$

$$e_2 = (3k-1, k),$$

$$e_3 = (k, 3k+1),$$

$$e_4 = (3k+1, 3k),$$

$$e_5 = (3k, 2k),$$

$$e_6 = (2k, 1),$$

$$e_7 = (1, k+1),$$

$$e_8 = (k+1, 3k+2),$$

$$e_9 = (3k+2, 3k+3).$$

Note that the edges $e_3$ and $e_8$ have length $2k+1$, the edges $e_2$ and $e_6$ have length $2k-1$, the edges $e_5$ and $e_7$ have length $k$ and the edges $e_1$, $e_4$ and $e_9$ have length 1. Thus, the cycle $C$ has the following distribution of edges:

$$\begin{pmatrix} k & 2k+1 \\ k & 2k-1 \\ k & k \\ k & 1 \end{pmatrix}.$$

Since $C$ is a $[1, 4k]$-graph, it is a minimal $k$-equitable representation of $C_{4k}$, and so $C_{4k}$ is minimally $k$-equitable. $\square$

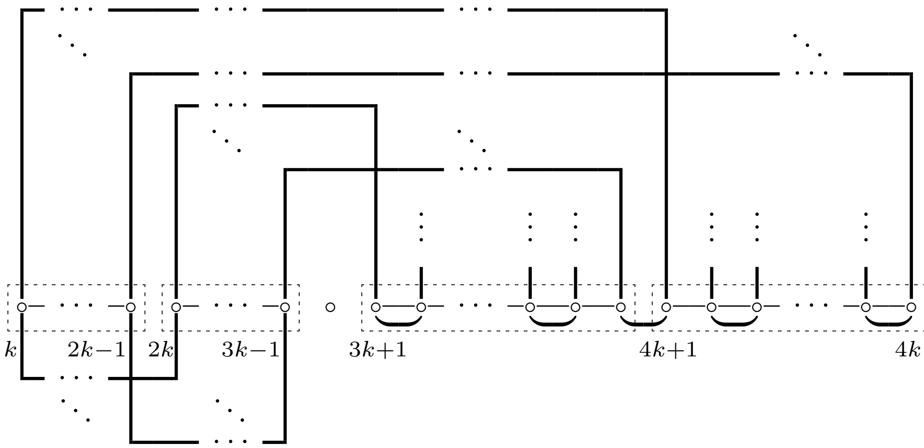


Fig. 5. The graph $G_3$.

The next lemma finishes the case $k$ odd.

**Lemma 3.3.** *If $k$ is an odd integer greater than 2, and $m > 4$, then the cycle $C_{mk}$ is minimally $k$-equitable.*

*Proof.* The construction for $m = 4$ in the proof of Lemma 3.2 cannot be extended by simple subdivision to give a minimal $k$-equitable representation of $C_{5m}$. We must again start from the begining. Let us start, similarly as for $m = 2, 3$, with $G_1 = C(3k + 1, 4k + 1, k)$ having the following distribution of edges:

$$\begin{pmatrix} k & k \\ k & 1 \end{pmatrix}.$$

Now, let us subdivide the edges of length $k$, but unlike in the case $k = 3$ in Lemma 3.2, let us get a graph with a gap of one integer inside it by defining $G_2$ to be the $(0, k; k; 2)$-extension of $G_1$. $G_2$ has, thus, the following distribution of edges:

$$\begin{pmatrix} k & 2k+1 \\ k & k+1 \\ k & 1 \end{pmatrix},$$

and is a $(0, k; 2k + 1)$-outer cycle. Continuing with another subdivision, this time of the edges of length $2k + 1$, let $G_3$ be the $(0, k; 2k + 1; 1)$-extension of $G_2$, see Fig. 5.

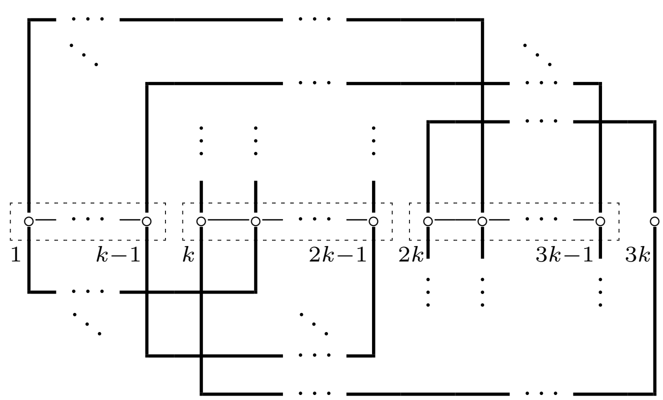


Fig. 6. The subdivision of $G_3$.

Note that the cycle $G_3$ has the following distribution of edges:

$$\begin{pmatrix} k & 3k+1 \\ k & k+1 \\ k & k \\ k & 1 \end{pmatrix}.$$

Now, we shall subdivide the edges of length $k$ in such a way that we remove the gap inside our graph. Let $C$ be obtained from $G_3$ by adding the set $[1, k-1] \cup \{3k\}$ to the set of vertices, subdividing the edge $(k, 2k)$ with the vertex $3k$ and subdividing every other edge of length $k$ with the apropriate vertex in the segment $[1, k-1]$ such that we get edges of lengths $k$ and $2k$, see Fig. 6.

Thus, the cycle $C$ has the following distribution of edges:

$$\begin{pmatrix} k & 3k+1 \\ k & 2k \\ k & k+1 \\ k & k \\ k & 1 \end{pmatrix}.$$

Since $C$ is a $[1, 5k]$-graph, it is a minimal $k$-equitable representation of $C_{5k}$.

Now, at last, we are at a point from which we can continue by induction. To finish the proof of the lemma, we shall construct by induction a family of cycles $C^{(m)}$, for $m = 5, 6, \ldots$, such that $C^{(m)}$ is a minimal $k$-equitable representation of $C_{km}$. To keep the induction going, we shall also make sure that the cycle $C^{(m)}$ has the additional property of being $(0, k; (m+1)k/2 + 1)$-outer for $m$ odd and

$(k, 0; (m-2)k/2-1)$-outer for $m$ even. It will also have the following distribution of edges:

$$\begin{pmatrix} k & 5k \\ k & 6k \\ \vdots & \vdots \\ k & (m-2)k \\ k & (m-1)k \\ k & (m+1)k/2+1 \\ k & 2k \\ k & k+1 \\ k & k \\ k & 1 \end{pmatrix}$$

when $m$ is odd and the following:

$$\begin{pmatrix} k & 5k \\ k & 6k \\ \vdots & \vdots \\ k & (m-2)k \\ k & (m-1)k \\ k & (m-2)k/2-1 \\ k & 2k \\ k & k+1 \\ k & k \\ k & 1 \end{pmatrix}$$

when $m$ is even.

Let $C^{(5)}$ be the cycle $C$ constructed above. It is clearly a $(0, k; 3k+1)$-outer cycle with the appropriate distribution of edges. We shall perform the induction by subdividing each edge of $C^{(m)}$ of length $(m+1)k/2+1$, for $m$ odd, to get two edges of lengths $mk = ((m+1)-1)k$ and $(m-1)k/2-1 = ((m+1)-2)k/2-1$. Analogously, for $m$ even, we shall subdivide each edge of $C^{(m)}$ of length $(m-2)k/2-1$ to get two edges of lengths $mk$ and $(m+2)k/2+1$.

So assume that the cycle $C^{(m)}$ with the required properties is constructed. If $m$ is odd, then $C^{(m)}$ is a $(0, k; (m+1)k/2+1)$-outer cycle. Let $C^{(m+1)}$ be the $(0, k; (m+1)k/2+1; 1)$-extension of $C^{(m)}$. Since $C^{(m)}$ does not have edges of length $(m-1)k/2-1 = ((m+1)-2)k/2-1$, it follows from (2) that $C^{(m+1)}$ is an $(k, 0; ((m+1)-2)k/2-1)$-outer cycle as required. Also by (2), $C^{(m+1)}$ has the required distribution of edges, and is thus a $k$-equitable representation of $C_{(m+1)k}$.

Since the last parameter in the extension operation by which $C^{(m+1)}$ is defined is equal to 1, $C^{(m+1)}$ is a $[j, j+(m+1)k-1]$-graph for a certain integer $j$ and hence a minimal $k$-equitable representation of $C_{(m+1)k}$.

If $m$ is even, then $C^{(m)}$ is a $(k, 0; (m-2)k/2)$-outer cycle, so let $C^{(m+1)}$ be the $(k, 0; (m-2)k/2; 1)$-extension of it. By an argument similar to the above one, $C^{(m+1)}$ has the required properties. $\square$

## §3.3. The case $k$ even

We shall break this case into two cases depending on the divisibility of $k$ by 4. Let us first consider the case $k \equiv 2 \bmod 4$.

**Lemma 3.4.** *If $k \equiv 2$ mod 4 and $m > 1$, then the cycle $C_{mk}$ is minimally $k$-equitable.*

*Proof.* Let us assume that $k$ is fixed. We shall prove the lemma by induction on $m$. Unlike in the case $k$ odd, we can start the induction from $m = 2$. Thus, we shall construct a family of cycles $C^{(m)}$, for $m = 2, 3, \ldots$, such that $C^{(m)}$ is a minimal $k$-equitable representation of $C_{mk}$. The cycle $C^{(m)}$ will have the additional property of being $(k/2, k/2; (m-1)k/2+1)$-outer for $m$ even and $(k/2, k/2; mk/2-1)$-outer for $m$ odd. It will also have the following distribution of edges:

$$\begin{pmatrix} k & 2k \\ k & 3k \\ \vdots & \vdots \\ k & (m-2)k \\ k & (m-1)k \\ k & (m-1)k/2+1 \\ k & 1 \end{pmatrix}$$

when $m$ is even and the following:

$$\begin{pmatrix} k & 2k \\ k & 3k \\ \vdots & \vdots \\ k & (m-2)k \\ k & (m-1)k \\ k & mk/2-1 \\ k & 1 \end{pmatrix}$$

when $m$ is odd.

The cycle $C^{(2)}$ is constructed as follows. If $k = 2$, then set $C^{(2)} = (1, 2, 4, 3)$. For $k > 2$, let us take two cycles $G_1 = C(1, k/2+2; k/2)$ and $G_2 = C(k, 3k/2+1; k/2)$. By (1), both $G_1$ and $G_2$ have the following distribution of edges:

$$\begin{pmatrix} k/2 & k/2+1 \\ k/2-1 & 1 \\ 1 & 2 \end{pmatrix}.$$

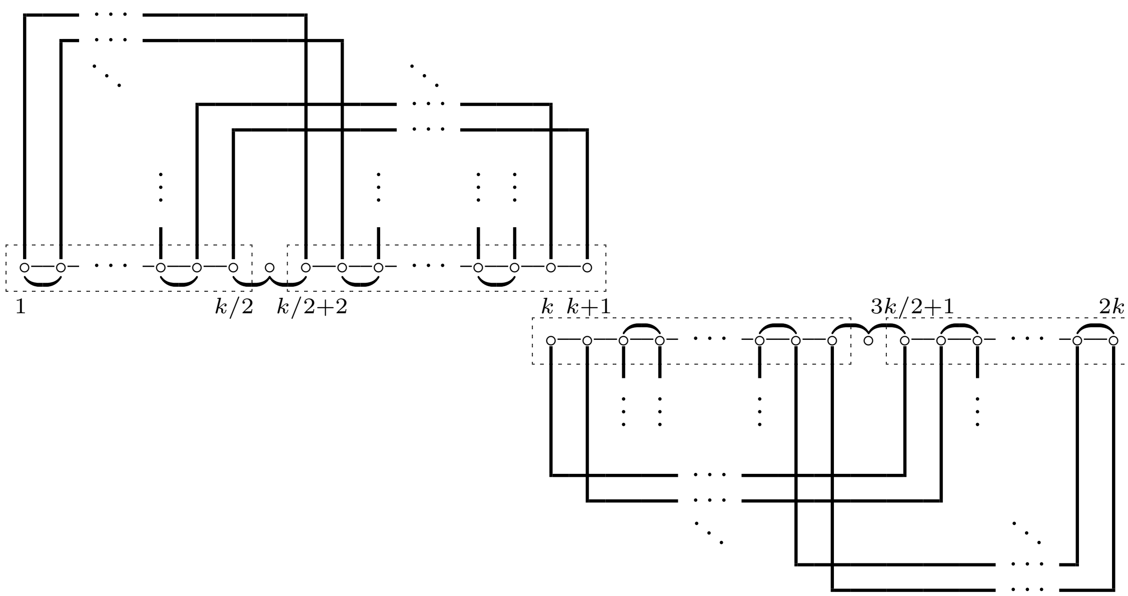


Fig. 7. The cycle $C^{(2)}$.

The cycles $G_1$ and $G_2$ have two common vertices $k$ and $k+1$ and one common edge $e_0 = (k, k+1)$. Let $C$ be the cycle obtained by taking the union of $G_1$ and $G_2$ with the edge $e_0$ removed. Thus, $C$ has the following distribution of edges.

$$\begin{pmatrix} k & k/2+1 \\ k-4 & 1 \\ 2 & 2 \end{pmatrix}$$

Let $C^{(2)}$ be obtained from $C$ by adding the set $\{k/2+1, 3k/2\}$ to the vertex set, and subdividing the edge $(k/2, k/2+2)$ with the vertex $k/2+1$ and the edge $(3k/2-1, 3k/2+1)$ with $3k/2$, see Fig. 7.

The cycle $C^{(2)}$ satisfies the required conditions because it is $(k/2, k/2; k/2+1)$-outer, it is a $[1, 2k]$-graph, and it has the following distribution of edges.

$$\begin{pmatrix} k & k/2+1 \\ k & 1 \end{pmatrix}$$

Let us assume that the cycle $C^{(m)}$ is constructed, and that it satisfies the required conditions. In the process of induction, for $m$ even, we shall subdivide each edge of $C^{(m)}$ of length $(m-1)k/2+1$ to get two edges of lengths $mk =$

$((m+1)-1)k$ and $(m+1)k/2-1$. Analogously, for $m$ odd, we shall subdivide each edge of $C^{(m)}$ of length $mk/2-1$ to get two edges of lengths $mk$ and $mk/2+1 = ((m+1)-1)k/2+1$.

If $m$ is even, then $C^{(m)}$ is a $(k/2, k/2; (m-1)k/2+1)$-outer cycle. Let $C^{(m+1)}$ be the $(k/2, k/2; (m-1)k/2+1; 1)$-extension of it. Since if $(m+1)k/2-1 \neq (m-1)k/2+1$, then $C^{(m)}$ does not have edges of length $(m+1)k/2-1$, $C^{(m+1)}$ is a $(k/2, k/2; (m+1)k/2-1)$-outer cycle by (2). It also follows from (2) that $C^{(m+1)}$ has the required distribution of edges and hence it is a $k$-equitable representation of $C_{(m+1)k}$. Clearly, $C^{(m+1)}$ is a $[j, j+(m+1)k-1]$-graph for a certain integer $j$, and so is a minimal $k$-equitable representation of $C_{(m+1)k}$.

If $m$ is odd, then let $C^{(m+1)}$ be the $(k/2, k/2; mk/2-1)$-extension of $C^{(m)}$. Similarly as above, it can be verified that all the required conditions are satisfied. $\square$

Now we shall consider the case $k \equiv 0 \bmod 4$.

**Lemma 3.5.** *If $k \equiv 0$ mod 4 and $m > 1$, then the cycle $C_{mk}$ is minimally $k$-equitable.*

*Proof.* Let us assume that $k$ is fixed. Similarly as in the proof of Lemma 3.4, we shall use induction on $m$, and we shall construct a family of cycles $C^{(m)}$, for $m = 2, 3, \ldots$, such that $C^{(m)}$ is a minimal $k$-equitable representation of $C_{mk}$. Now, the cycle $C^{(m)}$ will have the additional property of being $(k_1, k_2; (m-1)k/2+1)$-outer for $m$ even and $(k_2, k_1; mk/2-1)$-outer for $m$ odd, where $k_1 = k/2+1$ and $k_2 = k/2-1$. As in the construction used to prove Lemma 3.4, it will have the following distribution of edges:

$$\begin{pmatrix} k & 2k \\ k & 3k \\ \vdots & \vdots \\ k & (m-2)k \\ k & (m-1)k \\ k & (m-1)k/2+1 \\ k & 1 \end{pmatrix}$$

when $m$ is even and the following:

$$\begin{pmatrix} k & 2k \\ k & 3k \\ \vdots & \vdots \\ k & (m-2)k \\ k & (m-1)k \\ k & mk/2-1 \\ k & 1 \end{pmatrix}$$

when $m$ is odd.

The cycle $C^{(2)}$ is constructed in a way which is a slight modification of the method used to prove lemma 3.4. Let $G_1 = C(1, k/2+2; k_1)$ and $G_2 = C(k+1, 3k/2+2; k_2)$. Note that $k_1$ and $k_2$ are odd integers. By (1), $G_1$ has the following distribution of edges:

$$\begin{pmatrix} k/2+1 & k/2+1 \\ k/2+1 & 1 \end{pmatrix},$$

and $G_2$ the following:

$$\begin{pmatrix} k/2-1 & k/2+1 \\ k/2-2 & 1 \\ 1 & 3 \end{pmatrix}.$$

The cycles $G_1$ and $G_2$ have two common vertices $k+1$ and $k+2$ and one common edge $e_0 = (k+1, k+2)$. Let $C$ be the cycle obtained by taking the union of $G_1$ and $G_2$ with the edge $e_0$ removed. Thus, $C$ has the following distribution of edges.

$$\begin{pmatrix} k & k/2+1 \\ k-3 & 1 \\ 1 & 3 \end{pmatrix}$$

Let $C^{(2)}$ be obtained from $C$ by adding the set $\{3k/2, 3k/2+1\}$ to the vertex set, and subdividing the edge $(3k/2-1, 3k/2+2)$ with the vertices $3k/2$ and $3k/2+1$ as to get three edges of length 1, see Fig. 8.

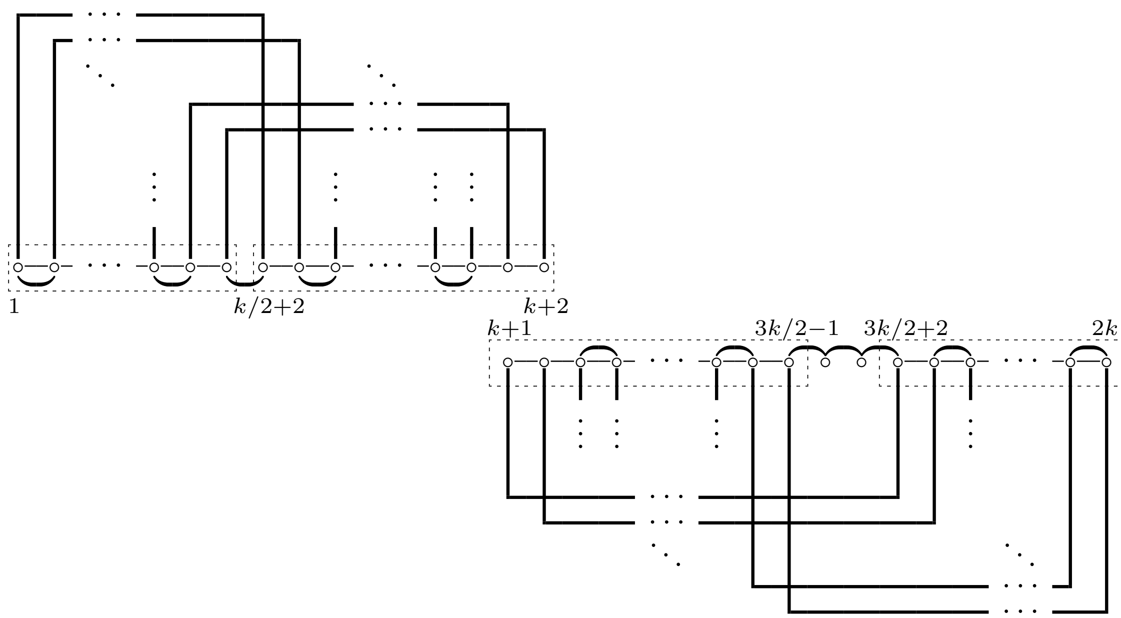


Fig. 8. The cycle $C^{(2)}$.

Hence, the cycle $C^{(2)}$ satisfies the required conditions because it is $(k_1, k_2; k/2+1)$-outer, it is a $[1, 2k]$-graph, and it has the following distribution of edges.

$$\begin{pmatrix} k & k/2+1 \\ k & 1 \end{pmatrix}$$

Let us assume that the cycle $C^{(m)}$ is constructed and that it satisfies the required conditions. The induction goes almost exactly as in the proof of Lemma 3.4. The only difference is that the endpoints of the edges to be subdivided do not lie symmetrically at both ends of the segment of all vertices, but there are $k_1$ of them at one end and $k_2$ of them at the other.

If $m$ is even, then $C^{(m)}$ is a $(k_1, k_2; (m-1)k/2) + 1)$-outer cycle thus let $C^{(m+1)}$ be the $(k_1, k_2; (m-1)k/2 + 1; 1)$-extension of it. If $m$ is odd, then $C^{(m)}$ is $(k_2, k_1; mk/2 - 1)$-outer so let $C^{(m+1)}$ be obtained by taking the $(k_2, k_1; mk/2 - 1; 1)$-extension of it. Similarly as in the proof of Lemma 3.4, it can be shown that all the required conditions are satisfied. $\square$

# CHAPTER 4

# SPLITTING NECKLACES AND A GENERALIZATION OF THE BORSUK-ULAM ANTIPODAL THEOREM

## §4.1. Introduction

Let $t$ be a natural number. An *opened $t$-coloured necklace* is a sequence of elements (*beads*) from the integer segment $[1, t]$. Let $N$ be an opened $t$-coloured necklace. A *splitting* of $N$ is a partition $N_1 \cup N_2 \cup \ldots \cup N_\ell$ of the set of beads of $N$ such that for every colour $i$, $1 \leq i \leq t$, the beads of colour $i$ are spread evenly between the sets $N_j$, i.e. all of the sets $N_j$ contain the same number of beads of colour $i$. A splitting of $N$ which is a partition into $k$ sets is called a *$k$-splitting*. The *size* of the splitting of $N$ is the minimal number of cutpoints of $N$ needed to partition it into segments preserved by the splitting.

Note that if the beads of each colour are consecutive in $N$, then any $k$-splitting cuts each segment of one colour beads at $k-1$ points at least, and hence has size at least $t(k-1)$. The following natural question arises: is this trivial lower bound also an upper bound? In other words, if $N$ is an opened $t$-coloured necklace admitting a $k$-splitting, does $N$ have a $k$-splitting of size $t(k-1)$? Somewhat surprisingly the answer to this question is 'yes'.

Let us now briefly describe the history of this problem. Bhatt and Leiserson [9] and Bhatt and Leighton [8] pointed out that this problem has some applications to VLSI circuit design. Goldberg and West [34] proved that for every $t$, an opened

$t$-coloured necklace admitting a 2-splitting has a 2-splitting of size $t$. They also raised the question about the general upper bound for $k$-splittings. Alon and West [5] gave a very short proof of the above upper bound for 2-splittings using the Borsuk-Ulam antipodal theorem; they also conjectured that $t(k-1)$ is an upper bound for $k$-splittings. Alon [4] proved the $t(k-1)$ upper bound for $k$-splittings using involved methods of algebraic topology. In this chapter we are going to give another proof of Alon's result. Our proof will be more elementary and will use a classical result of algebraic topology (Lemma 4.10) only as a starting point; after that the argument will be purely combinatorial.

**Theorem 4.1.** (N. Alon [4]) *Every necklace with $ka_i$ beads of colour $i$, $1 \leq i \leq t$, has a $k$-splitting of size at most $t(k-1)$.*

To prove Theorem 4.1 we shall formulate and prove a new, very natural generalization of the Borsuk-Ulam antipodal theorem. From this generalization we shall immediately obtain a continuous version of Theorem 4.1 implying, as in Alon [4], Theorem 4.1 itself.

To formulate our generalization of the Borsuk-Ulam antipodal theorem, we must introduce some more terminology. Let $\mathbb{R}_+$ be the metric space of nonnegative reals with the natural metric. Given a natural number $n$, let $\mathbb{R}_{+,n}$ be obtained by taking the product of $\mathbb{R}_+$ with the integer segment $[0, n-1] \subset \mathbb{N}$ and identifying the points $(0,0)$, $(0,1)$, ..., $(0, n-1)$ to a single point denoted 0. The metric $\mu$ on $\mathbb{R}_{+,n}$ is defined as follows:

$$\mu\big((x,i),(y,i)\big) = |x-y|$$

and

$$\mu\big((x,i),(y,j)\big) = x+y$$

for $x, y \in \mathbb{R}_+$, $0 \leq i, j \leq n-1$, and $i \neq j$. Thus $\mathbb{R}_{+,n}$ is the union of $n$ half-lines with a common endpoint and equipped with the natural metric.

Given a natural number $m$, let $\mathbb{R}^m_{+,n}$ be the product

$$\underbrace{\mathbb{R}_{+,n} \times \mathbb{R}_{+,n} \times \ldots \times \mathbb{R}_{+,n}}_{m \text{ times}}$$

with the metric $\mu$ defined by

$$\mu\Big(\big(x_1, x_2, \ldots, x_m\big), \big(y_1, y_2, \ldots, y_m\big)\Big) = \sum_{i=1}^{m} \mu\big(x_i, y_i\big).$$

Let $\mathbb{O}$ be the point $(0, 0, \ldots, 0) \in \mathbb{R}^m_{+,n}$, and let $\mathbb{S}^{m-1}_n$ be the unit sphere in $\mathbb{R}^m_{+,n}$ with the centre at $\mathbb{O}$, i.e. let

$$\mathbb{S}^{m-1}_n = \left\{x \in \mathbb{R}^m_{+,n} : \mu(x, \mathbb{O}) = 1\right\}.$$

Let $\eta : [0, n-1] \to [0, n-1]$ be the function of taking the cyclic successor, i.e. let $\eta(i) = (i+1) \bmod n$, $i = 0, 1, \ldots, n-1$. Let $\omega : \mathbb{S}^{m-1}_n \to \mathbb{S}^{m-1}_n$ be defined by

$$\omega\Big(\big(x_1, i_1\big), \big(x_2, i_2\big), \ldots, \big(x_m, i_m\big)\Big) = \Big(\big(x_1, \eta(i_1)\big), \big(x_2, \eta(i_2)\big), \ldots, \big(x_m, \eta(i_m)\big)\Big).$$

We are now ready to state our generalization of the Borsuk-Ulam's theorem.

**Theorem 4.2.** *If $p$ is a prime and $m$ is any natural number, then for any continuous map*

$$h : \mathbb{S}^{m(p-1)}_p \to \mathbb{R}^m$$

*there exists an $x \in \mathbb{S}^{m(p-1)}_p$ such that*

$$h(x) = h\big(\omega(x)\big) = \ldots = h\big(\omega^{p-1}(x)\big).$$

Note that for $p = 2$, $\mathbb{S}^{m(p-1)}_p$ is naturally homeomorphic to $\mathbb{S}^m$, the $\ell_1$-sphere in $\mathbb{R}^{m+1}$, with the map $\omega$ on $\mathbb{S}^m_2$ corresponding to the antipodal map on $\mathbb{S}^m$. Thus if $p = 2$, Theorem 4.2 is a reformulation of the Borsuk-Ulam antipodal theorem. In Section 4.4 (Lemma 4.12), we shall give another description of $\mathbb{S}^{m(p-1)}_p$ by defining a triangulation of it.

The rest of this chapter is partitioned as follows. In Section 4.2, we prove Theorem 4.1 using Theorem 4.2; in Section 4.3, we prove the main lemma needed in the proof of Theorem 4.2, whose proof is given in Section 4.4.

## §4.2. Continuous Splittings

In this section we shall prove Theorem 4.3, which easily implies Theorem 4.1, and is in fact a continuous version of it. We shall show that Theorem 4.3 follows immediately from Theorem 4.2. Now, let us introduce the terminology needed to formulate Theorem 4.3. Let $I = [0, 1]$ be the real unit interval. An *interval $m$-colouring* is a function from $I$ to the integer segment $[1, m]$ such that the set of points mapped to $i$, $1 \leq i \leq m$, is (Lebesgue) measurable. A *$k$-splitting of size $r$* of such a colouring is a partition $I = F_1 \cup \ldots \cup F_k$ satisfying the following conditions:

(i) There is a sequence of numbers $0 = y_0 \leq y_1 \leq \ldots \leq y_r \leq y_{r+1} = 1$ such that for each of the segments $(y_i, y_{i+1})$, $0 \leq i \leq m$, and each of the sets $F_j$, $1 \leq j \leq k$, $(y_i, y_{i+1})$ is either contained in $F_j$ or is disjoint from it.

(ii) The measure of the set of points mapped to $i$, $1 \leq i \leq m$, which are contained in $F_j$, $1 \leq j \leq k$, is precisely $1/k$ of the total measure of the points of the colour $i$.

**Theorem 4.3.** (Alon [4]) *If $p$ is a prime number, then every interval $m$-colouring has a $p$-splitting of size $m(p-1)$.*

The proof of this result given by Alon uses a generalization of the Borsuk-Ulam antipodal theorem due to Bárány, Shlosman and Szücs [7], and another topological result of Bárány, Shlosman and Szücs ([7] Statement A′). We shall show that our new generalization of the Borsuk-Ulam antipodal theorem is strong enough to imply Theorem 4.3 immediately.

*Proof of Theorem* 4.3. Let $f : I \to [1, m]$ be an interval $m$-colouring. We shall define a continuous map $h : \mathbb{S}_p^{m(p-1)} \to \mathbb{R}^m$ and apply Theorem 4.2. Let $q = m(p-1) + 1$. Given

$$x = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_q) \in \mathbb{S}_p^{m(p-1)}$$

where

$$\overline{x}_i = (x_i, k_i)\,,$$

$i = 1, 2, \ldots, q$, $x_i \in \mathbb{R}_+$, $0 \le k_i \le p-1$, let

$$I = F_0^{(x)} \cup F_1^{(x)} \cup \ldots \cup F_{p-1}^{(x)}$$

be a splitting of size $m(p-1)$ of $f$ defined as follows. Let $0 = y_0 \le y_1 \le \ldots \le y_{q-1} \le y_q = 1$ be the sequence or reals satisfying

$$y_i - y_{i-1} = x_i,$$

for $i = 1, \ldots, q$. Note that

$$\sum_{i=1}^{q} x_i = 1.$$

Let

$$J_s^{(x)} = \{ i : 1 \le i \le q, \ k_i = s \},$$

and

$$F_s^{(x)} = \bigcup_{i \in J_s^{(x)}} (y_{i+1}, y_i),$$

$s = 0, 1, \ldots, p-1$. In other words the partition $I = F_0^{(x)} \cup F_1^{(x)} \cup \ldots \cup F_{p-1}^{(x)}$ is obtained by cutting $I$ into consecutive segments of lengths $x_1, x_2, \ldots, x_q$ and putting the $i$-th segment into the set $F_{k_i}^{(x)}$. Let $h(x) = (r_1, r_2, \ldots, r_m) \in \mathbb{R}^m$ be such that $r_i$, $1 \le i \le m$, is the measure of the set of points contained in $F_0^{(x)}$ which are mapped to $i$ by $f$. Clearly $h$ is continuous.

By Theorem 4.2, there exists $x \in \mathbb{S}_p^{m(p-1)}$ such that

$$h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x)). \tag{1}$$

We claim that the partition $I = F_0^{(x)} \cup F_1^{(x)} \cup \ldots \cup F_{p-1}^{(x)}$ is a $p$-splitting of $f$. To prove the claim we shall show that

$$h(\omega^j(x)) = (r_1^{(j)}, r_2^{(j)}, \ldots, r_m^{(j)}),$$

$0 \le j \le p-1$, where $r_i^{(j)}$, $1 \le i \le m$, is the measure of the set of points contained in $F_{\eta^{-j}(0)}^{(x)}$ which are mapped to $i$ by $f$. This will finish the proof of the theorem since it follows from (1) that, for $1 \le i \le m$, we have

$$r_i^{(0)} = r_i^{(1)} = \ldots = r_i^{(p-1)}.$$

Note that for $j = 0, 1, \ldots, p-1$ we have

$$\omega^j(x) = \left(\overline{\overline{x}}_1, \overline{\overline{x}}_2, \ldots, \overline{\overline{x}}_q\right)$$

where

$$\overline{\overline{x}}_i = \left(x_i, \eta^j(k_i)\right),$$

$i = 1, 2, \ldots, q$. Thus

$$\begin{aligned} J_s^{(\omega^j(x))} &= \left\{i : 1 \le i \le q,\ \eta^j(k_i) = s\right\} \\ &= \left\{i : 1 \le i \le q,\ k_i = \eta^{-j}(s)\right\} = J^{(x)}_{\eta^{-j}(s)}. \end{aligned}$$

Therefore

$$F_0^{(\omega^j(x))} = \bigcup_{k \in J^{(y)}_{\eta^{-j}(0)}} (y_{k+1}, y_k) = F^{(x)}_{\eta^{-j}(0)},$$

and

$$h(\omega^j(x)) = (r_1^{(j)}, r_2^{(j)}, \ldots, r_m^{(j)}) \in \mathbb{R}^m$$

where $r_i$, $1 \le i \le m$, is the measure of the set of points contained in $F^{(x)}_{\eta^{-j}(0)}$ which are mapped to $i$ by $f$. This completes our proof. □

Note that in Theorem 4.3 we assume that $p$ is prime. Unlike in the case of Theorem 4.2 this assumption is not essential. We are now going to present Alon's proofs that Theorem 4.3 implies its generalized version, Corollary 4.4, and that Corollary 4.4 implies Theorem 4.1.

**Corollary 4.4.** (Alon [4]) *For any natural numbers $k$ and $m$, every interval $m$-colouring has a $k$-splitting of size $m(k-1)$.*

*Proof.* (Alon [4]) We shall use induction on the number of prime factors of $k$. If $k$ is prime then Corollary 4.4 follows from Theorem 4.3. Let $k = k_1 k_2$ where $k_1, k_2 \neq 1$, and assume that every interval $m$-colouring has a $k'$-splitting of size $m(k'-1)$ for any integer $k'$ having less primes in its factorization than $k$ has.

Let $f : I \to [1, m]$ be an interval $m$-colouring. We shall show that $f$ has a $k$-splitting of size $k(m-1)$. By our induction assumption $f$ has a $k_1$-splitting $I = F_1 \cup F_2 \cup \ldots \cup F_{k_1}$ of size $m(k_1 - 1)$. By point (i) of the definition of splittings for interval colourings there is a sequence of numbers $0 = y_0 \le y_1 \le \ldots \le y_{m(k_1-1)} \le y_{m(k_1-1)+1} = 1$ such that for each of the segments $I_i = (y_i, y_{i+1})$, $0 \le i \le m(k_1 - 1)$, and each of the sets $F_j$, $1 \le j \le k_1$, $I_i$ is either contained in $F_j$ or is disjoint from it. Clearly we can assume that all $I_i$ are nonempty since otherwise we can change our sequence of numbers $y_j$ by deleting repeating ones, and adding new.

For $j = 1, 2, \ldots, k_1$, let $f_j : I \to [1, m]$ be the interval $m$-colouring obtained as follows. Let us place the intervals $I_i$ contained in $F_j$ next to each other getting an interval $A_j$, and let $\alpha_j : I \to A_j$ be the affine map taking 0 to the smaller endpoint of $A_j$ and 1 to its bigger endpoint. Now set $f_j = f \circ \alpha_j$. By the inductive assumption there is a $k_2$-splitting of $f_j$ of size $m(k_2 - 1)$, $j = 1, 2, \ldots, k_1$. Transforming these $k_2$-splittings into partitions of $F_j$, for $j = 1, 2, \ldots, k_1$, we get a partition of $I$ into $k = k_1 k_2$ sets which is a $k$-splitting of $f$ of size

$$m(k_1 - 1) + k_1(m(k_2 - 1)) = m(k - 1).$$

Thus the proof of the theorem is complete. □

*Proof of Theorem* 4.1. (Alon [4]) Let $f : I \to [1, t]$ be the interval $t$-colouring obtained by partitioning $I$ into $s = k \sum_{i=1}^{t} a_i$ segments of equal length (called in the future by *small segments*) and colouring the $i$-th small segment with the colour of the $i$-th bead of the necklace. By Corollary 4.4 there is a $k$-splitting $I = F_1 \cup F_2 \cup \ldots \cup F_k$ of size $t(k-1)$ of $f$. This splitting can be transformed into a $k$-splitting of size $t(k-1)$ of the necklace provided that the cuts do not occur inside the small segments. We shall show by induction on the number of this 'bad' cuts that the $k$-splitting of size $t(k-1)$ of $f$ can be transformed into a $k$-splitting of size $t(k-1)$ of $f$ without any 'bad' cuts.

If there are no 'bad' cuts then we are done. Assume that there are $k > 0$ 'bad' cuts and that the result holds for any number $k' < k$ of 'bad' cuts. Let $i$, $1 \leq i \leq t$, be a colour such that the number of 'bad' cuts occuring inside small segments of colour $i$ is positive. Let us define a multigraph with $\{F_j : 1 \leq j \leq k\}$ as the vertex set and $(F_j, F_\ell)$ being an edge if there is a 'bad' cut occuring inside a small segment of colour $i$ and between a segment contained in $F_j$ and a segment contained in $F_\ell$. Since for every $j$, $1 \leq j \leq k$, the measure of points of colour $i$ contained in $F_j$ is a multiple of the length of a small segment, there are no vertices of degree 1 in our multigraph. Therefore it contains a cycle. By shifting the cuts corresponding to this cycle along the small segments in which they occur, we can decrease the number of 'bad' cuts at least by 1 getting again a $k$-splitting of $f$ of size $t(k-1)$. This completes the proof of the induction step, and hence the proof of the theorem. $\square$

## §4.3. The Main Lemma

Our aim in this section is to prove Lemma 4.11 from which we shall deduce Theorem 4.2 in the next section. First, let us introduce some more terminology. If $x_0, x_1, \ldots, x_k$ are points in $\mathbb{R}^m$ such that $\{x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0\}$ is a linearly independent set of $k$ vectors in $\mathbb{R}^m$, then we say that these points are *affinely independent*. Let $0 \leq k \leq m$, and $x_0, x_1, \ldots, x_k$ be affinely independent points in $\mathbb{R}^m$. The *$k$-simplex* $\Delta = (x_0, x_1, \ldots, x_k)$ is the following subset of $\mathbb{R}^m$:

$$\left\{ x = \sum_{i=0}^{k} \mu_i x_i : \sum_{i=0}^{k} \mu_i = 1,\ \mu_i > 0 \right\}. \tag{2}$$

Since the points $x_0, x_1, \ldots, x_k$ are affinely independent, the reals $\mu_i$, $0 \leq i \leq k$, are uniquely determined by $x$ and $x_0, x_1, \ldots, x_k$. We shall call the sum in (2) the barycentric representation of $x$ with respect to $(x_0, x_1, \ldots, x_k)$. The points $x_0, \ldots, x_k$ are the *vertices* of $\Delta$; the *skeleton* of $\Delta$ is the set of all its vertices, and $k$ is the dimension of $\Delta$. A simplex $\Delta_1$ is a *face* of a simplex $\Delta_2$ if the skeleton of

$\Delta_1$ is a subset of the skeleton of $\Delta_2$.

A *simplicial complex* $K$ is a finite set of disjoint simplices such that every face of every simplex of $K$ is also a simplex of $K$. The *body* $|K|$ of the simplicial complex $K$ is the union of all its simplices; the complex $K$ is then also called a *simplicial decomposition* of $|K|$.

If $\{x_1, x_2, \ldots, x_k\}$ is the set of vertices of the simplicial complex $K$ and $x \in |K|$, then there are unique reals $\mu_1, \mu_2, \ldots, \mu_k$ such that

$$x = \sum_{i=1}^{k} \mu_i x_i, \tag{3}$$

where $\mu_i \geq 0$ for every $i = 1, 2, \ldots, k$,

$$\sum_{i=1}^{k} \mu_i = 1,$$

and the set $\{x_i : \mu_i > 0\}$ is a simplex of $K$. We shall call the sum (3) the *barycentric representation* of $x$ with respect to $K$, or just the barycentric representation of $x$ if the complex is clear from the context.

The simplicial complex $K'$ is a *subcomplex* of the simplicial complex $K$ if the set of simplices of $K'$ is a subset of the set of simplices of $K$, in particular the set of vertices of $K'$ is a subset of the set of vertices of $K$.

Let $\omega$ be a continuous function from a subset $X$ of $\mathbb{R}^m$ to itself, and $k$ be a natural number. We shall say that $\omega$ is a *$Z_k$-action* if the set $\{\omega^0, \omega, \omega^2, \ldots, \omega^{k-1}\}$, where $\omega^0$ is the identity map on $X$, is a $k$-element cyclic group under composition. We shall also say that such an action is *free* if for every $x \in X$ all the elements $x$, $\omega(x)$, $\omega^2(x), \ldots, \omega^{k-1}(x)$ are different.

Let $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ be the $\ell_1$-norm on $\mathbb{R}^m$, namely for $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, let

$$\|x\| = \sum_{i=1}^{m} |x_i|.$$

Let

$$\mathbb{B}^m = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$$

be the $m$-*dimensional unit ball* and let

$$\mathbb{S}^m = \left\{x \in \mathbb{R}^{m+1} : \|x\| = 1\right\}$$

be the $m$-*dimensional unit sphere*.

Let $p$ be a fixed prime number. For each natural number $n$, we are going to define a simplicial complex $X_{n,p}$ such that $|X_{n,p}|$ is homeomorphic to the topological space obtained by identifying the boundaries of $p$ disjoint copies of the ball $\mathbb{B}^{(p-1)n}$. Also, each of the complexes $X_{n,p}$ will be equipped with a free $Z_p$-action $\omega$. We shall prove that for any continuous map $h : |X_{n,p}| \to \mathbb{R}^n$, there exists an $x \in |X_{n,p}|$ such that $h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x))$.

Before we define the family of complexes $X_{n,p}$, let us define the family of complexes $Y_{n,p}$ in $\mathbb{R}^{(p-1)n}$. For a given positive integer $n$ and $i = 1, \ldots, n$, let

$$x^0_{n,i} = (\underbrace{0, 0, \ldots, 0}_{(p-1)(i-1)}, \underbrace{-1, -1, \ldots, -1}_{p-1}, \underbrace{0, 0, \ldots, 0}_{(p-1)(n-i)}) \in \mathbb{R}^{(p-1)n},$$

and

$$x^j_{n,i} = (\underbrace{0, 0, \ldots, 0}_{(p-1)(i-1)}, \underbrace{0, 0, \ldots, 0}_{j-1}, 1, \underbrace{0, 0, \ldots, 0}_{p-j-1}, \underbrace{0, 0, \ldots, 0}_{(p-1)(n-i)}) \in \mathbb{R}^{(p-1)n},$$

for $j = 1, 2, \ldots, p-1$. Set

$$T_{n,i} = \left\{x^j_{n,i} : j = 0, 1, \ldots, p-1\right\}$$

and let $\Delta_{n,i}$ be the simplex with the skeleton $T_{n,i}$, $i = 1, \ldots, n$, and let

$$\bigcup_{i=1}^{n} T_{n,i}$$

be the set of vertices of $Y_{n,p}$. Let $T$ be the skeleton of a simplex of $Y_{n,p}$ if and only if for every $i = 1, \ldots, n$ we have

$$|T \cap T_{n,i}| \leq p - 1. \tag{4}$$

The elements of $Y_{n,p}$ are indeed simplices since for any set $T$ satisfying (4), the elements of $T$ are affinely independent.

Our aim now is to show that $Y_{n,p}$ is a simplicial decomposition of a subset of $\mathbb{R}^{(p-1)n}$ which is homeomorphic to the sphere $\mathbb{S}^{(p-1)n-1}$. Let us first prove the following lemma.

**Lemma 4.5.** *$Y_{n,p}$ is a simplicial complex.*

*Proof.* To prove that $Y_{n,p}$ is a simplicial complex it is enough to show that the simplices of $Y_{n,p}$ are pairwise disjoint. Let $\Delta_1$ and $\Delta_2$ be a pair of distinct simplices of $Y_{n,p}$, and suppose that there is an $a \in \Delta_1 \cap \Delta_2$. Let $T_1$ and $T_2$ be the skeletons of $\Delta_1$ and $\Delta_2$ respectively. As $a \in \Delta_1$ we have

$$a = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_{i,j} x_{n,i}^{j} \tag{5}$$

where $\mu_{i,j} \geq 0$, $1 \leq i \leq n$, $0 \leq j \leq p-1$,

$$\sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_{i,j} = 1,$$

and

$$T_1 = \left\{ x_{n,i}^{j} : 1 \leq i \leq n,\ 0 \leq j \leq p-1,\ \mu_{i,j} > 0 \right\}.$$

Analogously, as $a \in \Delta_2$ we have

$$a = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu'_{i,j} x_{n,i}^{j} \tag{6}$$

where $\mu'_{i,j} \geq 0$, $1 \leq i \leq n$, $0 \leq j \leq p-1$,

$$\sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu'_{i,j} = 1,$$

and

$$T_2 = \left\{ x_{n,i}^{j} : 1 \leq i \leq n,\ 0 \leq j \leq p-1,\ \mu'_{i,j} > 0 \right\}.$$

Thus, (5) and (6) are the barycentric representations of $a$ with respect to $\Delta_1$ and $\Delta_2$ respectively. Since $\Delta_1 \neq \Delta_2$, we have $T_1 \neq T_2$, and thus there are $i_0$ and $j_0$, $1 \leq i_0 \leq n$, $0 \leq j_0 \leq p-1$, such that $\mu_{i_0,j_0} \neq \mu'_{i_0,j_0}$.

Assume $a = (a_1, a_2, \ldots, a_{(p-1)n}) \in \mathbb{R}^{(p-1)n}$, and let

$$b = (b_1, \ldots, b_{p-1}) = (a_{(p-1)(i_0-1)+1}, a_{(p-1)(i_0-1)+2}, \ldots, a_{(p-1)i_0})$$

be the image of $a$ under the projection onto the $i_0$-th component of $\mathbb{R}^{(p-1)n} = \mathbb{R}^{p-1} \times \ldots \times \mathbb{R}^{p-1}$. We have

$$b = \sum_{j=0}^{p-1} \mu_{i_0,j} x_{n,i_0}^j = \sum_{j=0}^{p-1} \mu'_{i_0,j} x_{n,i_0}^j.$$

We shall obtain a contradiction by showing that $\mu_{i_0,j} = \mu'_{i_0,j}$, $0 \le j \le p-1$.

By the definition of $Y_{n,p}$, not all of $\mu_{i_0,j}$, $0 \le j \le p-1$, can be positive since $\Delta_1$ is a simplex of $Y_{n,p}$, and hence

$$\mu_{i_0,0} = -\min\{0, b_1, b_2, \ldots, b_{p-1}\},$$

and

$$\mu_{i_0,j} = b_j + \mu_{i_0,0},$$

for $j = 1, \ldots, p-1$. Analogously, since $\Delta_2$ is a simplex of $Y_{n,p}$, not all of $\mu'_{i_0,j}$, $0 \le j \le p-1$, can be positive and we have

$$\mu'_{i_0,0} = -\min\{0, b_1, b_2, \ldots, b_{p-1}\},$$

and

$$\mu'_{i_0,j} = b_j + \mu'_{i_0,0},$$

for $j = 1, \ldots, p-1$. Thus $\mu_{i_0,j} = \mu'_{i_0,j}$, $j = 0, \ldots, p-1$, as required. $\square$

Let $X_{n,p}$ be the subcomplex of $Y_{n+1,p}$ such that $T$ is the skeleton of a simplex of $X_{n,p}$ if and only if

$$|T \cap T_{n+1,n+1}| \le 1.$$

Now, we are going to prove that $|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$ which implies that $|X_{n,p}|$ is homeomorphic to the topological space obtained by identifying the boundaries of $p$ disjoint copies of the ball $\mathbb{B}^{(p-1)n}$.

In the proof we shall need the following two lemmas. Let $K$ be a simplicial complex and let $x$ be a vertex of $K$. We say that $K$ is an $x$-*cone* if for every simplex $\Delta$ of $K$ with skeleton $T$, say, $T \cup \{x\}$ is also the skeleton of a simplex of $K$. Furthermore, for an $x$-cone $K$ let $K'$ be the simplicial complex such that $\Delta$ is a simplex of $K'$ if $\Delta$ is a simplex of $K$ and $x$ is not a vertex of $\Delta$. Then, we shall say that $K$ is an *$x$-cone over $K'$*. Lemmas 4.6 and 4.7 clearly hold.

**Lemma 4.6.** *If $K$ is an $x$-cone over $K'$, and $|K'|$ is homeomorphic to the sphere $\mathbb{S}^k$ or to the ball $\mathbb{B}^k$, then $|K|$ is homeomorphic to $\mathbb{B}^{k+1}$.* □

**Lemma 4.7.** *Let $K_1$ and $K_2$ be simplicial complexes such that $|K_1|$ and $|K_2|$ are both homeomorphic to the ball $\mathbb{B}^{k+1}$, $K_1 \cup K_2$ is a simplicial complex and $|K_1 \cap K_2|$ is homeomorphic to the sphere $\mathbb{S}^k$. Then $|K_1 \cup K_2|$ is homeomorphic to the sphere $\mathbb{S}^{k+1}$.* □

We can now prove the following lemma.

**Lemma 4.8.** *$|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$.*

*Proof.* We shall use induction on $n$. For $n = 1$, $|Y_{n,p}|$ is the boundary of a $(p-1)$-dimensional simplex so $Y_{n,p}$ is homeomorphic to $\mathbb{S}^{p-2}$.

Given $n \geq 1$, assume that $|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$. Let $Y_{n,p}^{(\alpha)}$, $\alpha = 0, 1, \ldots, p-1$, and $\overline{Y}_{n,p}^{(\alpha)}$, $\alpha = 0, 1, \ldots, p-2$, be subcomplexes of $Y_{n+1,p}$ defined as follows. Let

$$\left\{x_{n+1,i}^{j} : i = 1, \ldots, n,\ j = 0, \ldots, p-1\right\} \cup \left\{x_{n+1,n+1}^{j} : j = 0, \ldots, \alpha\right\}$$

be the set of vertices of both $Y_{n,p}^{(\alpha)}$ and $\overline{Y}_{n,p}^{(\alpha)}$. Let $T$ be the skeleton of a simplex of $Y_{n,p}^{(\alpha)}$ if and only if

$$\left|T \cap \left\{x_{n+1,n+1}^{j} : j = 0, \ldots, \alpha\right\}\right| \leq \alpha,$$

and $\Delta$ be a simplex of $\overline{Y}_{n,p}^{(\alpha)}$ if and only if $\Delta$ is a simplex of $Y_{n+1,p}$. Note that $Y_{n,p}^{(p-1)} = Y_{n+1,p}$. We shall show that $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to the ball $\mathbb{B}^{(p-1)n+\alpha}$, $\alpha = 0, \ldots, p-2$, and $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to the sphere $\mathbb{S}^{(p-1)n+\alpha-1}$, $\alpha =$

$0,\ldots,p-1$, thus in particular that $|Y_{n+1,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)(n+1)-1}$. We shall use induction on $\alpha$.

Let us consider the case $\alpha=0$. Clearly, $\overline{Y}_{n,p}^{(0)}$ is an $x_{n+1,n+1}^{0}$-cone over $Y_{n,p}$. Hence, by Lemma 4.6, $|\overline{Y}_{n,p}^{(0)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n}$ since $|Y_{n,p}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$. $|Y_{n,p}^{(0)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n-1}$ since $Y_{n,p}^{(0)}=Y_{n,p}$.

Given $\alpha$, $0\le\alpha\le p-3$, assume that $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$. Clearly, $\overline{Y}_{n,p}^{(\alpha+1)}$ is an $x_{n+1,n+1}^{\alpha+1}$-cone over $\overline{Y}_{n,p}^{(\alpha)}$. Hence, by Lemma 4.6, $|\overline{Y}_{n,p}^{(\alpha+1)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha+1}$ since $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$. Thus, we get that $|\overline{Y}_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$ for all $\alpha=0,\ldots,p-2$.

Now, given $\alpha$, $0\le\alpha\le p-2$, assume that $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$. Let $K$ be a subcomplex of $Y_{n,p}^{(\alpha+1)}$ with the same set of vertices and such that $T$ is the skeleton of a simplex of $K$ if and only if

$$\left|T\cap\left\{x_{n+1,n+1}^{j}:j=0,\ldots,\alpha\right\}\right|\le\alpha.$$

We claim that $|K|$ is homeomorphic to the ball $\mathbb{B}^{(p-1)n+\alpha}$. Indeed, $K$ is an $x_{n+1,n+1}^{\alpha+1}$-cone over $Y_{n,p}^{(\alpha)}$. Thus, by Lemma 4.6, $|K|$ is homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$ since $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$.

Now, observe that

$$Y_{n,p}^{(\alpha+1)}=\overline{Y}_{n,p}^{(\alpha)}\cup K,$$

and

$$\overline{Y}_{n,p}^{(\alpha)}\cap K=Y_{n,p}^{(\alpha)}.$$

Thus, by Lemma 4.7, $|Y_{n,p}^{(\alpha+1)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha}$ since $|\overline{Y}_{n,p}^{(\alpha)}|$ and $|K|$ are both homeomorphic to $\mathbb{B}^{(p-1)n+\alpha}$ and $|\overline{Y}_{n,p}^{(\alpha)}\cap K|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$. Therefore, $|Y_{n,p}^{(\alpha)}|$ is homeomorphic to $\mathbb{S}^{(p-1)n+\alpha-1}$ for all $\alpha=0,\ldots,p-1$ and the lemma is proved. □

By using Lemmas 4.6 and 4.7, it is straightforward to verify that $|X_{n,p}|$ is homeomorphic to the topological space obtained by identifying the boundaries of $p$ disjoint copies of the ball $\mathbb{B}^{(p-1)n}$.

Let us now define a free $Z_p$-action $\omega_n$ on the complex $Y_{n,p}$. Assume that $y \in |Y_{n,p}|$ has the following barycentric representation:

$$y = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_i^j x_{n,i}^j.$$

Then set

$$\omega_n(y) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_i^{(j+1) \bmod p} x_{n,i}^j.$$

Note that if $x_{n,i}^j$ is a vertex of $Y_{n,p}$, then

$$\omega_n(x_{n,i}^j) = x_{n,i}^{(j-1) \bmod p}.$$

The map $\omega_n$ is clearly a $Z_p$-action; moreover we have the following lemma.

**Lemma 4.9.** *The map $\omega_n$ is a free action.*

*Proof.* Since $p$ is a prime, it is enough to show that $\omega_n(y) \neq y$ for all $y \in |Y_{n,p}|$. Suppose there is a $y \in |Y_{n,p}|$ such that $\omega_n(y) = y$. Let $T$ be the skeleton of the simplex $\Delta$ containing $y$, and let $T_{n,i}$ have a nonempty intersection with $T$. By the definition of $Y_{n,p}$, $T \cap T_{n,i}$ has at most $p-1$ elements. Since $p$ is prime, $\omega(T) \cap T_{n,i} = \omega(T \cap T_{n,i}) \neq T \cap T_{n,i}$. Hence the simplices of $Y_{n,p}$ containing $y$ and $\omega(y)$ are different. This contradiction completes the proof of the lemma. $\square$

Note that $\omega_{n+1}$ restricted to the complex $X_{n,p}$ is a $Z_p$-free action on $X_{n,p}$. In the sequel, we shall drop the subscript from $\omega_n$ when the domain is clear from the context.

Let $M_1$ and $M_2$ be two metric spaces and let $\alpha_1, \alpha_2 : M_1 \to M_2$ be continuous maps. If

$$H : M_1 \times [0,1] \to M_2$$

is a continuous map such that

$$H(x, 0) = \alpha_1(x)$$

and

$$H(x, 1) = \alpha_2(x),$$

for all $x \in M_1$, then we say that $H$ is a *homotopy* from $\alpha_1$ to $\alpha_2$. If there is a homotopy from $\alpha_1$ to a constant map, then we say that $\alpha_1$ is *null homotopic*. If $\theta$ is a free action on the sphere $\mathbb{S}^k$ and $\alpha$ is a map from $\mathbb{S}^k$ to $\mathbb{S}^k$, then we say that $\alpha$ is *equivariant* with respect to $\theta$ if $\alpha \circ \theta = \theta \circ \alpha$. The following lemma ([43] Theorem 8.3, p.42, and [7] Lemma 2) will be needed in the proof of the main result of this section, Lemma 4.11.

**Lemma 4.10.** *Suppose that $k \geq 1$, $p \geq 2$, and we are given a free $Z_p$-action on the sphere $\mathbb{S}^k$. Then there is no equivariant map $\alpha : \mathbb{S}^k \to \mathbb{S}^k$ which is null homotopic.* □

The following lemma is analogous to the generalization of the Borsuk-Ulam antipodal theorem due to Bárány, Shlosman and Szücs. The difference is in the definition of the action $\omega$, and the proof given here is more elementary as well.

**Lemma 4.11.** *For any continuous map $h : |X_{n,p}| \to \mathbb{R}^n$, there exists an $x \in |X_{n,p}|$ such that $h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x))$.*

*Proof.* Suppose there is a continuous map $h : |X_{n,p}| \to \mathbb{R}^n$ such that for no $x \in |X_{n,p}|$ we have $h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x))$. We shall get a contradiction with Lemmas 4.8 and 4.10 by obtaining a map $\alpha : |Y_{n,p}| \to |Y_{n,p}|$ equivariant with respect to $\omega$ and null homotopic.

Let us first define a map $f : |X_{n,p}| \to |Y_{n,p}|$. For $x \in |X_{n,p}|$, assume

$$\begin{aligned} h(x) &= (r_1^0, \ldots, r_n^0), \\ h(\omega(x)) &= (r_1^1, \ldots, r_n^1), \\ &\vdots \\ h(\omega^{p-1}(x)) &= (r_1^{p-1}, \ldots, r_n^{p-1}). \end{aligned}$$

For $i = 1, \dots, n$, set $r_i = \min\{r_i^0, \dots, r_i^{p-1}\}$ and let

$$r = \sum_{i=1}^{n} \sum_{j=0}^{p-1} (r_i^j - r_i).$$

By our assumption about $h$, $r > 0$ and hence we can set $s_i^j = (r_i^j - r_i)/r$. Let $f(x)$ be defined as follows:

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^j x_{n,i}^j.$$

Since for all $i$ and $j$, $1 \leq i \leq n$, $0 \leq j \leq p-1$, we have $s_i^j \geq 0$ and

$$\sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^j = 1,$$

to show that $f(x) \in |Y_{n,p}|$ it is clearly enough to show that

$$T = \{x_{n,i}^j : 1 \leq i \leq n,\ 0 \leq j \leq p-1,\ s_i^j > 0\}$$

is the skeleton of a simplex of $Y_{n,p}$. But we indeed have that $|T \cap T_{n,i}| \leq p-1$ for every $i = 1, 2, \dots, n$, since $r_i$ is one of $r_i^0, \dots, r_i^{p-1}$ and hence at least one of $s_i^0, \dots, s_i^{p-1}$ must be equal to 0.

Let $\alpha$ be the restriction of $f$ to $|Y_{n,p}|$. We shall show that $\alpha$ is equivariant with respect to $\omega$. Let $x \in |Y_{n,p}|$ and assume that $\alpha(x) \in |Y_{n,p}|$ has the following barycentric representation:

$$\alpha(x) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^j x_{n,i}^j.$$

By the definition of $\omega$, we have that

$$\omega(\alpha(x)) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} s_i^{(j+1) \bmod p} x_{n,i}^j. \tag{7}$$

Assume that $\alpha(\omega(x)) \in |Y_{n,p}|$ has the following barycentric representation:

$$\alpha(\omega(x)) = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \overline{s}_i^j x_{n,i}^j. \tag{8}$$

From the definition of $f$ it follows that

$$\overline{s}_i^j = s_i^{(j+1) \bmod p},$$

for $i = 1, \ldots, n$ and $j = 0, \ldots, p-1$. Therefore (7) and (8) imply that $\omega(\alpha(x)) = \alpha(\omega(x))$, and $\alpha$ is equivariant with respect to $\omega$.

To finish the proof of Lemma 4.11, it is enough to show that $\alpha$ is null homotopic. We shall define a homotopy from $\alpha$ to a constant map using the extension $f$ of $\alpha$. Let $H : |Y_{n,p}| \times [0,1] \to |Y_{n,p}|$ be defined as follows. For $y \in |Y_{n,p}|$ with the following barycentric representation

$$y = \sum_{i=1}^{n} \sum_{j=0}^{p-1} \mu_i^j x_{n,i}^j,$$

and $t \in [0,1]$, set

$$H(y,t) = f\left[\sum_{i=1}^{n} \sum_{j=0}^{p-1} (1-t)\mu_i^j x_{n+1,n+1}^j + t x_{n+1,i}^0\right].$$

Thus

$$H(y,0) = f(y) = \alpha(y),$$

and

$$H(y,1) = f(x_{n+1,n+1}^0)$$

for all $y \in |Y_{n,p}|$. So $H$ is a homotopy from $\alpha$ to a constant map proving that $\alpha$ is null homotopic. $\square$

## §4.4. Proof of the Generalization of the Borsuk-Ulam Theorem

In this section we are going to prove Theorem 4.2. We shall define an equivariant map

$$\zeta : |X_{m,p}| \to \mathbb{S}_p^{m(p-1)}$$

and apply Lemma 4.10.

Given a positive integer $m$, let $q = (p-1)m+1$ and let $Z_{m,p}$ be the subcomplex of the simplicial complex $Y_{q,p}$ such that $T$ is the skeleton of a simplex of $Z_{m,p}$ if and only if

$$|T \cap T_{q,i}| \le 1$$

for every $i = 1, \ldots, q$. It is clear that if we restrict the action $\omega$ on $Y_{q,p}$ to $Z_{m,p}$, we get a free $Z_p$-action on $Z_{m,p}$. We shall denote it also by $\omega$.

We shall define the function $\zeta$ as the composition of two equivariant maps

$$\gamma : |X_{m,p}| \to |Z_{m,p}|,$$

and

$$g : |Z_{m,p}| \to \mathbb{S}_p^{m(p-1)}.$$

The map $g$ is easy to define because there is a straightforward equivariant map from $|Z_{m,p}|$ to $\mathbb{S}_p^{m(p-1)}$ which happens to be a homeomorphism. The hard part is to define the function $\gamma$.

**Lemma 4.12.** *There exists a homeomorphism*

$$g : |Z_{m,p}| \to \mathbb{S}_p^{m(p-1)}$$

*which is equivariant with respect to the action $\omega$ on $|Z_{m,p}|$ and $\omega$ on $\mathbb{S}_p^{m(p-1)}$.*

*Proof.* The map $g$ we are to define has to satisfy $g \circ \omega = \omega \circ g$ where $\omega$ acts on $|Z_{m,p}|$ on the left-hand side and on $\mathbb{S}_p^{m(p-1)}$ on the right-hand side. Let $x \in |Z_{m,p}|$ have the following barycentric representation with respect to $Z_{m,p}$:

$$x = \sum_{i=1}^{q} \sum_{j=0}^{p-1} \mu_{i,j} x_{q,i}^{j}.$$

It follows from the definition of $Z_{m,p}$ that for every $i$, $1 \leq i \leq q$, there is at most one $j$, $0 \leq j \leq p-1$, such that $\mu_{i,j} > 0$. Set

$$g(x) = \Big( (\mu_{1,j_1}, j_1), (\mu_{2,j_2}, j_2), \ldots, (\mu_{q,j_q}, j_q) \Big) \in \mathbb{R}^q_{+,p}$$

where $j_i$, $1 \leq i \leq q$, is such that $\mu_{i,j} = 0$ for all $j \neq j_i$, $0 \leq j \leq p-1$.

Since

$$\sum_{i=1}^{q} \mu_{i,j_i} = 1,$$

we have $g(x) \in \mathbb{S}_p^{m(p-1)}$. It is straightforward to verify that $g$ is a homeomorphism and that $g \circ \omega = \omega \circ g$. Thus the lemma is proved. □

Before we define the function $\gamma$, we need some more preliminary lemmas. Given a prime $p$, let

$$P = 2^{[0,p-1]} \setminus \{\emptyset, [0,p-1]\}$$

be the set of all subsets of $[0,p-1] \subset \mathbb{N}$ which are nonempty and different from $[0,p-1]$.

Let $\eta : [0,p-1] \to [0,p-1]$ be the function defined in Section 4.1; $\eta(i) = (i+1) \bmod p$ and let $\Theta : P \to P$ be defined by

$$\Theta(A) = \{\eta(a) : a \in A\}.$$

We are going to define a function $\varphi : P \to [0,p-1]$ satisfying

$$\varphi(\Theta(A)) = \eta(\varphi(A)).$$

If $A \in P$, then set

$$\xi(A) = \sum_{i \in A} 2^i,$$

and let

$$B_A = \left\{\xi(A), \xi(\Theta(A)), \xi(\Theta^2(A)), \ldots, \xi(\Theta^{p-1}(A))\right\}.$$

The following lemma holds.

**Lemma 4.13.** *$B_A$ contains $p$ different numbers.*

*Proof.* Suppose that $\xi(\Theta^j(A)) = \xi(\Theta^{j+k}(A))$ and $1 \le k \le p-1$. Since $p$ is a prime, $k$ is relatively prime to $p$, and hence $\xi(A) = \xi(\Theta(A))$. But this is possible only when $A = \emptyset$ or $A = [0, p-1]$. Since $1 \le |A| \le p-1$, the resulting contradiction finishes the proof of the lemma. $\square$

We can now define $\varphi$. Let

$$\varphi(A) = \eta^{-j}(\max(\Theta^j(A)))$$

where $j$ is such that

$$\xi(\Theta^j(A)) = \max(B_A).$$

By Lemma 4.13, $\varphi$ is well defined; also we have the folowing lemma.

**Lemma 4.14.** *The function $\varphi$ is such that for all $A \in P$ we have*

$$\varphi(\Theta(A)) = \eta(\varphi(A)).$$

*Proof.* We have

$$\varphi(A) = \eta^{-j}(\max(\Theta^j(A)))$$

where $j$ satisfies

$$\xi(\Theta^j(A)) = \max(B_A).$$

We also have

$$\varphi(\Theta(A)) = \eta^{-j'}(\max(\Theta^{j'}(\Theta(A))))$$

where $j'$ satisfies

$$\xi(\Theta^{j'}(\Theta(A))) = \max(B_{\Theta(A)}).$$

Since $B_{\Theta(A)} = B_A$, we have $j' = (j-1) \bmod p$ and hence

$$\varphi(\Theta(A)) = \eta^{-j+1}(\max(\Theta^j(A))) = \eta(\varphi(A)).$$

Thus the proof of the lemma is complete. $\square$

If $K$ is a simplicial complex, then the *barycentric subdivision* $K'$ of $K$ is the simplicial decomposition of $|K|$ obtained as follows. For a simplex $\Delta = (x_0, \ldots, x_k) \in K$, let

$$c_\Delta = \frac{1}{k+1} \sum_{i=0}^{k} x_i$$

be the *barycentre* of $\Delta$. Let $K'$ consist of all simplices $(c_{\Delta_0}, \ldots, c_{\Delta_k})$ such that $\Delta_i \in K$, $i = 0, 1, \ldots, k$, and $\Delta_i$ is a proper face of $\Delta_{i+1}$, $i = 0, 1, \ldots, k-1$.

We are now going to define the *quasi barycentric subdivision* $X'_{m,p}$ of $X_{m,p}$. Let $\mathcal{A}_i$, $1 \le i \le m+1$, be the set of simplices $\Delta$ of $X_{m,p}$ such that the vertices of $\Delta$ are contained in $T_{m+1,i}$. Let

$$\left\{ c_\Delta : \Delta \in \bigcup_{i=1}^{m+1} \mathcal{A}_i \right\}$$

be the set of vertices of $X'_{m,p}$ where $c_\Delta$ is the barycentre of $\Delta$. Let $T$ be the skeleton of a simplex of $X'_{m,p}$ if and only if for every $i$, $1 \le i \le m+1$, we have

$$\{c_\Delta \in T : \Delta \in \mathcal{A}_i\} = \{c_{\Delta_0}, \ldots, c_{\Delta_k}\}$$

where $\Delta_i$ is a proper face of $\Delta_{i+1}$, $i = 0, 1, \ldots, k-1$. It is straightforward to verify the following lemma.

**Lemma 4.15.** *$X'_{m,p}$ is a simplicial decomposition of $|X_{m,p}|$.* □

We shall define $\gamma : |X_{m,p}| \to |Z_{m,p}|$ on the vertices of $X'_{m,p}$ first. The map $\gamma$ restricted to the vertices of $X'_{m,p}$ will take its values in the set of vertices of $Z_{m,p}$. Given a vertex $c_\Delta$ of $X'_{m,p}$, let $i$, $1 \le i \le m+1$, be such that $\Delta \in \mathcal{A}_i$. Let $T$ be the skeleton of $\Delta$ and

$$A = \left\{ j : 0 \le j \le p-1,\ x^j_{m+1,i} \in T \right\}.$$

By the definition of $X_{m,p}$, we have

$$1 \le |A| \le p-1$$

if $1 \le i \le m$, and

$$|A| = 1$$

if $i = m+1$. Set

$$\gamma(c_\Delta) = x_{q,(p-1)(i-1)+|A|}^{\varphi(A)} .$$

We shall now show that $\gamma$ maps skeletons of simplices of $X'_{m,p}$ to skeletons of simplices of $Z_{m,p}$.

**Lemma 4.16.** *If $(c_{\Delta_0}, \ldots, c_{\Delta_k})$ is a simplex of $X'_{m,p}$, then $(\gamma(c_{\Delta_0}), \ldots, \gamma(c_{\Delta_k}))$ is a simplex of $Z_{m,p}$.*

*Proof.* Assume that $(c_{\Delta_0}, \ldots, c_{\Delta_k})$ is a simplex of $X'_{m,p}$ and

$$\gamma(c_{\Delta_i}) = x_{q,r_i}^{a_i}$$

for $i = 0, \ldots, k$. By the definition of $Z_{m,p}$, to prove that $(x_{q,r_0}^{a_0}, \ldots, x_{q,r_k}^{a_k})$ is a simplex of $Z_{m,p}$ we have to show that all $r_0, \ldots, r_k$ are distinct. Suppose $r_j = r_\ell$ and $0 \le j \le \ell \le k$. There is exactly one $i$ and one $s$, $1 \le i \le m+1$, $1 \le s \le p-1$, such that

$$r_j = r_\ell = (p-1)(i-1) + s.$$

Hence, by the definition of $\gamma$, we have

$$c_{\Delta_j}, c_{\Delta_\ell} \in \mathcal{A}_i$$

and

$$|T_j| = |T_\ell| = s$$

where $T_j$ and $T_\ell$ are skeletons of $\Delta_j$ and $\Delta_\ell$ respectively. This contradicts the definition of $X'_{m,p}$ since, according to this definition, $\Delta_j$ is a proper face of $\Delta_\ell$. Thus the lemma is proved. □

We now extend $\gamma$ linearly to $|X'_{m,p}|$. If $x \in (c_{\Delta_0}, \ldots, c_{\Delta_k}) \in X'_{m,k}$ has the following barycentric representation

$$x = \sum_{i=0}^{k} \mu_i c_{\Delta_i},$$

then let

$$\gamma(x) = \sum_{i=0}^{k} \mu_i \gamma(c_{\Delta_i}).$$

**Lemma 4.17.** *The map $\gamma$ is equivariant with respect to $\omega$.*

*Proof.* We have to show that for every $x \in |X_{m,p}|$ we have

$$\gamma(\omega(x)) = \omega(\gamma(x)).$$

It is enough to prove this equality for $x$ being a vertex of $X'_{m,p}$. Let $x = c_\Delta$ be the barycentre of $\Delta \in \mathcal{A}_i$, let $T$ be the skeleton of $\Delta$ and set

$$A = \left\{ j : 0 \le j \le p-1,\ x^j_{m+1,i} \in T \right\}.$$

If $\omega(c_\Delta) = c_{\Delta'}$, then by the definition of $\omega$, we have $\Delta' \in \mathcal{A}_i$. If $T'$ is the skeleton of $\Delta'$, then

$$\left\{ j : 0 \le j \le p-1,\ x^j_{m+1,i} \in T' \right\} = \Theta(A).$$

Therefore we have

$$\gamma(\omega(x)) = x^{\varphi(\Theta(A))}_{q,(p-1)(i-1)+|\Theta(A)|}\ .$$

Since $|\Theta(A)| = |A|$ and $\varphi(\Theta(A)) = \eta(\varphi(A))$, we have

$$\gamma(\omega(x)) = x^{\eta(\varphi(A))}_{q,(p-1)(i-1)+|A|}\ .$$

We also have

$$\omega(\gamma(x)) = x^{\eta(\varphi(A))}_{q,(p-1)(i-1)+|A|}\ ,$$

so

$$\gamma(\omega(x)) = \omega(\gamma(x)),$$

as required. □

We are now ready to prove Theorem 4.2.

*Proof of Theorem 4.2.* Let $h : \mathbb{S}_p^{m(p-1)} \to \mathbb{R}^m$ be a continuous function. Let $g : |Z_{m,p}| \to \mathbb{S}_p^{m(p-1)}$ be a homeomorphism satisfying $g \circ \omega = \omega \circ g$ (see Lemma 4.12). Let us consider the function

$$h \circ g \circ \gamma : |X_{m,p}| \to \mathbb{R}^m.$$

By Lemma 4.11 there exists $y \in |X_{m,p}|$ satisfying

$$h \circ g \circ \gamma(y) = h \circ g \circ \gamma(\omega(y)) = \ldots = h \circ g \circ \gamma(\omega^{p-1}(y)).$$

Let $x = g \circ \gamma(y)$. Since $g \circ \gamma(\omega(y)) = \omega(x)$, we have

$$h(x) = h(\omega(x)) = \ldots = h(\omega^{p-1}(x)),$$

and the proof of Theorem 4.2 is complete. $\square$

## §4.5. Concluding remark

Although our proof of Theorem 4.1 is much more combinatorial than the original one given by Alon [4], it is still based upon a result from algebraic topology. It would be desirable to find a purely combinatorial proof. Probably the way to give such a proof would be to find a purely combinatorial proof of our generalization of the Borsuk-Ulam antipodal theorem (Theorem 4.2). Recall that the Borsuk-Ulam theorem has a purely combinatorial proof which perhaps could be generalized.

# CHAPTER 5

# REMARKS ON A GENERALIZATION OF RADON'S THEOREM

The well-known theorem of Radon [36] says that, for any $A \subset \mathbb{R}^n$ satisfying $|A| \geq n+2$, there are disjoint subsets $B$ and $C$ of $A$ such that their convex hulls have nonempty intersection. Since, for any $A \subset \mathbb{R}^n$ satisfying $|A| = n+2$ the convex hull of $A$ is the image of the closure of an $(n+1)$-dimensional simplex under a linear map, Radon's theorem is an immediate corollary to the following theorem. The terms used in this chapter are defined in Chapter 4.

**Theorem 5.1.** *Let $\Delta \subset \mathbb{R}^{n+1}$ be an $(n+1)$-dimensional simplex and let $K$ be the simplicial complex containing all faces of $\Delta$. If $f : |K| \to \mathbb{R}^n$ is a linear map, then there are two disjoint faces $\Delta_1$, $\Delta_2$ of $\Delta$ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.*

Thus the following theorem of Bajmóczy and Bárány [6] can be thought of as a generalization of Radon's theorem.

**Theorem 5.2.** *Let $\Delta$ and $K$ be as in Theorem 5.1. If $f : |K| \to \mathbb{R}^n$ is a continuous map, then there are two disjoint faces $\Delta_1$, $\Delta_2$ of $\Delta$ such that $f(\Delta_1) \cap f(\Delta_2) \neq \emptyset$.*

Bajmóczy and Bárány use the following antipodal theorem of Borsuk and Ulam [13] in their proof.

**Theorem 5.3.** *For any continuous map $h : \mathbb{S}^n \to \mathbb{R}^n$ there exists $x \in \mathbb{S}^n$ with $h(x) = h(-x)$.*

Theorem 5.2 follows immediately from Theorem 5.3 and the following theorem.

**Theorem 5.4.** *Let $\Delta$ and $K$ be as in Theorem 5.1. There exists a continuous map $g : \mathbb{S}^n \to |K|$ such that for every $x \in \mathbb{S}^n$ the supports of $g(x)$ and $g(-x)$ are disjoint.*

In this brief chapter we are going to give a new very simple proof of Theorem 5.4. We present in it an explicit construction of the function $g$.

*Proof of Theorem* 5.4. Assume that $\Delta = (x_0, \dots, x_{n+1})$. Let $K_1$ be the simplicial complex with $\{x_0, \dots, x_{n+1}\}$ as its set of vertices and all proper faces of $\Delta$ as its simplices. Let $K_2$ be the barycentric subdivision of $K_1$. Let $\omega : |K_2| \to |K_2|$ be a free $Z_2$-action defined as follows. If $T \subset \{x_0, \dots, x_{n+1}\}$ is the skeleton of a simplex $\sigma$ of $K_1$ and $c_\sigma$ is the barycentre of $\sigma$, then let

$$\omega(c_\sigma) = c_{\sigma'} \,,$$

where $\sigma'$ is the simplex of $K_1$ whose skeleton $T'$ is the complement of $T$, that is

$$T' = \{x_0, \dots, x_{n+1}\} \setminus T.$$

Thus, we have defined $\omega$ on the vertices of $K_2$. Let us extend $\omega$ linearly to $|K_2|$, that is for any $x \in (c_{\sigma_1}, \dots, c_{\sigma_r}) \in K_2$ having the following barycentric representation

$$x = \sum_{i=1}^{r} \mu_i c_{\sigma_i},$$

let

$$\omega\left(x\right) = \sum_{i=1}^{r} \mu_i \omega(c_{\sigma_i}).$$

Clearly, $\omega$ is well defined and there is a homeomorphism $f : \mathbb{S}^n \to |K_2|$ which is equivariant with respect to the antipodal map on $\mathbb{S}^n$ and $\omega$ on $|K_2|$, that is such that for every $x \in \mathbb{S}^n$ the following equality holds:

$$f(-x) = \omega(f(x)).$$

Therefore, to prove our theorem, it is enough to show the existence of a continuous map $h : |K_2| \to |K|$ such that for every $x \in |K_2|$ the supports of $h(x)$ and $h(\omega(x))$ are disjoint.

Let $K_3$ be the barycentric subdivision of $K_2$. We shall define $h$ on the vertices of $K_3$ first. Let $d_A$ be the barycentre of the simplex $A = (c_{\sigma_1}, \ldots, c_{\sigma_r})$ of $K_2$. Since $A$ is a simplex of $K_2$, we can assume that $\sigma_i$ is a proper face of $\sigma_{i+1}$, $i = 1, \ldots, r-1$. Define

$$h(d_A) = c_{\sigma_1}.$$

Now, let us extend $h$ linearly to $|K_3| = |K_2|$, that is for $x \in (d_{A_1}, \ldots, d_{A_s}) \in K_3$ with the following barycentric representation

$$x = \sum_{i=1}^{s} \mu_i d_{A_i},$$

let

$$h\left(x\right) = \sum_{i=1}^{s} \mu_i h\left(d_{A_i}\right).$$

Now, we shall show that for every $x \in |K_2|$ the supports of $h(x)$ and $h(\omega(x))$ in $K$ are disjoint. Note first that if $d_A$ is the barycentre of a simplex $A = (c_{\sigma_1}, \ldots, c_{\sigma_r})$, then

$$\omega(d_A) = \omega\left(\frac{1}{r}\sum_{i=1}^{r} c_{\sigma_i}\right) = \frac{1}{r}\sum_{i=1}^{r}\omega(c_{\sigma_i}) = d_B$$

where

$$B = \left(\omega(c_{\sigma_1}), \ldots, \omega(c_{\sigma_r})\right).$$

For $x \in |K_2|$, let

$$\{A_1, \ldots, A_r\}$$

be the support of $x$ in $K_3$ and

$$\{B_1, \ldots, B_r\}$$

be the support of $\omega(x)$ in $K_3$ where $B_i = \omega(A_i)$, $i = 1, \ldots, r$. Let

$$\{\sigma_{i,1}, \ldots, \sigma_{i,s_i}\}$$

be the skeleton of $A_i$, $i = 1, \ldots, r$, where $\sigma_{i,j}$ is a proper face of $\sigma_{i,j+1}$, $j = 1, \ldots, s_i - 1$. Now let

$$\{\sigma'_{i,1}, \ldots, \sigma'_{i,s_i}\}$$

be the skeleton of $B_i$, $i = 1, \ldots, r$. Since the skeleton of $\sigma'_{i,j}$ is the complement of the skeleton of $\sigma_{i,j}$, the simplex $\sigma'_{i,j+1}$ is a proper face of $\sigma'_{i,j}$, for all $i = 1, \ldots, r$ and $j = 1, \ldots, s_i - 1$.

Since $h(A_i) = \sigma_{i,1}$, $i = 1, \ldots, r$, the support of $h(x)$ in $K_2$ is the set

$$\{\sigma_{1,1}, \sigma_{2,1}, \ldots, \sigma_{r,1}\},$$

and since $h(B_i) = \sigma'_{i,s_i}$, $i = 1, \ldots, r$, the support of $h(\omega(x))$ in $K_2$ is the set

$$\{\sigma'_{1,s_1}, \sigma'_{2,s_2}, \ldots, \sigma'_{r,s_r}\}.$$

We can assume that $A_i$ is a proper face of $A_{i+1}$, $i = 1, \ldots, r$. Then $\sigma_{i+1,1}$ is a (not necessarily proper) face of $\sigma_{i,1}$, $i = 1, \ldots, r$, and thus the support of $h(x)$ in $\Delta^{n+1}$ is the skeleton of $\sigma_{1,1}$. Since $A_i$ is a proper face of $A_{i+1}$, the simplex $B_i$ is a proper face of $B_{i+1}$, $i = 1, \ldots, r$. Therefore, $\sigma'_{i+1,s_{i+1}}$ is a face of $\sigma'_{i,s_i}$, $i = 1, \ldots, r$, and thus the support of $h(\omega(x))$ in $K$ is the skeleton of $\sigma'_{1,s_1}$. Now recall that the skeleton of $\sigma'_{1,s_1}$ is a complement of the skeleton of $\sigma_{1,s_1}$. But the skeleton of $\sigma_{1,1}$ is contained in the skeleton of $\sigma_{1,s_1}$ so the supports of $h(x)$ and $h(\omega(x))$ in $K$ are disjoint, and the theorem is proved. $\square$

# CHAPTER 6

## AN OBSERVATION ON INTERSECTION DIGRAPHS OF CONVEX SETS IN THE PLANE

Given a finite family of sets, its *intersection graph* has a vertex corresponding to each set, with edges between vertices corresponding to non-disjoint sets. The notion of intersection graphs is well studied—see an issue of *Discrete Mathematics* [20] which is dedicated to papers on this subject. Maehara [45] introduced and studied a class of *intersection digraphs*; this notion was later generalized by Sen, Das, Roy and West [50]. Let $D = (V, E)$ be a digraph and $\{(S_v, T_v) : v \in V\}$ be a family of ordered pairs of sets. Sen, Das, Roy and West define $D$ to be the intersection digraph of this family if $E = \{\overrightarrow{uv} : S_u \cap T_v \neq \emptyset\}$. Note that this definition allows loops in our digraph.

By assigning to a vertex of a graph the set of edges incident with it, it is easy to see that every graph is an intersection graph of finite sets. Let the *intersection number* $i\#(G)$ of a graph $G$ be the minimum size of a set $U$ such that $G$ is the intersection graph of subsets of $U$. Erdös, Goodman and Pósa [24] proved that $i\#(G)$ is equal to the minimum number of complete subgraphs needed to cover all its edges and that

$$i\#(G) \leq \lfloor n^2/4 \rfloor \tag{1}$$

for an $n$-vertex graph $G$. Equality in (1) is achieved by the complete bipartite graph $G = K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$.

An analogous construction shows also that every digraph is an intersection digraph of finite sets. To get a representation of a digraph as an intersection digraph, it is enough to assign to a vertex $v$ a pair of subsets $(S_v, T_v)$ of the edge-set, where $S_v$ is the set of edges having their 'starting point' at $v$ and $T_v$ is the set of edges with their 'terminal point' at $v$. By analogy to graphs, Sen, Das, Roy and West [50] define the *intersection number* $i\#(D)$ of a digraph $D$ as the minimum size of a set $U$ such that $D$ is the intersection digraph of ordered pairs of subsets of $U$. They also define a *generalized complete bipartite subgraph* (GBS) of a digraph $D$ as a subdigraph whose vertex-set can be expressed as $X \cup Y$ ($X$ and $Y$ need not be disjoint) and whose edge-set is equal to $\{\overrightarrow{xy} : x \in X,\ y \in Y\}$. An easy result of Sen, Das, Roy and West characterizes the intersection number $i\#(D)$ as the minimum number of GBS's required to cover the edges of $D$. They also give the best possible upper bound on the intersection number of digraphs:

$$i\#(D) \leq n,$$

for an $n$-vertex digraph $D$.

Given a family of sets, a natural question to ask about intersection graphs and digraphs is whether all graphs (all digraphs) are intersection graphs (digraphs) of sets (ordered pairs of sets) from this family. Of special interest are intersection graphs and digraphs where the sets are required to be convex sets in the Euclidean space. If the space is one-dimensional then we get interval graphs, characterized in [28], [29], [44], and interval digraphs, characterized in [50]. In three dimensions all graphs and digraphs can be represented. With two-dimensional convex sets not all graphs can be obtained. Wegner [59] gave an example of a graph which is not an intersection graph of convex sets in the plane. The graph is obtained from $K_5$ by subdividing each edge. For digraphs Sen, Das, Roy and West [50] observed that an analogue of Wegner's counter-example fails and posed the question whether every digraph is the intersection digraph of ordered pairs of convex sets in the plane. In this brief chapter we present a simple observation allowing us to give a positive answer to this question.

**Theorem 6.1.** *Let $D = (V, E)$ be a digraph. Then there is a family $\mathcal{A} = \{(S_v, T_v) : v \in V\}$ of pairs of convex sets in $\mathbb{R}^2$ such that $D$ is the intersection digraph of $\mathcal{A}$.*

*Proof.* Set $n = |V|$. Let $A \subset \mathbb{R}^2$ be a set of $n$ points on a circle, and let $f : V \to A$ be a bijection. Set, for each $v \in V$,

$$S_v = \{f(v)\},$$
$$T_v = \mathrm{conv}(\{f(u) : \overrightarrow{uv} \in E\}),$$

where for $B \subset \mathbb{R}^2$, $\mathrm{conv}(B)$ is the convex hull of $B$, that is the smallest convex set containing $B$. So, for each vertex $v$ of $D$, the 'source set' of $v$ contains one element of $A$, namely the one that corresponds to $v$ under $f$, and the 'terminal set' of $v$ is the convex hull of the elements of $A$ corresponding to all predecessors of $v$. It is easy to see that $T_v$ does not contain any other elements of $A$. So $S_u \cap T_v \neq \emptyset$ if and only if $f(u) \in T_v$, which holds if and only if $\overrightarrow{uv} \in E$. Therefore $D$ is the intersection digraph of the family $\{(S_v, T_v) : v \in V\}$. Of course for each $v \in V$, the sets $S_v$ and $T_v$ are convex subsets of $\mathbb{R}^2$ so the theorem is proved. $\square$

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