WHEN IS A CATEGORY OF ADHERENCE-DETERMINED CONVERGENCES SIMPLE?

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Abstract. We provide a characterization of classes of filters $\mathbb{D}$ for which the full subcategory $\operatorname{fix} A_{\mathbb{D}}$ of Conv formed by convergences determined by the adherence of filters of the class $\mathbb{D}$ is simple in Conv. Along the way, we also elucidate when two classes of filters result in the same category of adherence-determined convergences. As an application of the main result, we show that the category of hypotopologies is not simple, thus answering a question from [25].

1. Introduction

It is well known that a topological space X is Tychonoff (completely regular and Hausdorff) if and only if X can be embedded into a product $\mathbb{R}^I$ of some copies of the real line $\mathbb{R}$. Using the concept of reflectivity in category theory, we say that the Tychonoff spaces form the epireflective hull of $\mathbb{R}$ or that the class of Tychonoff spaces contains an initially dense member $\mathbb{R}$. We also say that the subcategory of Tychonoff spaces is *simple* in the category of all topological spaces.

In this paper, we will consider fundamental subcategories of the category Conv of convergences and continuous functions. For a systematic study of convergence spaces, see [9, 10, 11]. A convergence is a relation between points of a set X and the filters on X. Each topological space X induces a convergence on the set X by relating a point x to all filters that converge to x (all filters that refine the neighborhood filter at x). Convergence theory studies this relation in greater generality and considers the topological convergence only as a special case. The need to study non-topological convergences was pointed out by Gustave Choquet in his fundamental paper [5]. It turns out that considering topological problems in the larger setting of convergence spaces is often illuminating, in a way that can be compared to using complex numbers to solve a problem formulated in the reals.

The special subclasses of convergence spaces formed by pretopologies and by pseudotopologies have long been recognized as fundamental: they were already introduced by Choquet in the pioneering paper [5] and the category PreTop of pretopological spaces and PsTop of pseudotopological spaces have special structural roles with respect to Top: in categorical terms, PreTop is the extensional hull of Top while PsTop is the quasitopos hull of Top (See, e.g., [17] for details). These two categories turn out to fit in a larger useful classification introduced by Szymon Dolecki: *adherence-determined convergences* [6]. Beyond its categorical unifying power, the notion turns out to be the key to convergence-theoretic characterizations of various types of quotient maps (e.g., biquotient, countably biquotient, hereditarily quotient) and of various topological notions (e.g., bisequential, countably bisequential, Fréchet-Urysohn, etc) in terms of functorial inequalities, which in turn allows to unify, generalize, and refine a large spectrum of topological results on preservation under maps [6], under products (e.g., [12]), on compactness and its variants e.g, [7, 8, 9, 24], etc. The categories ParaTop and HypoTop, of paratopologies and of hypotopologies respectively, appear naturally within this classification and fill important gaps left by the traditional categorical approach in the quest to interpret classical topological notions categorically.

We are going to investigate the question of simplicity of subcategories of Conv. In particular, the category Top of topological spaces is simple in Conv. Several subcategories of Top are also simple in Conv, see for example [13, 14, 15, 16]. Simplicity also holds when we enlarge Top to the adherence-determined subcategories PreTop and ParaTop (see Antoine [2] and Bourdaud [3, 4] for PreTop and [25] for ParaTop). However, when we enlarge PreTop to PsTop or to the whole Conv, then Eva and Robert Lowen showed [21] that simplicity fails. The category HypoTop was shown in [25] not to be simple, under the assumption that measurable cardinals form a proper class.

In this paper, we give a complete characterization of when one of the fundamental categories of adherence-determined convergences is simple. Namely, we give a condition on a class $\mathbb{D}$ of filters that characterizes the simplicity of the full subcategory $\operatorname{fix} \mathrm{A}_{\mathbb{D}}$ of Conv formed by convergences determined by the adherence of filters of the class $\mathbb{D}$. When $\mathbb{D}$ is respectively the class of principal filters, of countably-based filters, of countably complete filters, and of all filters then $\operatorname{fix} \mathrm{A}_{\mathbb{D}}$ is respectively PreTop, ParaTop, HypoTop and PsTop. As a result, we recover results of [2, 4, 3] on the simplicity of PreTop, of [25] on the simplicity of ParaTop and of [21] on the non-simplicity of PsTop. Moreover, we answer the question raised in [25] and prove in ZFC that HypoTop (and more generally the category of μ-hypotopologies where μ is an infinite cardinal) is not a simple subcategory of Conv (Corollary 15). To summarize

class $\mathbb{D}$ of filters	$\mathbb{D}$-adherence-determined conv.	category	simple
$\mathbb{F}$ of all	pseudotopology	PsTop	X
$\mathbb{F}_1$ of countably based	paratopology	ParaTop	✓
$\mathbb{F}_{\wedge 1}$ of countably complete	hypotopology	HypoTop	X
$\mathbb{F}_0$ of principal	pretopology	PreTop	✓

2. Basic definitions

2.1. **Categorical terminology.** Let us first introduce the general concept of a simple subcategory in general and in particular for topological categories. For more details, we refer the reader to [1, 18, 22].

Let $\mathcal{A}$ be a full and isomorphism-closed subcategory of a category $\mathcal{B}$ with the embedding functor $E : \mathcal{A} \to \mathcal{B}$. Given an object B of $\mathcal{B}$, a pair (u, A), (where A is an object of $\mathcal{A}$ and $u : B \to E(A)$ is a morphism of $\mathcal{B}$) is called an $\mathcal{A}$-*reflection* of B provided that for each object A' of $\mathcal{A}$ and each morphism $f : B \to E(A')$ there exists a unique morphism $g : A \to A'$ such that $f = E(g) \circ u$. The subcategory $\mathcal{A}$ is *epireflective* in $\mathcal{B}$ provided that for every object B of $\mathcal{B}$ there exists an $\mathcal{A}$-reflection (u, A) of B with u being an epimorphism of $\mathcal{B}$. We say that $\mathcal{A}$ is *simple* in $\mathcal{B}$ provided that $\mathcal{A}$ is epireflective in $\mathcal{B}$ and there exists an object A of $\mathcal{A}$ such that every epireflective (full and isomorphism-closed) subcategory of $\mathcal{B}$ containing A must contain $\mathcal{A}$. We say then that $\mathcal{A}$ is the *epireflective hull* of A in $\mathcal{B}$.

For example, the epireflective hull of the closed interval $[0,1]$ in the category of Hausdorff spaces (and continuous functions), is the subcategory of compact Hausdorff spaces, however the epireflective hull of the same interval in the category of all topological spaces (and continuous functions) is the subcategory of Tychonoff spaces (completely regular and Hausdorff). This difference is caused by the fact that in the category of Hausdorff spaces a morphism is an epimorphism if and only if its image is dense, while in the category of all topological spaces being an epimorphism is equivalent to being surjective.

Let $\mathcal{B}$ be a concrete category over the category Sets of sets and functions, with the forgetful functor $U : \mathcal{B} \to \mathsf{Sets}$. A class indexed family $(f_i : B \to B_i)_{i\in I}$ of morphisms of $\mathcal{B}$ is an *initial source* provided that if B' is an object of $\mathcal{B}$ and $f : U(B') \to U(B)$ is a function such that $f_i \circ f : B' \to B_i$ is a morphism of $\mathcal{B}$, then f is also a morphism of $\mathcal{B}$. We say that $\mathcal{B}$ is *topological* provided that each structured source (a class indexed family $(f_i : X \to U(B_i))_{i\in I}$ of functions) has a unique initial lift, that is, there exists a unique object B of $\mathcal{B}$ with $U(B) = X$ such that $f_i : B \to B_i$ is a morphism of $\mathcal{B}$ for each $i \in I$ and $(f_i : B \to B_i)_{i\in I}$ is an initial source in $\mathcal{B}$.

For example, the category of all topological spaces is topological, but the category of Hausdorff spaces is not ([1]).

Assume that $\mathcal{B}$ is a topological category. In such a category epimorphisms are exactly those morphisms that are surjective functions and a (full and isomorphism-closed) subcategory $\mathcal{A}$ is epireflective in $\mathcal{B}$ if and only if $\mathcal{A}$ is closed under the formation of products and extremal subobjects ((f, A') is an extremal subobject of A iff $f : A' \to A$ is an embedding). Moreover, for any object A of $\mathcal{B}$, there exists the epireflective hull of A in $\mathcal{B}$ obtained by taking all extremal subobjects of the powers of A. An explicit condition for $\mathcal{A}$ to be simple in $\mathcal{B}$ is that there exists an object A_0 of $\mathcal{A}$ such that for any object A of $\mathcal{A}$ there exists an initial source $(f_i : A \to A_0)_{i\in I}$ in $\mathcal{A}$.

In this paper we will be concerned with simplicity of some (full and isomorphism-closed) subcategories of the category Conv of convergences (and continuous functions).

2.2. **Convergence spaces.** The context of this paper is that of the category Conv of convergence spaces and continuous maps. We use the terminology and notations of [11]. In particular, a *convergence* ξ on a set X is a relation between points of X and filters on X, denoted

$$x \in \lim_\xi \mathcal{F}$$

whenever x and $\mathcal{F}$ are ξ-related, subjected to two simple axioms: $x \in \lim_\xi \{x\}^\uparrow$ for every $x \in X$, where $\{x\}^\uparrow$ denotes the principal ultrafilter including $\{x\}$, and $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$ whenever $\mathcal{G}$ is a filter finer than the filter $\mathcal{F}$. If (X, ξ) and (Y, τ) are two convergence spaces, a map $f : X \to Y$ is *continuous* (from ξ to τ), in symbols $f \in C(\xi, \tau)$, if

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f[\mathcal{F}],$$

[1]To see that, note that the empty structured source on a set X with at least 2 elements has no initial lift. Otherwise, there would be a topology on X such that for any Hausdorff space Y any function $f : Y \to X$ is continuous. Such topology would have to be antidiscrete, hence not Hausdorff.

where $f[\mathcal{F}]=\{B\subset Y: f^{-}(B)\in\mathcal{F}\}$ is the image filter. Of course, every topology τ can be seen as a convergence given by $x\in\lim_{\tau}\mathcal{F}$ if and only if $\mathcal{F}\geq\mathcal{N}_{\tau}(x)$, where $\mathcal{N}_{\tau}(x)$ denotes the neighborhood filter of x in the topology τ. This turns the category Top of topological spaces and continuous maps into a full subcategory of Conv.

We denote by $|\cdot|:\mathsf{Conv}\to\mathsf{Sets}$ the forgetful functor, so that $|\xi|$ denotes the underlying set of a convergence ξ and $|f|$ is the underlying function of a morphism, though we will normally not distinguish notationally the morphism and the underlying function and denote them both by f. If $|\xi|=|\tau|$, we say that ξ is *finer than* τ or that τ is *coarser than* ξ, in symbols, $\xi\geq\tau$, if the identity map of their underlying set belongs to $C(\xi,\tau)$. This order turns the set of convergences on a given set into a complete lattice whose greatest element is the discrete topology, whose least element is the antidiscrete topology, and whose suprema and infima are given by

$$\lim_{\bigvee_{\xi\in\Xi}\xi}\mathcal{F}=\bigcap_{\xi\in\Xi}\lim_{\xi}\mathcal{F}\text{ and }\lim_{\bigwedge_{\xi\in\Xi}\xi}\mathcal{F}=\bigcup_{\xi\in\Xi}\lim_{\xi}\mathcal{F}. \tag{2.1}$$

A point x of a convergence space (X,ξ) is *isolated* if $\{x\}^{\uparrow}$ is the only filter converging to x in ξ. A convergence is called *prime* if it has at most one non-isolated point.

Conv is a concrete topological category; in particular, for every $f:X\to|\tau|$, there is the coarsest convergence on X, called the *initial convergence* for (f,τ) and denoted $f^{-}\tau$, making f continuous (to τ), and for every $f:|\xi|\to Y$, there is the finest convergence on Y, called the *final convergence for* (f,ξ) and denoted $f\xi$, making f continuous (from ξ). Note that with these notations

$$f\in C(\xi,\tau)\iff\xi\geq f^{-}\tau\iff f\xi\geq\tau. \tag{2.2}$$

Moreover, the initial lift on X of a structured source $(f_i:X\to|\tau_i|)_{i\in I}$ turns out to be $\bigvee_{i\in I}f_i^{-}\tau_i$ and the final lift on Y of a structured sink $(f_i:|\xi_i|\to Y)_{i\in I}$ turns out to be $\bigwedge_{i\in I}f_i\xi_i$.

Products, subspaces, coproducts (sums) and quotients are then defined as usual via initial and final structures.

Let Φ be a class of convergences. We say that a convergence η is *initially dense* in Φ if and only if for each $\xi\in\Phi$ there exists a set A of functions from $|\xi|$ to $|\eta|$ such that $\xi=\bigvee_{f\in A}f^{-}\eta$. Note that if η is initially dense in Φ, then $\eta\in\Phi$ and $\xi=\bigvee_{f\in C(\xi,\eta)}f^{-}\eta$ for every $\xi\in\Phi$ ([2]). Note that if Φ is the class of objects of some full and isomorphism closed subcategory $\mathcal{A}$ of Conv, then η is initially dense in Φ if and only if $\mathcal{A}$ is the epireflective hull of η in Conv. A class of convergences is *simple* provided it includes an initially dense convergence.

2.3. **Filters and classes of filters.** If $\mathcal{P}(X)$ denotes the powerset of X and $\mathcal{A}\subset\mathcal{P}(X)$ then we write

$$\begin{aligned}\mathcal{A}^{\uparrow_X}=\mathcal{A}^{\uparrow} &:= \{B\subset X:\exists A\in\mathcal{A},A\subset B\}\\ \mathcal{A}^{\#} &:= \{H\subset X:\forall A\in\mathcal{A},H\cap A\neq\emptyset\}.\end{aligned}$$

[2]If $\xi=\bigvee_{f\in A}f^{-}\eta$ then in particular $\xi\geq f^{-}\eta$ for every $f\in A$ so that, inview of (2.2), $A\subset C(\xi,\eta)$. Since $\xi\geq\bigvee_{f\in C(\xi,\eta)}f^{-}\eta\geq\bigvee_{f\in A}f^{-}\eta$ is always true, $\xi=\bigvee_{f\in A}f^{-}\eta$ for some A if and only if $\xi=\bigvee_{f\in C(\xi,\eta)}f^{-}\eta$.

The set $\mathbb{F}X$ of filters on X is ordered by inclusion. The infimum of a family $\mathbb{D} \subset \mathbb{F}X$ of filters always exists and is $\bigcap_{\mathcal{D}\in\mathbb{D}} \mathcal{D}$. On the other hand, the supremum of a pair of filters may fail to exist in $\mathbb{F}X$. We say that two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh,* in symbols $\mathcal{A}\#\mathcal{B}$, if $\mathcal{A} \subset \mathcal{B}^{\#}$, equivalently, $\mathcal{B} \subset \mathcal{A}^{\#}$. Given $\mathcal{F}, \mathcal{G} \in \mathbb{F}X$ the supremum $\mathcal{F} \vee \mathcal{G}$ of the two filters exist (in $\mathbb{F}X$) if and only if $\mathcal{F}\#\mathcal{G}$.

Recall that the powerset $\mathcal{P}(X) = \{\emptyset\}^{\uparrow_X}$ is the *degenerate filter* on X and we denote by $\overline{\mathbb{F}X}$ the set of all (degenerate or proper) filters on X. Then inclusion turns $\overline{\mathbb{F}X}$ into a complete lattice in which $\mathcal{F} \vee \mathcal{G} = \mathcal{P}(X)$ whenever $\mathcal{F}$ and $\mathcal{G}$ do not mesh. Note that, denoting by Rel the category of sets with relations as morphisms, $\overline{\mathbb{F}} : \mathsf{Rel} \to \mathsf{Rel}$ is a functor that associates with a set $X \in \mathsf{Ob}(\mathsf{Rel})$ the set $\overline{\mathbb{F}X}$ and with a relation $R \subset X \times Y$ the relation $\overline{\mathbb{F}}R : \overline{\mathbb{F}X} \to \overline{\mathbb{F}Y}$ defined by

$$(\overline{\mathbb{F}}R)(\mathcal{F}) = R[\mathcal{F}] = \{R(F) : F \in \mathcal{F}\}^{\uparrow_Y}.$$

We will denote by $\mathbb{D} \subset \overline{\mathbb{F}}$ the fact that $\mathbb{D}$ is a subfunctor, that is, $\mathbb{D}X \subset \overline{\mathbb{F}X}$ for every set X and $\overline{\mathbb{F}}R(\mathcal{D}) \in \mathbb{D}Y$ for every $\mathcal{D} \in \mathbb{D}X$ and every relation $R \subset X \times Y$. In the terminology of [20, 11], we say that $\mathbb{D}$ is an $\mathbb{F}_0$-*composable class of filters.* Such a class must contain all principal filters, in particular every principal ultrafilter. Moreover, for such a class, if $\mathcal{D}, \mathcal{L} \in \mathbb{D}X$ with $\mathcal{D}\#\mathcal{L}$ then $\mathcal{D} \vee \mathcal{L} \in \mathbb{D}X$, and if $\mathcal{D} \in \mathbb{D}X$ and $X \subset Y$, then $\mathcal{D}^{\uparrow_Y} \in \mathbb{D}Y$ (See e.g., [20],[11, Lemma XIV.3.7] for this and other properties of $\mathbb{F}_0$-composable classes). Among such classes, we distinguish the class $\mathbb{F}_0$ of principal filters, $\mathbb{F}_1$ of countably based filters and more generally $\mathbb{F}_\kappa$ of filters with a filter-base of cardinality less than $\aleph_\kappa$, $\mathbb{F}_{\wedge\kappa}$ of $\aleph_\kappa$-complete filters. In contrast, the class $\mathbb{U}$ of ultrafilters and the class $\mathbb{E}$ of filters generated by a sequence are not $\mathbb{F}_0$-composable.

Given $\mathcal{F} \in \mathbb{F}X$ and $\mathbb{D}$ a class of filters, we write

$$\mathbb{D}(\mathcal{F}) := \{\mathcal{D} \in \mathbb{D}X : \mathcal{D} \geq \mathcal{F}\}.$$

Accordingly, $\mathbb{U}(\mathcal{F}) \neq \emptyset$ for every filter $\mathcal{F}$ and $\mathcal{F} = \bigcap_{\mathcal{U}\in\mathbb{U}(\mathcal{F})} \mathcal{U}$ while $\mathcal{F}^{\#} = \bigcup_{\mathcal{U}\in\mathbb{U}(\mathcal{F})} \mathcal{U}$.

Let ξ be a convergence on a set X and $\mathcal{H}$ be a filter on X. We say that $\mathcal{H}$ *adheres* to $x \in X$ (and write $x \in \mathrm{adh}_\xi \mathcal{H}$) if there exists a filter $\mathcal{G}$ that refines $\mathcal{H}$ with $x \in \lim_\xi \mathcal{G}$. In other words,

$$\mathrm{adh}_\xi \mathcal{H} := \bigcup_{\mathbb{F}X \ni \mathcal{G} \geq \mathcal{H}} \lim_\xi \mathcal{G} = \bigcup_{\mathbb{F}X \ni \mathcal{F}\#\mathcal{H}} \lim_\xi \mathcal{F} = \bigcup_{\mathcal{U}\in\mathbb{U}(\mathcal{H})} \lim_\xi \mathcal{U}. \tag{2.3}$$

Let $\mathbb{D}$ be a class of filters. A convergence ξ is $\mathbb{D}$-*adherence-determined* if $x \in \lim_\xi \mathcal{F}$ whenever $x \in \mathrm{adh}_\xi \mathcal{D}$ for each filter $\mathcal{D} \in \mathbb{D}$ such that $\mathcal{D}$ is a filter on $|\xi|$ and $\mathcal{D}\#\mathcal{F}$.

If $\mathbb{D}$ is an $\mathbb{F}_0$-composable class of filters, then $\mathrm{A}_\mathbb{D}$ defined on objects by

$$\lim_{\mathrm{A}_\mathbb{D}\,\xi} \mathcal{F} = \bigcap_{\mathbb{D} \ni \mathcal{D}\#\mathcal{F}} \mathrm{adh}_\xi \mathcal{D}$$

is a concrete reflector and $\mathrm{fix}\,\mathrm{A}_\mathbb{D} = \{\xi \in \mathsf{Ob}(\mathsf{Conv}) : \xi = \mathrm{A}_\mathbb{D}\,\xi\}$ is the subcategory of $\mathbb{D}$-adherence-determined convergences.

$\mathbb{D}$	$\mathrm{A}_{\mathbb{D}}$	$\xi = \mathrm{A}_{\mathbb{D}}\,\xi$ is a	category fix $\mathrm{A}_{\mathbb{D}}$
$\mathbb{F}$	S	pseudotopology	PsTop
$\mathbb{F}_1$	S_1	paratopology	ParaTop
$\mathbb{F}_{\wedge 1}$	$\mathrm{S}_{\wedge 1}$	hypotopology	HypoTop
$\mathbb{F}_0$	S_0	pretopology	PreTop

3. Main results

3.1. **For what classes $\mathbb{D}$ and $\mathbb{H}$ do we have $\mathrm{A}_{\mathbb{D}} = \mathrm{A}_{\mathbb{H}}$?([3]).** Given a class $\mathbb{D}$, define for each set X

$$\widehat{\mathbb{D}}X := \{\mathcal{H} \in \mathbb{F}X : \forall \mathcal{U} \in \mathbb{U}(\mathcal{H}) \exists \mathcal{D} \in \mathbb{D}X : \mathcal{H} \leq \mathcal{D} \leq \mathcal{U}\}$$

thus defining a new class $\widehat{\mathbb{D}}$ ([4]). By definition $\mathbb{D} \subset \widehat{\mathbb{D}}$, and

$$\mathbb{D} \subset \mathbb{J} \Rightarrow \widehat{\mathbb{D}} \subset \widehat{\mathbb{J}}. \tag{3.1}$$

Moreover,

Lemma 1. *Given a class $\mathbb{D}$ of filters, $\widehat{\mathbb{D}} = \widehat{\widehat{\mathbb{D}}}$.*

Proof. As $\mathbb{D} \subset \widehat{\mathbb{D}}$, (3.1) implies that $\widehat{\mathbb{D}} \subset \widehat{\widehat{\mathbb{D}}}$. If $\mathcal{H} \in \widehat{\widehat{\mathbb{D}}}$ then for every $\mathcal{U} \in \mathbb{U}(\mathcal{H})$ there is $\mathcal{F} \in \widehat{\mathbb{D}}$ with $\mathcal{H} \leq \mathcal{F} \leq \mathcal{U}$. As $\mathcal{F} \in \widehat{\mathbb{D}}$ and $\mathcal{U} \in \mathbb{U}(\mathcal{F})$ there is $\mathcal{D} \in \mathbb{D}$ with $\mathcal{H} \leq \mathcal{F} \leq \mathcal{D} \leq \mathcal{U}$ and thus $\mathcal{H} \in \widehat{\mathbb{D}}$. $\square$

Theorem 2. *Given two classes of filters $\mathbb{D}$ and $\mathbb{H}$,*

$$\mathrm{A}_{\mathbb{D}} \geq \mathrm{A}_{\mathbb{H}} \iff \mathbb{H} \subset \widehat{\mathbb{D}}.$$

Proof. Assume that $\mathbb{H} \subset \widehat{\mathbb{D}}$ and let $x \in \lim_{\mathrm{A}_{\mathbb{D}}\,\xi} \mathcal{F}$ and $\mathcal{H} \in \mathbb{H}$ with $\mathcal{H} \# \mathcal{F}$. Because $\mathcal{H} \in \widehat{\mathbb{D}}$, there is $\mathcal{D} \in \mathbb{D}(\mathcal{H})$ with $\mathcal{D} \# \mathcal{F}$. As $\mathcal{D} \in \mathbb{D}$ and $\mathcal{D} \# \mathcal{F}$, $x \in \mathrm{adh}_\xi \mathcal{D}$. As $\mathcal{D} \geq \mathcal{H}$, $\mathrm{adh}_\xi \mathcal{D} \subset \mathrm{adh}_\xi \mathcal{H}$. Hence $x \in \lim_{\mathrm{A}_{\mathbb{H}}\,\xi} \mathcal{F}$.

Assume conversely that $\mathbb{H} \not\subset \widehat{\mathbb{D}}$, that is, there is $\mathcal{H}_0 \in \mathbb{H}$ with $\mathcal{H}_0 \notin \widehat{\mathbb{D}}$, so that there is $\mathcal{U}_0 \in \mathbb{U}(\mathcal{H}_0)$ such that $\mathcal{H}_0 \nleq \mathcal{D}$ whenever $\mathcal{D} \in \mathbb{D}$ and $\mathcal{D} \leq \mathcal{U}_0$. Consider the prime convergence σ on $X \cup \{\infty\}$ in which $\infty \in \lim_\sigma \mathcal{F}$ if and only if $\mathcal{F}$ and $\mathcal{H}_0$ do not mesh. Then $\sigma = \mathrm{A}_{\mathbb{H}}\,\sigma$ because $\mathcal{H}_0 \in \mathbb{H}$, but $\mathrm{A}_{\mathbb{D}}\,\sigma \ngeq \sigma$. Indeed, by definition $\infty \notin \lim_\sigma \mathcal{U}_0$ because $\mathcal{H}_0 \# \mathcal{U}_0$ but $\infty \in \lim_{\mathrm{A}_{\mathbb{D}}\,\sigma} \mathcal{U}_0$. Indeed, if $\mathcal{D} \in \mathbb{D}$ and $\mathcal{D} \# \mathcal{U}_0$, equivalently, $\mathcal{D} \leq \mathcal{U}_0$ then $\mathcal{H}_0 \nleq \mathcal{D}$, that is, there is $H \in \mathcal{H}_0$ with $H^c \in \mathcal{D}^\#$. As $\{H^c\}^\uparrow$ does not mesh with $\mathcal{H}_0$, $\infty \in \lim_\sigma \{H^c\}^\uparrow$ and thus $\infty \in \mathrm{adh}_\sigma \mathcal{D}$, which completes the proof that $\mathrm{A}_{\mathbb{D}}\,\sigma \ngeq \sigma$. As a result, $\mathrm{A}_{\mathbb{D}} \ngeq \mathrm{A}_{\mathbb{H}}$. $\square$

Corollary 3. *Given two classes of filters $\mathbb{D}$ and $\mathbb{H}$,*

$$\mathrm{A}_{\mathbb{D}} = \mathrm{A}_{\mathbb{H}} \iff \widehat{\mathbb{H}} = \widehat{\mathbb{D}}.$$

Proof. Assume $\mathrm{A}_{\mathbb{D}} = \mathrm{A}_{\mathbb{H}}$. In view of Theorem 2, $\mathbb{H} \subset \widehat{\mathbb{D}}$ and $\mathbb{D} \subset \widehat{\mathbb{H}}$. In view of (3.1) and Lemma 1, $\widehat{\mathbb{H}} \subset \widehat{\widehat{\mathbb{D}}} = \widehat{\mathbb{D}}$ and $\widehat{\mathbb{D}} \subset \widehat{\widehat{\mathbb{H}}} = \widehat{\mathbb{H}}$ so that $\widehat{\mathbb{H}} = \widehat{\mathbb{D}}$.

Conversely, if $\widehat{\mathbb{H}} = \widehat{\mathbb{D}}$ then $\mathbb{D} \subset \widehat{\mathbb{D}} = \widehat{\mathbb{H}}$ and $\mathbb{H} \subset \widehat{\mathbb{H}} = \widehat{\mathbb{D}}$ so that $\mathrm{A}_{\mathbb{D}} = \mathrm{A}_{\mathbb{H}}$ by Theorem 2. $\square$

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[3]We would like to thank Emilio Angulo-Perkins, Fadoua Chigr, and Jesús González Sandoval for helpful discussions around the results in this subsection.

[4]Note that

$$\widehat{\mathbb{D}}X = \{\mathcal{H} \in \mathbb{F}X : \forall \mathcal{F} \in \mathbb{F}X \,(\mathcal{F} \# \mathcal{H} \Rightarrow \exists \mathcal{D} \in \mathbb{D}(\mathcal{H}), \mathcal{D} \# \mathcal{F})\}.$$

Example 4. Of course, $\widehat{\mathbb{U}}=\mathbb{F}=\widehat{\mathbb{F}}$. Moreover, $\widehat{\mathbb{F}_0}=\mathbb{F}_0$. To see the latter, assume that $\mathcal{H}\notin\mathbb{F}_0$, so that $(\ker\mathcal{H})^c\in\mathcal{H}^\#$. Then there is $\mathcal{U}\in\mathbb{U}(\mathcal{H}\vee(\ker\mathcal{H})^c)$. Note that any $\mathcal{D}\in\mathbb{F}_0(\mathcal{H})$ is of the form $\{D\}^\uparrow$ for $D\subset\ker\mathcal{H}$, so that $D\notin\mathcal{U}$. Hence $\mathcal{H}\notin\widehat{\mathbb{F}_0}$.

Example 5. A topological (or convergence) space (X,ξ) is called *bisequential* if for every ultrafilter $\mathcal{U}$ with $x\in\lim_\xi\mathcal{U}$ there is $\mathcal{D}\in\mathbb{F}_1$ with $\mathcal{D}\leq\mathcal{U}$ and $x\in\lim_\xi\mathcal{D}$. In other words, $\xi=\mathrm{T}\,\xi$ is bisequential if and only if each neighborhood filter $\mathcal{N}_\xi(x)$ is in $\widehat{\mathbb{F}_1}$. Naturally, we call filters of $\widehat{\mathbb{F}_1}$ *bisequential filters.* As there are bisequential topological spaces that are not first-countable ([5]), $\mathbb{F}_1$ is a proper subclass of $\widehat{\mathbb{F}_1}$ but

$$\mathrm{A}_{\mathbb{F}_1}=\mathrm{A}_{\widehat{\mathbb{F}_1}}=\mathrm{S}_1\,.$$

3.2. For what class $\mathbb{D}$ is $\operatorname{fix}\mathrm{A}_{\mathbb{D}}$ simple?

Definition 6. A class $\mathbb{D}$ of filters is called *refinable* if there is a set Y such that for any set X and every $\mathcal{D}\in\mathbb{D}X$ and $\mathcal{F}\in\mathbb{F}X$ with $\mathcal{F}\#\mathcal{D}$ there is $\mathcal{L}\in\mathbb{D}(\mathcal{D})$ with $\mathcal{L}\#\mathcal{F}$ and there is $f:X\to Y$ with $\mathcal{D}\leq f^-[f[\mathcal{L}]]$.

Lemma 7. *If $\mathcal{D}\leq f^-[f[\mathcal{L}]]$ then*

$$\mathcal{H}\neg\#\mathcal{D}\Rightarrow f[\mathcal{H}]\neg\# f[\mathcal{L}].$$

Proof. If $f[\mathcal{H}]\# f[\mathcal{L}]$, equivalently, $\mathcal{H}\# f^-[f[\mathcal{L}]]$ then, in particular, $\mathcal{H}\#\mathcal{D}$ because $\mathcal{D}\leq f^-[f[\mathcal{L}]]$. □

If $\mathbb{D}$ is refinable, given Y as in Definition 6, define the convergence space $\daleth_{\mathbb{D}}$ defined on

$$Y_\infty=Y\cup\{\infty_0\}\cup\{y_\mathcal{G}:\mathcal{G}\in\mathbb{D}Y\},$$

by

$$Y\cup\{\infty_0\}\subset\lim\nolimits_{\daleth_{\mathbb{D}}}\mathcal{F}$$

for every $\mathcal{F}\in\mathbb{F}(Y_\infty)$ and $y_\mathcal{G}\in\lim_{\daleth_{\mathbb{D}}}\mathcal{F}$ if $\mathcal{F}$ and $\mathcal{G}^\uparrow\wedge\{\infty_0\}^\uparrow$ do not mesh, where $\mathcal{G}^\uparrow$ is the filter generated on Y_∞ by $\mathcal{G}$. Note that, by definition, for every $\mathcal{G}\in\mathbb{D}Y$,

$$y_\mathcal{G}\notin\operatorname{adh}_{\daleth_{\mathbb{D}}}\mathcal{G}^\uparrow. \tag{3.2}$$

Theorem 8. *Let $\mathbb{D}$ be an $\mathbb{F}_0$-composable class of filters. The category* $\operatorname{fix}\mathrm{A}_{\mathbb{D}}$ *is simple (in* Conv*) if and only if $\mathbb{D}$ is refinable.*

Proof. Assume that $\mathbb{D}$ is not refinable and let (Y,τ_0) with $\tau_0=\mathrm{A}_{\mathbb{D}}\,\tau_0$. Because $\mathbb{D}$ is not refinable, there is X, $\mathcal{D}_0\in\mathbb{D}X$ and $\mathcal{F}_0\in\mathbb{F}X$ with $\mathcal{D}_0\#\mathcal{F}_0$ and for every $\mathcal{L}\in\mathbb{D}(\mathcal{D}_0)$ with $\mathcal{L}\#\mathcal{F}_0$ and every $f\in Y^X$, we have $\mathcal{D}\nleq f^-[f[\mathcal{L}]]$, that is, there is $D_{\mathcal{L},f}\in\mathcal{D}_0$ with $D_{\mathcal{L},f}\notin f^-[f[\mathcal{L}]]$, that is, $D^c_{\mathcal{L},f}:=X\setminus D_{\mathcal{L},f}$ belongs to $(f^-[f[\mathcal{L}]])^\#$, equivalently,

$$\exists D_{\mathcal{L},f}\in\mathcal{D}_0: f(D^c_{\mathcal{L},f})\in(f[\mathcal{L}])^\#. \tag{3.3}$$

Let ξ be the prime convergence on $X\cup\{\infty\}$ defined by $\infty\in\lim_\xi\mathcal{F}$ if and only if $\mathcal{F}$ and $\mathcal{D}_0$ do not mesh. Note that by definition $\infty\notin\operatorname{adh}_\xi\mathcal{D}_0$. This is easily seen to be a convergence. Moreover, $\xi=\mathrm{A}_{\mathbb{D}}\,\xi$ because if $\infty\in\operatorname{adh}_\xi\mathcal{L}$ for every $\mathcal{L}\in\mathbb{D}X$ with $\mathcal{L}\#\mathcal{F}$ then $\mathcal{D}_0\neg\#\mathcal{F}$ (for otherwise $\infty\in\operatorname{adh}_\xi\mathcal{D}_0$ because $\mathcal{D}_0\in\mathbb{D}X$).

In particular, $\infty\notin\lim_\xi\mathcal{F}_0$. We will see that $\infty\in\lim_{\bigvee_{f\in C(\xi,\tau_0)}f^-\tau_0}\mathcal{F}_0$ so that $\xi\neq\bigvee_{f\in C(\xi,\tau_0)}f^-\tau_0$ and as a result $\operatorname{fix}\mathrm{A}_{\mathbb{D}}$ is not simple.

[5]Take for instance the one-point compactification of a discrete set X of cardinality that is not measurable. See [23, Example 10.15] for details.

To see this, let $f\in C(\xi,\tau_0)$ and $\mathcal{G}\in\mathbb{D}Y$ with $\mathcal{G}\#f[\mathcal{F}_0]$. Then $f^-[\mathcal{G}]\#\mathcal{F}_0$. Either $f^-[\mathcal{G}]$ and $\mathcal{D}_0$ do not mesh or they do. In the former case, $\infty\in\lim_\xi f^-[\mathcal{G}]$ and by continuity, $f(\infty)\in\lim_{\tau_0} f[f^-[\mathcal{G}]]$. As $f[f^-[\mathcal{G}]]\#\mathcal{G}$, $\infty\in\mathrm{adh}_{\tau_0}\mathcal{G}$. In the later case, the filter $\mathcal{L}:=f^-[\mathcal{G}]\vee\mathcal{D}_0$ belongs to $\mathbb{D}X$ because $\mathbb{D}$ is $\mathbb{F}_0$-composable. By (3.3) and continuity of f, $f(\infty)\in\mathrm{adh}_{\tau_0}f[\mathcal{L}]$ because $\infty\in\lim_\xi\{D^c_{\mathcal{L},f}\}^\uparrow$. Moreover, $f[\mathcal{L}]\geq f[f^-[\mathcal{G}]]\geq\mathcal{G}$ so that $f(\infty)\in\mathrm{adh}_\xi\mathcal{G}$. Hence $f(\infty)\in\lim_{\tau_0} f[\mathcal{F}_0]$ for every $f\in C(\xi,\tau_0)$, that is, $\infty\in\lim_{\bigvee_{f\in C(\xi,\tau_0)}f^-\tau_0}\mathcal{F}_0$.

Assume now that $\mathbb{D}$ is refinable. We will show that $\daleth_\mathbb{D}$ is initially dense in $\mathrm{fix}\,\mathrm{A}_\mathbb{D}$.

We first check that $\daleth_\mathbb{D}=\mathrm{A}_\mathbb{D}\daleth_\mathbb{D}$. If $y\in Y_\infty\setminus\lim_{\daleth_\mathbb{D}}\mathcal{F}$ then $y=y_\mathcal{D}$ for some $\mathcal{D}\in\mathbb{D}Y$ and, and $\mathcal{F}\#(\mathcal{D}^\uparrow\wedge\{\infty_0\}^\uparrow)$ so that $\infty_0\in\bigcap_{F\in\mathcal{F},F\notin\mathcal{D}^\#}F$. If $\infty_0\in\ker\mathcal{F}$ then $\{\infty_0\}^\uparrow\in\mathbb{D}Y_\infty$, $\{\infty_0\}^\uparrow\#\mathcal{F}$ and $y_\mathcal{D}\notin\mathrm{adh}_{\daleth_\mathbb{D}}\{\infty_0\}^\uparrow$. Else, there is $F_0\in\mathcal{F}$ with $\infty_0\notin F_0$. Since $\mathcal{F}\#(\mathcal{D}^\uparrow\wedge\{\infty_0\}^\uparrow)$ Then $\mathcal{D}^\uparrow\in\mathbb{D}Y_\infty$, $\mathcal{D}^\uparrow\#\mathcal{F}$ and $y_\mathcal{D}\notin\mathrm{adh}_{\daleth_\mathbb{D}}\mathcal{D}^\uparrow$.

To see that $\daleth_\mathbb{D}$ is initially dense in $\mathrm{fix}\,\mathrm{A}_\mathbb{D}$, consider $\xi=\mathrm{A}_\mathbb{D}\,\xi$ on X and suppose that $x\notin\lim_\xi\mathcal{F}$, so that there is $\mathcal{D}\in\mathbb{D}X$ with $\mathcal{F}\#\mathcal{D}$ and $x\notin\mathrm{adh}_\xi\mathcal{D}$. By refinability, there is $\mathcal{L}\in\mathbb{D}(\mathcal{D})$ (so that $x\notin\mathrm{adh}_\xi\mathcal{L}$) with $\mathcal{L}\#\mathcal{F}$ and there is $f_0:X\to Y$ with $\mathcal{D}\leq f_0^-[f_0[\mathcal{L}]]$.

Let $h:X\to Y_\infty$ be defined by $h(t)=f_0(t)$ for $t\notin\{x\}\cup\ker\mathcal{D}$, $h(x)=y_\mathcal{G}$ for $\mathcal{G}:=f_0[\mathcal{L}]\in\mathbb{D}Y$ and $h(t)=\infty_0$ for $t\in\ker\mathcal{D}$. We show that $h\in C(\xi,\daleth_\mathbb{D})$ and $h(x)\notin\lim_{\daleth_\mathbb{D}}h[\mathcal{F}]$ so that $\xi\leq\bigvee_{h\in C(\xi,\daleth_\mathbb{D})}h^-\daleth_\mathbb{D}$.

To see that $h\in C(\xi,\daleth_\mathbb{D})$, note that if $t\in\lim_\xi\mathcal{H}$ and $t\neq x$ then

$$h(t)\in Y\cup\{\infty_0\}\subset\lim_{\daleth_\mathbb{D}}h[\mathcal{H}],$$

hence we only need to consider the case $x\in\lim_\xi\mathcal{H}$. Then $\mathcal{D}$ and $\mathcal{H}$ do not mesh because $x\notin\mathrm{adh}_\xi\mathcal{D}$. In particular $\ker\mathcal{D}\notin\mathcal{H}^\#$. Moreover $h(x)=y_\mathcal{G}$. If $x\notin\ker\mathcal{H}$, $h[\mathcal{H}]=f_0[\mathcal{H}]$. In view of Lemma 7, $f_0[\mathcal{H}]$ and $\mathcal{G}$ do not mesh so $y_\mathcal{G}=h(x)\in\lim_{\daleth_\mathbb{D}}h[\mathcal{H}]$.

To see that $h(x)\notin\lim_{\daleth_\mathbb{D}}h[\mathcal{F}]$ note that as $\mathcal{F}\#\mathcal{L}$ then $f_0[\mathcal{F}]\#\mathcal{G}$. Moreover, if $\ker\mathcal{D}\in\mathcal{F}$ then $h[\mathcal{F}]=\{\infty_0\}^\uparrow$ does not converge to $y_\mathcal{G}$. Else $(\ker\mathcal{D})^c\in\mathcal{F}^\#$ and $h[\mathcal{F}]=f_0[\mathcal{F}]$ meshes with $\mathcal{G}$ so $h(x)\notin\lim_{\daleth_\mathbb{D}}h[\mathcal{F}]$. □

A class $\mathbb{D}$ of filters is called *fiber-stable* if there is a set Y such that for every set X and every $\mathcal{D}\in\mathbb{D}X$ there is $f:X\to Y$ with $\mathcal{D}\leq f^-[f[\mathcal{D}]]$. Of course, every fiber-stable class is also refinable, as we can then take $\mathcal{L}=\mathcal{D}$ in the definition of a refinable class. Hence, it is sufficient for $\mathrm{fix}\,\mathrm{A}_\mathbb{D}$ to be simple that the class $\mathbb{D}$ be fiber-stable.

Example 9. The category PreTop of pretopologies is $\mathrm{fix}\,\mathrm{A}_{\mathbb{F}_0}$ and is simple because $\mathbb{F}_0$ is fiber-stable, hence refinable. Indeed, taking $Y=\{0,1\}$, then for every X and $\{A\}^\uparrow\in\mathbb{F}X$, $\{A\}^\uparrow\leq f^-[f[\{A\}^\uparrow]]$ where $f(x)=1$ if and only if $x\in A$. That Prtop is simple is known from [3, II.2]. Note that $\daleth_{\mathbb{F}_0}$ is an initially dense object of Prtop that is different from the Bourdaud pretopology on 3 points. Indeed, it has 6 points.

Example 10. The category ParaTop of paratopologies is $\mathrm{fix}\,\mathrm{A}_{\mathbb{F}_1}$ and is simple because $\mathbb{F}_1$ is fiber-stable, hence refinable. Take $Y=\omega$. Given X and $\mathcal{D}\in\mathbb{F}_1X$, we can deal with $\mathcal{D}$ with a two-valued map as in Example 9 if $\mathcal{D}$ is principal. Otherwise, $\mathcal{D}\vee(\ker\mathcal{D})^c$ is a non-degenerate free countably based filter and thus has a decreasing filter base $(H_n)_{n\in\omega}$ with $H_1=X\setminus\ker\mathcal{D}$. Consider $f:X\to Y$ defined by $f(x)=1$ if $x\in\ker\mathcal{D}$ and $f(x)=n>1$ if $x\in H_{n-1}\setminus H_n$. Then $f^-[f[\mathcal{D}]]\geq\mathcal{D}$.

That Partop is simple is [25, Theorem 1] and $\daleth_{\mathbb{F}_1}$ is a slight simplification of the initially dense object $\daleth$ used in [25].

Though fiber-stability is often more practical to check, there are refinable classes that are not fiber-stable:

Example 11. The class $\widehat{\mathbb{F}_1}$ is refinable but not fiber-stable. $\widehat{\mathbb{F}_1}$ is refinable because $\mathrm{A}_{\mathbb{F}_1} = \mathrm{A}_{\widehat{\mathbb{F}_1}}$ is simple. To see that $\widehat{\mathbb{F}_1}$ is not fiber-stable, given any set Y let X be a set of non-measurable cardinality $\operatorname{card} X > \operatorname{card} Y$. The cofinite filter $\mathcal{H}$ on X is then a bisequential filter (See [23, Example 10.15]), that is, $\mathcal{H} \in \widehat{\mathbb{F}_1}$. On the other hand, there is $y \in Y$ with an infinite fiber $A = f^-(y)$, so that $A\#\mathcal{H}$ and thus $f(A) = \{y\} \in (f[\mathcal{H}])^\#$. Hence, if $x \in A$ then $X \setminus \{x\} \in \mathcal{H}$ but $X \setminus \{x\} \notin f^-[f[\mathcal{H}]]$.

4. Non-simplicity of the class of μ-hypotopologies

For background on set theory we refer the reader to [19]. Let μ be an infinite cardinal. A filter $\mathcal{H}$ is *μ-complete* provided $\bigcap \mathcal{H}' \in \mathcal{H}$ for every $\mathcal{H}' \subseteq \mathcal{H}$ with $|\mathcal{H}'| < \mu$. Note that each filter is $\aleph_0$-complete. A convergence ξ on X is a *μ-hypotopology* iff it is $\mathbb{H}$-adherence-determined, where $\mathbb{H}$ is the class of all μ-complete filters. In particular, a convergence ξ is a *pseudotopology* if and only if it is an $\aleph_0$-hypotopology (is $\mathbb{F}$-adherence-determined with $\mathbb{F}$ being the class of all filters) and ξ is a *hypotopology* if and only if it is an $\aleph_1$-hypotopology (is $\mathbb{H}$-adherence-determined for $\mathbb{H}$ consisting of all countably complete filters).

Let λ is a regular uncountable cardinal and $A \subseteq \lambda$. We say that A is *unbounded* in λ if there no upper bound on A in λ and we say that A is closed in λ if $\sup A' \in A$ for any A' that is bounded in λ (this is equivalent to A being closed in the order topology on λ). The closed unbounded subsets of λ form a filter base and the filter on λ generated by them is called the *closed unbounded filter* on λ. This filter is λ-complete.

Lemma 12. *Let Y be a set, $\lambda > \operatorname{card} Y$ be an uncountable regular cardinal and $\mathcal{C}$ be the closed unbounded filter on λ. Then for each $f : \lambda \to Y$ there exists a uniform λ-complete filter $\mathcal{F}_f$ on λ such that $f[\mathcal{F}_f] = f[\mathcal{C}]$ and $\mathcal{F}_f$ does not mesh with $\mathcal{C}$.*

Proof. Let $f : \lambda \to Y$ be arbitrary. Define $\mathcal{P} := \{f^-(y) : y \in f[\lambda]\}$ to be the family of fibers of f with

$$\mathcal{P}_0 := \{P \in \mathcal{P} : \operatorname{card} P < \lambda\}$$

and $\mathcal{P}_1 := \mathcal{P} \setminus \mathcal{P}_0$. Note that the regularity of λ implies that $\operatorname{card} P_0 < \lambda$, where

$$P_0 := \bigcup_{P \in \mathcal{P}_0} P$$

and so $\mathcal{P}_1$ is not empty. We claim that there exists $C \in \mathcal{C}$ such that both $P \cap C$ and $P \setminus C$ have cardinality λ for every $P \in \mathcal{P}_1$. Let $\mathcal{P}_1$ be enumerated as $\{P_\xi : \xi < \kappa\}$ for some cardinal $\kappa \leq \operatorname{card} Y$.

We will use transfinite induction to construct two sequences $(A_\alpha)_{\alpha<\lambda}$ and $(B_\alpha)_{\alpha<\lambda}$ of subsets of λ such that

- any two distinct members of the family $\{A_\alpha : \alpha < \lambda\} \cup \{B_\alpha : \alpha < \lambda\}$ are disjoint.
- for any $\alpha < \lambda$ we have $A_\alpha = \{\gamma_{\alpha,\xi} : \xi < \kappa\}$ and $B_\alpha = \{\delta_{\alpha,\xi} : \xi < \kappa\}$ with $\gamma_{\alpha,\xi}, \delta_{\alpha,\xi} \in P_\xi$ for every $\xi < \kappa$

- $\mathrm{cl}\left(\bigcup_{\alpha<\lambda} A_\alpha\right) \cap \bigcup_{\alpha<\lambda} B_\alpha = \varnothing$, where cl is the closure operation in the order topology on λ.

Taking $C := \mathrm{cl}\left(\bigcup_{\alpha<\lambda} A_\alpha\right)$ satisfies the requirements.

Suppose that $\beta < \lambda$ is an ordinal such that A_α and B_α are defined for each $\alpha < \beta$ and that

- any two distinct members of the family $\{A_\alpha : \alpha < \beta\} \cup \{B_\alpha : \alpha < \beta\}$ are disjoint.
- for any $\alpha < \beta$ we have $A_\alpha = \{\gamma_{\alpha,\xi} : \xi < \kappa\}$ and $B_\alpha = \{\delta_{\alpha,\xi} : \xi < \kappa\}$ with $\gamma_{\alpha,\xi}, \delta_{\alpha,\xi} \in P_\xi$ for every $\xi < \kappa$
- $\mathrm{cl}\left(\bigcup_{\alpha<\beta} A_\alpha\right) \cap \bigcup_{\alpha<\beta} B_\alpha = \varnothing$.

For each $\xi < \kappa$, the set

$$P'_\xi := \{\gamma_{\alpha,\xi} : \alpha < \beta\} \cup \{\delta_{\alpha,\xi} : \alpha < \beta\}$$

is a subset of P_ξ of cardinality $< \lambda$ so there is $\gamma_{\beta,\xi} \in P_\xi$ with $\gamma_{\beta,\xi} > \sup P'_\xi$. Let $A_\beta := \{\gamma_{\beta,\xi} : \xi < \kappa\}$.

For each $\xi < \kappa$, let $\delta_{\beta,\xi} \in P_\xi$ be such that $\delta_{\beta,\xi} > \sup A_\beta$. Let $B_\beta := \{\delta_{\beta,\xi} : \xi < \beta\}$. It is clear that the obtained sequences $(A_\alpha)_{\alpha<\lambda}$ and $(B_\alpha)_{\alpha<\lambda}$ satisfy the requirements.

Let $C \in \mathcal{C}$ be such that both $P \cap C$ and $P \setminus C$ have cardinality λ for every $P \in \mathcal{P}_1$.

Since $\mathrm{card}\, P_0 < \lambda$, it follows that $P_1 := \bigcup_{P\in\mathcal{P}_1} \in \mathcal{C}$ so $P_1 \cap C \in \mathcal{C}$. Let $h : P_1 \cap C \to P_1 \setminus C$ be a bijection such that if $x \in P \cap C$ for some $P \in \mathcal{P}_1$, then $h(x) \in P \setminus C$. Extend h to a bijection $h : \lambda \to \lambda$ by declaring that $h(x) := x$ whenever $x \in P_0$. Let $\mathcal{F}_f := h[\mathcal{C}]$. Since h is a bijection and $\mathcal{C}$ is a uniform λ-complete filter on λ, it follows that $\mathcal{F}_f$ is uniform and λ-complete. It is clear that $f[\mathcal{F}_f] = f[\mathcal{C}]$. Since $P_1 \cap C \in \mathcal{C}$ and $P_1 \setminus C \in \mathcal{F}_f$, it follows that $\mathcal{F}_f$ does not mesh with $\mathcal{C}$. □

Theorem 13. *For any infinite cardinal μ the class of μ-hypotopologies is not simple.*

Proof. We show that the class $\mathbb{D}$ of μ-complete filters is not refinable, that is, for every Y there is X (with $X = \lambda \geq \mu$ a regular uncountable cardinal), and there is $\mathcal{D} \in \mathbb{D}X$ and $\mathcal{F}\#\mathcal{D}$ (taking $\mathcal{D} = \mathcal{F} = \mathcal{C}$ the closed unbounded filter on λ) such that for every $\mathcal{L} \in \mathbb{D}(\mathcal{D})$ with $\mathcal{L}\#\mathcal{F}$ and every $f : X \to Y$, $\mathcal{D} \not\leq f^-[f[\mathcal{L}]]$. Indeed, if $\mathcal{L} \geq \mathcal{D}$ then for every $f : X \to Y$, take $\mathcal{F}_f$ as in Lemma 12 to the effect that$f[\mathcal{L}] \geq f[\mathcal{D}] = f[\mathcal{F}_f]$ and thus $f[\mathcal{L}]\#f[\mathcal{F}_f]$, equivalently, $\mathcal{F}_f\#f^-[f[\mathcal{L}]]$. As $\mathcal{F}_f$ does not mesh with $\mathcal{D}$, $f^-[f[\mathcal{L}]] \ngeq \mathcal{D}$. The conclusion follows from Theorem 8. □

Remark 14. Note that Lemma 12 is the key to Theorem 13 and a direct proof based on this lemma rather than through Theorem 8 is relatively easy: for λ and $\mathcal{C}$ as in Lemma 12, let ξ be a convergence on λ defined by $\alpha \in \lim_\xi \mathcal{F}$ iff $\mathcal{F} = \{\alpha\}^\uparrow$ for $\alpha > 0$ and $0 \in \lim_\xi \mathcal{F}$ iff $\bigcap \mathcal{F} \subseteq \{0\}$ and $\mathcal{F}$ does not mesh with $\mathcal{C}$. This convergence can be shown to be a μ-hypotopology. Now, for every convergence space (Y, τ) and infinite μ, pick $\lambda \geq \mu$ uncountable and non-measurable. It is not difficult to verify, using Lemma 12, that the corresponding convergence ξ satisfies $\xi \neq \bigvee_{f\in C(\xi,\tau)} f^-\tau$.

The following answers affirmatively [25, Problem 3].

Corollary 15. *The class of hypotopologies is not simple.*

Proof. Hypotopologies are $\aleph_1$-hypotopologies. □

Moreover, we recover the main result of [21].

Corollary 16. *The class of pseudotopologies is not simple.*

Proof. Pseudotopologies are $\aleph_0$-hypotopologies. □

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