Three problems in convergence theory

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Abstract

In this note it is proved that the class of paratopologies is simple and that under the assumption that the measurable cardinals form a proper class, the class of hypotopologies is not simple. Moreover, an example is given of a Hausdorff convergence with idempotent set adherence (subdiagonal convergence) that is not weakly diagonal.

1 Introduction.

One way to describe a topological space is to consider the neighborhood filters of points and the convergence relation between points and filters defined using the neighborhood filters. Convergence theory studies this relation in greater generality and considers the topological convergence only as a special case. The need to study non-topological convergences was pointed out by Gustave Choquet in his fundamental paper [5], where he investigates natural convergences on the family of closed subsets of a topological space and concludes that some of them are not topological unless the underlying topology is locally compact.

The exact collection of axioms required for a convergence space to satisfy varies in the literature. We follow the definition of Dolecki in [9] (see also [11, 10]). A *convergence* ξ on a nonempty set X is a relation between the elements of X and the filters on X. Given a filter $\mathcal{F}$ on X and $x \in X$, we write $x \in \lim_\xi \mathcal{F}$ when $(x, \mathcal{F}) \in \xi$ and we require that $\lim_\xi \mathcal{F} \subseteq \lim_\xi \mathcal{G}$ whenever $\mathcal{F} \subseteq \mathcal{G}$ and that $x \in \lim_\xi \{x\}^\uparrow$ for every $x \in X$, where $\{x\}^\uparrow := \{A \subseteq X : x \in A\}$ is the principal ultrafilter generated by x. In particular, any topology on a set X induces a convergence τ defined by $x \in \lim_\tau \mathcal{F}$ if and only if $U \in \mathcal{F}$ for every open set $U \subseteq X$ with $x \in U$. Any convergence obtained in such a way is called *topological* or just a *topology*.

Convergences more general than topologies, called pretopologies, had been already considered by Hausdorff [12], Sierpiński [17] and Čech [4]. A convergence is a *pretopology* when filters convergent to a point x are refinements of a single vicinity filter at x. However, a breakthrough was made by Choquet in [5] who introduced a still larger class of *pseudotopologies*, by requiring that $x \in \lim_\xi \mathcal{F}$ whenever $x \in \lim_\xi \mathcal{U}$ for every ultrafilter $\mathcal{U}$ containing $\mathcal{F}$.

As discovered by Dolecki [7], pseudotopologies arise in a natural way when we consider the property of compactness. Analogous considerations for the property of countable compactness lead to paratopologies and for the Lindelöf property to hypotopologies (see [7]).

Those classes of convergences are defined as $\mathbb{H}$-adherence-determined convergences for suitably chosen classes $\mathbb{H}$ of filters.

Let ξ be a convergence on a set X and $\mathcal{H}$ be a filter on X. We say that $\mathcal{H}$ *adheres* to $x \in X$ (and write $x \in \mathrm{adh}_\xi \mathcal{H}$) if there exists a filter $\mathcal{G}$ that refines $\mathcal{H}$ with $x \in \lim_\xi \mathcal{G}$. Given filters $\mathcal{F}$ and $\mathcal{H}$, we say that $\mathcal{F}$ and $\mathcal{H}$ *mesh* (in symbols $\mathcal{F} \# \mathcal{H}$) iff $F \cap H \neq \varnothing$ for every $F \in \mathcal{F}$ and $H \in \mathcal{H}$. Note that if ξ is a convergence on some set X, then X is uniquely determined by ξ. We will denote such X by $|\xi|$.

Let $\mathbb{H}$ be a class of filters including all principal filters. A convergence ξ is $\mathbb{H}$-*adherence-determined* if $x \in \lim_\xi \mathcal{F}$ whenever $x \in \mathrm{adh}_\xi \mathcal{H}$ for each filter $\mathcal{H} \in \mathbb{H}$ such that $\mathcal{H}$ is a filter on $|\xi|$ and $\mathcal{H} \# \mathcal{F}$. Note that a convergence is a pseudotopology if and only if it is $\mathbb{H}$-adherence-determined, where $\mathbb{H}$ is the class of all filters and it is a pretopology when we consider the smallest possible class $\mathbb{H}$ consisting of principal filters only.

If we take $\mathbb{H}$ to be the class of all countably based filters, then we get the class of *paratopologies*. This class was introduced by Dolecki [6] to enable a unification (with the aid of one formula) of various classes of quotient maps (corresponding to parameters in the formula). In particular, the class of hereditary quotient maps corresponds to the class of pretopologies and the class of biquotient maps to pseudotopologies. The class of paratopologies is obtained in this correspondence when we consider countably biquotient maps.

If $\mathbb{H}$ is the class of all countably complete filters (those that are closed under countable intersections), then the obtained class of $\mathbb{H}$-adherence-determined convergences is the class of *hypotopologies*. This class of convergences was introduced by Dolecki [7] to enable another unification procedure.

It turns out that each topology τ can be represented as the initial convergence with respect to continuous maps from τ to the Sierpiński topology on a set with two elements. The same is true for pretopologies ξ when we consider maps from ξ to the Bourdaud pretopology (see Antoine [1] and Bourdaud [2, 3]). However, as proved by Eva and Robert Lowen [16], there is no initially dense pseudotopology and the same negative result holds for the class of all convergences. In this paper we will investigate this property for the classes of paratopologies and hypotopologies.

To formally define the concept of an initial convergence, let's recall that if ξ and η are convergences and $f : |\xi| \to |\eta|$ (where $|\xi|$ and $|\eta|$ are the ground sets for the convergences ξ and η, respectively) then f is *continuous* from ξ to η (we write $f \in C(\xi, \eta)$) if $f(x) \in \lim_\eta f[\mathcal{F}]$ for every filter $\mathcal{F}$ on X and every $x \in \lim_\xi \mathcal{F}$. If η is a convergence, X is a set and $f : X \to |\eta|$, then the relation $f^-\eta$ between X and $\mathbb{F}X$ (all the filters on X) relating x to $\mathcal{F}$ if and only if $f(x) \in \lim_\xi f[\mathcal{F}]$ is a convergence on X. In other words, $\xi := f^-\eta$ is the coarsest convergence on X that makes f continuous from ξ to η.

Let Φ be a class of convergences. We say that a convergence η is *initially dense* in Φ iff for each $\xi \in \Phi$ there exists a set A of continuous functions from ξ to η such that $\xi = \bigcap_{f \in A} f^-\eta$. Note that if η is initially dense in Φ, then $\eta \in \Phi$ and that $\xi = \bigcap_{f \in A} f^-\eta$ means that ξ is the coarsest convergence on $|\xi|$ for which all functions in A are continuous. A class of convergences is *simple* provided it includes an initially dense convergence.

As a well known example, recall that a topological space is completely regular if and only if it is homeomorphic to a subspace of $\mathbb{R}^X$ for some set X. Using the terminology introduced above, that is equivalent to saying that the standard topology on the set of real numbers is initially dense in the class of completely regular topologies. In particular, the class of completely regular topologies is simple.

We will prove that:

Theorem 1. *The class of paratopologies is simple.*

To state our result about hypotopologies, we need to recall the concept of a measurable cardinal (see [13]). A *measurable cardinal* is a cardinal κ admitting a κ-complete free ultrafilter (a free ultrafilter closed under intersections of fewer than κ members).

We are going to show:

Theorem 2. *Assume that for each cardinal there exists a larger measurable cardinal. Then the class of hypotopologies is not simple.*

The assumption that measurable cardinals form a proper class is a very strong set-theoretic assumption. It would be desirable to find a proof requiring weaker assumptions. In particular, Theorem 2 suggests the following question.

Problem 3. Can it be proved in ZFC that the class of hypotopologies is not simple?

Another property of convergences studied in this paper is diagonality. Diagonal convergences were defined by Kowalsky [14] (see also [9, 10, 11]). This property is important since a topology can be characterized as a diagonal pretopology. Another way to characterize topologies is to say that a topology is a pretopology ξ with idempotent set adherence, that is, such that

$$\mathrm{adh}_\xi(\mathrm{adh}_\xi A) = \mathrm{adh}_\xi A$$

for every $A \subseteq |\xi|$, where

$$\mathrm{adh}_\xi A := \mathrm{adh}_\xi \{F \subseteq X : A \subseteq F\}.$$

The convergences with idempotent set adherence are called *subdiagonal* by Dolecki [10]. It is true in general that each diagonal convergence is subdiagonal.

In [15], Eva Lowen-Colebunders introduced and investigated convergences ξ such that every filter has a closed adherence, that is, such that

$$\mathrm{adh}_\xi(\mathrm{adh}_\xi \mathcal{F}) = \mathrm{adh}_\xi \mathcal{F}$$

for every filter $\mathcal{F}$ on $|\xi|$. She formulated a condition, called *weak diagonality*, which is a weakening of the diagonality property of Kowalsky and proved that a convergence is weakly diagonal if and only if filters have closed adherences.

Note that each weakly diagonal convergence is subdiagonal. Moreover, the definition of weak diagonality implies that diagonal convergences are weakly diagonal. Thus if ξ is a pretopology, then all three of these concepts are equivalent to ξ being a topology. Example 1.5 in [15] shows that there exists a subdiagonal convergence which is not weakly diagonal. In that example, however, the convergence is not Hausdorff. We say that a convergence is *Hausdorff* provided that each filter has at most one limit.

We will prove the following result.

Theorem 4. *There exists a Hausdorff subdiagonal convergence that is not weakly diagonal.*

2 Proof of Theorem 1.

Let $X = \omega \cup \{\infty_0, \infty_1, \infty_2\}$, where $\infty_0, \infty_1, \infty_2$ are distinct and do not belong to ω. Let $\daleth$ be the convergence on X such that a filter $\mathcal{F}$ converges to x iff either $x \neq \infty_2$ or $K \cup \{\infty_1, \infty_2\} \in \mathcal{F}$, for some finite $K \subseteq \omega$. Note that, in particular, $\infty_2 \in \lim_\daleth \mathcal{F}$ if and only if there is finite $F \in \mathcal{F}$ with $\infty_0 \notin F$.

We verify that $\daleth$ is a paratopology. Indeed, assume $x \in X \setminus \lim_\daleth \mathcal{F}$. Then $x = \infty_2$ and $\infty_0 \in F$ for every finite $F \in \mathcal{F}$. We want a countably based filter $\mathcal{H}$ such that $\mathcal{F} \# \mathcal{H}$ and $x \notin \operatorname{adh}_\daleth \mathcal{H}$. If $\infty_0 \in \bigcap \mathcal{F}$, then $\mathcal{H} := \{\infty_0\}^\uparrow$ satisfies the requirements. Otherwise, all sets in $\mathcal{F}$ are infinite and we can use the cofinite filter on X as $\mathcal{H}$.

We will show that $\daleth$ is initially dense in the class Φ of paratopologies. Let η be a paratopology on a set Y. We need to find a set C of continuous functions from η to $\daleth$ such that $\eta = \bigcap_{f \in C} f^- \daleth$.

Let $\mathbf{B}$ be the collection of all countable families

$$\mathcal{B} = \{B_0, B_1, \dots\},$$

of subsets of Y, where $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$. For each $\mathcal{B} \in \mathbf{B}$, let $f_\mathcal{B} : Y \to X$ be defined by

$$f_\mathcal{B}(y) = \begin{cases} n & \text{if } y \in B_n \setminus B_{n+1} \text{ for some } n \in \omega \\ \infty_0 & \text{if } y \in B_n \text{ for every } n \in \omega \\ \infty_1 & \text{if } y \in \operatorname{adh}_\eta \mathcal{H} \setminus \bigcup_{n \in \omega} B_n, \text{ where } \mathcal{H} \text{ is the filter generated by } \mathcal{B} \\ \infty_2 & \text{otherwise.} \end{cases}$$

Let

$$C := \{f_\mathcal{B} : \mathcal{B} \in \mathbf{B}\}.$$

It remains to verify that $\eta = \bigcap_{f \in C} f^- \daleth$. Let $\mathcal{F}$ be a filter on Y and $y \in Y$. It suffices to show that $y \in \lim_\eta \mathcal{F}$ if and only if $f(y) \in \lim_\daleth f[\mathcal{F}]$ for every $f \in C$.

Claim. Assume that $y \in \lim_\eta \mathcal{F}$. Then $f(y) \in \lim_\daleth f[\mathcal{F}]$ for every $f \in C$.

Proof. Let $f = f_\mathcal{B} \in C$ with

$$\mathcal{B} = \{B_0, B_1, \dots\} \in \mathbf{B},$$

where $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$, and let $\mathcal{H}$ be the filter generated by $\mathcal{B}$. If $\mathcal{F}$ and $\mathcal{H}$ do not mesh, then there is $F \in \mathcal{F}$ and $n \in \omega$ such that $F \cap B_m = \varnothing$ for every $m > n$. Taking $K := \{k \in \omega : k \leq n\}$ we have $K \cup \{\infty_1, \infty_2\} \in f[\mathcal{F}]$ so $f(y) \in \lim_\daleth f[\mathcal{F}]$. If $\mathcal{F} \# \mathcal{H}$, then $y \in \operatorname{adh}_\daleth \mathcal{H}$ so

$$f(y) \in X \setminus \{\infty_2\} \subseteq \lim_\daleth f[\mathcal{F}].$$

□

Claim. Assume that $y \notin \lim_\eta \mathcal{F}$. Then $f(y) \notin \lim_\daleth f[\mathcal{F}]$ for some $f \in C$.

Proof. Let $\mathcal{H}$ be a countably based filter such that $\mathcal{F} \# \mathcal{H}$ and $y \notin \operatorname{adh}_\eta \mathcal{H}$. Since $y \notin \operatorname{adh}_\eta \mathcal{H}$, there is $H \in \mathcal{H}$ with $y \notin H$. Let

$$\mathcal{B} = \{B_0, B_1, \dots\} \in \mathbf{B},$$

with $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$, be a base for $\mathcal{H}$ with $B_0 \subseteq H$. Since $y \notin B_0 \cup \operatorname{adh}_\eta \mathcal{H}$, it follows that $f_{\mathcal{B}}(y) = \infty_2$.

It remains to show that $\infty_2 \notin \lim_\daleth f_{\mathcal{B}}[\mathcal{F}]$. Suppose, for a contradiction, that $\infty_2 \in \lim_\daleth f_{\mathcal{B}}[\mathcal{F}]$. Then there is $F \in \mathcal{F}$ such that $\infty_0 \notin f_{\mathcal{B}}(F)$ and $f_{\mathcal{B}}(F) \cap \omega$ is finite.

Let $B := \bigcap_{n\in\mathbb{N}} B_n$. Since $\infty_0 \notin f_{\mathcal{B}}(F)$, it follows that $F \cap B = \varnothing$. Since $f_{\mathcal{B}}(F) \cap \omega$ is finite, there is $n \in \omega$ such that $m \notin f_{\mathcal{B}}(F)$ for every $m \geq n$. Since $\mathcal{F} \# \mathcal{H}$, we have $F \cap B_n \neq \varnothing$. As $F \cap B = \varnothing$, it follows that there is $m \geq n$ with $F \cap (B_m \setminus B_{m+1}) \neq \varnothing$. Since $m \in f_{\mathcal{B}}(F)$ for such m, we get a contradiction. □

3 Proof of Theorem 2.

We modify the argument from [16]. Assume that for each cardinal there exists a larger measurable cardinal. A filter $\mathcal{F}$ on a set X is *uniform* iff each member of $\mathcal{F}$ has the same cardinality as X.

Lemma 5. *Let X and Y be sets such that X is uncountable and* $\operatorname{card} X > \operatorname{card} Y$. *Then for each uniform countably complete ultrafilter $\mathcal{U}$ on X and each $f : X \to Y$ there exists a uniform countably complete ultrafilter $\mathcal{W}$ on X such that $\mathcal{W} \neq \mathcal{U}$ and $f[\mathcal{U}] = f[\mathcal{W}]$.*

Proof. Let $\mathcal{P} := \{f^-(y) : y \in f(X)\}$ with $\mathcal{P}_0 := \{A \in \mathcal{P} : \operatorname{card} A \leq \aleph_0\}$ and $\mathcal{P}_1 := \mathcal{P} \setminus \mathcal{P}_0$. Note that $\operatorname{card} P_0 < \operatorname{card} X$, where

$$P_0 := \bigcup_{P\in\mathcal{P}_0} P$$

and so $\mathcal{P}_1$ is not empty.

For each $P \in \mathcal{P}_1$, let $P = A_P \cup B_P$ with $A_P \cap B_P = \varnothing$ and $\operatorname{card} A_P = \operatorname{card} B_P$. Let

$$A := P_0 \cup \bigcup_{P\in\mathcal{P}_1} A_P \qquad \text{and} \qquad B := P_0 \cup \bigcup_{P\in\mathcal{P}_1} B_P.$$

Then $A \cup B = X \in \mathcal{U}$ and $A \cap B = P_0 \notin \mathcal{U}$ since $\mathcal{U}$ is uniform. Thus exactly one of A and B belongs to $\mathcal{U}$. Let $h : A \to B$ be a bijection such that $h(x) = x$ for each $x \in P_0$ and h maps A_P onto B_P for each $P \in \mathcal{P}_1$. Then $\mathcal{W} := h[\mathcal{U}]$ is a countably complete ultrafilter on X with $\mathcal{U} \neq \mathcal{W}$ and $f[\mathcal{U}] = f[\mathcal{W}]$ as required. □

Lemma 6. *Let X be a set of measurable cardinality. Then there exists a uniform countably complete ultrafilter on X.*

Proof. Let $\kappa := \operatorname{card} X$. Since κ is measurable, there exists a free κ-complete ultrafilter $\mathcal{U}$ on X. Since $\mathcal{U}$ is free, $X \setminus \{x\} \in \mathcal{U}$ for every $x \in X$. Since $\mathcal{U}$ is κ-complete, if $A \subseteq X$ has cardinality smaller than κ, then

$$X \setminus A = \bigcap_{x\in A} (X \setminus \{x\}) \in \mathcal{U},$$

so $A \notin \mathcal{U}$. Thus $\mathcal{U}$ is uniform. □

Lemma 7. *Let $\mathcal{U}$ be a uniform countably complete ultrafilter on an uncountable set X and $x_\infty \in X$. Define a convergence $\xi = \xi(\mathcal{U}, x_\infty)$ on X by $x \in \lim_\xi \mathcal{F}$ iff $\mathcal{F} = \{x\}^\uparrow$ for $x \in X \setminus \{x_\infty\}$ and $x_\infty \in \lim_\xi \mathcal{F}$ iff $\bigcap \mathcal{F} \subseteq \{x_\infty\}$ and $\mathcal{F} \not\subseteq \mathcal{U}$. Then ξ is a hypotopology.*

Proof. Let $\mathcal{F}$ be a filter on X and $x \in X \setminus \lim_\xi \mathcal{F}$. We need to find a countably complete filter $\mathcal{H}$ on X such that $\mathcal{H} \# \mathcal{F}$ but $x \notin \mathrm{adh}_\xi \mathcal{H}$. If $x \neq x_\infty$, then $\mathcal{F} \neq \{x\}^\uparrow$. Taking any $A \in \{x\}^\uparrow \setminus \mathcal{F}$ makes $\mathcal{H} := (X \setminus A)^\uparrow$ to be as required.

Assume $x = x_\infty$. If there is $y \in \bigcap \mathcal{F}$ with $y \neq x$, then $\mathcal{H} := \{y\}^\uparrow$ satisfies the requirements. If $\bigcap \mathcal{F} \subseteq \{x\}$, then $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{H} := \mathcal{U}$ satisfies the requirements. □

Proof of Theorem 2. Let η be a convergence on a set Y. Let X be such that $\mathrm{card}\, X$ is a measurable cardinal with $\mathrm{card}\, X > \mathrm{card}\, Y$ and let $\mathcal{U}$ be a uniform countably complete ultrafilter on X. Let $\xi := \xi(\mathcal{U}, x_\infty)$ be the hypotopology on X as in Lemma 7 for some $x_\infty \in X$. By Lemma 5, for every map $f \in Y^X$ there is a uniform countably complete ultrafilter $\mathcal{W}_f$ on X with $\mathcal{W}_f \neq \mathcal{U}$ and $f[\mathcal{U}] = f[\mathcal{W}_f]$. Then $\lim_\xi \mathcal{U} = \varnothing$ and $x_\infty \in \lim_\xi \mathcal{W}_f$ for each $f \in Y^X$. Thus

$$\xi \neq \bigcap_{f \in \mathscr{C}(\xi, \tau)} f^- \eta.$$

4 Proof of Theorem 4.

Let X_n be a countably infinite set for each $n \in \omega$ with $X_n \cap X_m = \varnothing$ whenever $n \neq m$. Let $y_0, y_1, \ldots$ be distinct with $\{y_0, y_1, \ldots\} \cap X_n = \varnothing$ for each $n \in \omega$. Let

$$z \notin \{y_0, y_1, \ldots\} \cup \bigcup_{n \in \omega} X_n$$

and

$$X := \bigcup_{n \in \omega} X_n \cup \{y_0, y_1, \ldots\} \cup \{z\}.$$

We define a Hausdorff pseudotopology ξ on X as follows. If $x \in \bigcup_{n \in \omega} X_n$, then the principal ultrafilter $\{x\}^\uparrow$ is the only filter on X that converge to x. A free ultrafilter $\mathcal{U}$ on X converges to y_n for $n \in \omega$ iff $\mathcal{U}$ refines the cofinite filter on X_n. A free ultrafilter $\mathcal{U}$ on X converges to z iff $\mathcal{U}$ refines the cofinite filter on $\{y_0, y_1, \ldots\}$ or there exists a sequence $x_0, x_1, \ldots$ with $x_n \in X_n$ for each $n \in \omega$ and $\mathcal{U}$ refines the cofinite filter on $\{x_0, x_1, \ldots\}$.

Claim. The convergence ξ is subdiagonal.

Proof. Let $A \subseteq X$ and $x \in X$ be such that $x \in \mathrm{adh}_\xi (\mathrm{adh}_\xi A)$. If $x \in \bigcup_{n \in \omega} X_n$, then $x \in \mathrm{adh}_\xi A$.

Assume that $x = y_n$ for some $n \in \omega$. Since $x \in \mathrm{adh}_\xi (\mathrm{adh}_\xi A)$, it follows that $x \in \mathrm{adh}_\xi A$ or $X_n \cap \mathrm{adh}_\xi A$ is infinite. Note that

$$X_n \cap \mathrm{adh}_\xi A = X_n \cap A,$$

and that if $X_n \cap A$ is infinite, then $x \in \mathrm{adh}_\xi A$. Therefore $x \in \mathrm{adh}_\xi A$.

Assume that $x = z$ and $x \notin \mathrm{adh}_\xi A$. Then the set $\{n \in \omega : A \cap X_n \neq \varnothing\}$ is finite, implying that the set $\{n \in \omega : y_n \in \mathrm{adh}_\xi A\}$ is also finite which contradicts $x \in \mathrm{adh}_\xi (\mathrm{adh}_\xi A)$. □

Claim. The convergence ξ is not weakly diagonal.

Proof. Let $\mathcal{F}$ be the free filter on X such that for $A \subseteq X$ we have $A \in \mathcal{F}$ iff $X_n \setminus A$ finite for every $n \in \omega$. We show that

$$z \in \mathrm{adh}_\xi \left(\mathrm{adh}_\xi \mathcal{F}\right) \setminus \mathrm{adh}_\xi \mathcal{F}.$$

Indeed, $z \in \mathrm{adh}_\xi \left(\mathrm{adh}_\xi \mathcal{F}\right)$ since $y_n \in \mathrm{adh}_\xi \mathcal{F}$ for each $n \in \omega$ and $z \in \mathrm{adh}_\xi \{y_0, y_1, \dots\}$. However, $z \notin \mathrm{adh}_\xi \mathcal{F}$ since for every ultrafilter $\mathcal{U}$ that converges to z either $\{y_0, y_1, \dots\} \in \mathcal{U}$ or there is a sequence $x_0, x_1, \dots$ with $x_n \in X_n$ for every $n \in \omega$ and $\{x_0, x_1, \dots\} \in \mathcal{U}$. If the former holds, then $\mathcal{U}$ does not refine $\mathcal{F}$. If the latter holds, then $\mathcal{U}$ does not refine $\mathcal{F}$ either since

$$\bigcup_{n \in \omega} \left(X_n \setminus \{x_n\}\right) \in \mathcal{F} \setminus \mathcal{U}.$$

Thus ξ is not weakly diagonal. □

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