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Edge-bandwidth of grids and tori

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Abstract

The *edge-bandwidth* of a graph G is the smallest number B' for which there is a bijective labeling of $E(G)$ with $\{1, \dots, e(G)\}$ such that the difference between the labels at any adjacent edges is at most B'. Here we compute the edge-bandwidth for rectangular grids:

$$B'(P_m \oplus P_n) = 2\min(m, n) - 1 \quad \text{if} \quad \max(m, n) \geqslant 3,$$

where $\oplus$ is the Cartesian product and P_n denotes the path on n vertices. This settles a conjecture of Calamoneri et al. [New results on edge-bandwidth, Theoret. Comput. Sci. 307 (2003) 503–513]. We also compute the edge-bandwidth of any torus (a product of two cycles) within an additive error of 5.

Keywords: Edge bandwidth; Graph products; Grids

1. Introduction

We will use the standard notation and terminology on graphs, see e.g. [3]. Also, we denote $[n] = \{1, \dots, n\}$.

Let G be a graph with n vertices. The *bandwidth* of G is $B(G) = \min_\eta\{B(\eta)\}$, where the minimum is taken over all bijections $\eta : V(G) \to [n]$ and $B(\eta)$ is the maximum of $|\eta(x) - \eta(y)|$ over all adjacent vertices x, y.

This classical problem was introduced by Harary [12, Problem 16, p. 167] and Harper [14]. It has been extensively studied due to its connections to isoperimetric inequalities [6], VLSI design and other layout problems [10], multicasting [4], multi-channel transmission of data with noise [2], graph searching [13], and others. (For each area, we mentioned a sample recent paper containing further pointers; also we refer the reader to the older surveys by Chinn et al. [7] and Chung [8].)

As a simple example, let us show how graph bandwidth appears in some multi-channel transmission problems. Suppose we want to encode each element $l \in [mn]$ as a pair $(l_1, l_2) \in [m] \times [n]$ to be transmitted over two channels. We want to minimize b such that if one of the channels fails (and we are told which one) then knowing the remaining

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part l_i, we can find an interval of length b containing all possible inputs l. Then the smallest possible b is precisely the bandwidth of the Cartesian product of the cliques K_m and K_n, see [2].

The *edge-bandwidth* $B'(G)$ is the bandwidth of the line graph of G. In other words, it is the smallest integer B' for which there is a bijection between $E(G)$ and $\{1, \dots, e(G)\}$ such that the difference between the labels at any two adjacent edges is at most B'. This parameter was introduced by Hwang and Lagarias [15]. Being just a special case of bandwidth, it is far less studied but recent years witnessed an increase of activity in this area [17,16,11,5,1].

Let us consider $P_m \oplus P_n$, the $m \times n$-grid, where P_n denotes the path of order n and $\oplus$ is the Cartesian product. Computing the (edge-) bandwidth for grids is of interest because these graphs epitomize the two-dimensional nature of many real world problems. Chvátalová [9] proved that $B(P_m \oplus P_n) = \min(m, n)$ if $\max(m, n) \geqslant 2$. Calamoneri et al. [5, p. 512] conjectured that

$$B'(P_m \oplus P_n) = 2\min(m, n) - 1. \tag{1}$$

The upper bound (an example of an edge labeling) is easy to produce (see Lemma 4 here). Balogh et al. [1] proved that

$$B'(P_n \oplus P_n) \geqslant 2n - \sqrt{n} - 1, \quad n \geqslant 2.$$

Here we completely settle the conjecture by proving the following results.

Theorem 1. *Let F be an arbitrary connected graph of order m and size l. If $n \geqslant \max(l + 2 + 3)$, then*

$$B'(F \oplus P_n) = l + m. \tag{2}$$

Theorem 2. *For any $n \geqslant 3$, we have $B'(P_n \oplus P_n) = 2n - 1$.*

Theorems 1 and 2 imply (1) for any positive m, n except for the pairs (1, 1), (1, 2), and (2, 2). The first two cases do not make much sense (namely, $P_m \oplus P_n$ has at most one edge) while the last case is an exception to (1): $B'(P_2 \oplus P_2) = 2$.

We believe that the restriction $n \geqslant l + 2$ in Theorem 1 can be weakened to $n \geqslant l + 1$ by appropriately modifying our proof of Theorem 2. However, the argument becomes far messier and its length seems to increase considerably. So, in order to keep this paper short and readable, we do the case $F = P_n$ only.

Tori, that is, Cartesian products of two cycles, were studied by Li et al. [18] who computed $B(C_m \oplus C_n)$ for all m, n. Balogh et al. [1] considered the edge bandwidth of the torus $C_n \oplus C_n$ and established the following bounds:

$$4n - 2\sqrt{2n} - 1 \leqslant B'(C_n \oplus C_n) \leqslant 4n, \quad n \geqslant 3. \tag{3}$$

We have been able to reduce the gap in (3):

Theorem 3. *For any $m \geqslant n \geqslant 3$, we have*

$$4n - 5 \leqslant B'(C_m \oplus C_n) \leqslant 4n. \tag{4}$$

Our proof techniques for Theorems 1–3 are built upon those from [1].

Independently of us, Akhtar, Jiang, Miller, and Pritikin report to have obtained new bounds on the edge-bandwidth of various graph products, in particular the following:

$$B'(P_n \oplus P_n) \geqslant 2n - 2 \quad \text{and} \quad B'(C_n \oplus C_n) \geqslant 4n - 5,$$

as well as the asymptotic result for $B'(P_n^{\oplus d})$, for any fixed $d \geqslant 3$.

Our paper is organized as follows. In Section 2 we provide some further notation and auxiliary results that we will need. The (easy) upper bounds of Theorems 1 and 2 are proved in Lemma 4. Sections 3 and 4 are dedicated to proving the corresponding lower bounds. Theorem 3 is proved in Section 5. Some open problems are mentioned in Section 6.

2. Notation and basics

Let us set up the notation that we will use for the *Cartesian product* $G = F \oplus H$ of any two graphs F and H of orders m and n, respectively. We will usually assume that $V(F) = [m]$ and $V(H) = [n]$. Thus, G has the vertex set

$$V(G) = \{(i, j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$$

and edges

$$\{r_{i,D} : 1 \leqslant i \leqslant m,\ D \in E(H)\} \cup \{c_{D,j} : D \in E(F),\ 1 \leqslant j \leqslant n\}$$

with $r_{i,xy}$ incident to (i, x) and (i, y) and $c_{xy,j}$ incident to (x, j) and (y, j). (We will abbreviate $\{x, y\}$ to xy sometimes.) The edges of the form $r_{i,D}$ are called *horizontal* and the edges $c_{D,j}$ are *vertical*. For $i = 1, \ldots, m$, the ith *row* is

$$R_i = \{r_{i,D} : D \in E(H)\},$$

and, for $j = 1, \ldots, n$, the jth *column* is

$$C_j = \{c_{D,j} : D \in E(F)\}.$$

(Thus, we use matrix-type coordinates.) An edge $D \in E(F)$ gives us the *quasi-row*

$$R'_D = \{c_{D,j} : 1 \leqslant j \leqslant n\},$$

and an edge $D \in E(H)$ gives us the *quasi-column*

$$C'_D = \{r_{i,D} : 1 \leqslant i \leqslant m\}.$$

A *line* is a row or a column. A *quasi-line* is a quasi-row or a quasi-column.

If one of the graphs is a path or a cycle, then we assume that it traverses its vertex set in the natural order. For example, the cycle C_n visits its vertices in this order: $1, 2, \ldots, n-1, n, 1$. If $H = P_n$ is a path, then we will denote $r_{i,j} = r_{i,\{j,j+1\}}$ and $C'_j = C'_{\{j,j+1\}}$ for $i \in [m]$ and $j \in [n-1]$. If $H = C_n$ is a cycle, then we additionally let $r_{i,n} = r_{i,\{n,1\}}$ and $C'_n = C'_{\{n,1\}}$. Likewise we define $c_{i,j}$ and R'_i if F is a cycle or a path. Since we use different letters R and C, corresponding to the rows and columns, this will not cause any clashes in notation.

For example, for $G = P_3 \oplus P_3$ we see the following picture:

$$\begin{array}{cccccc}
 & C_1 & C'_1 & C_2 & C'_2 & C_3 \\
R_1 & (1,1) & \overset{r_{1,1}}{\rule{3em}{0.4pt}} & (1,2) & \overset{r_{1,2}}{\rule{3em}{0.4pt}} & (1,3) \\
R'_1 & \Big|\, c_{1,1} & & \Big|\, c_{1,2} & & \Big|\, c_{1,3} \\
R_2 & (2,1) & \overset{r_{2,1}}{\rule{3em}{0.4pt}} & (2,2) & \overset{r_{2,2}}{\rule{3em}{0.4pt}} & (2,3) \\
R'_2 & \Big|\, c_{2,1} & & \Big|\, c_{2,2} & & \Big|\, c_{2,3} \\
R_3 & (3,1) & \overset{r_{3,1}}{\rule{3em}{0.4pt}} & (3,2) & \overset{r_{3,2}}{\rule{3em}{0.4pt}} & (3,3)
\end{array}$$

Having introduced the notation we are ready to prove the upper bound in Theorems 1 and 2.

Lemma 4. *If F is a graph of order m and size l, then*

$$B'(F \oplus P_n) \leqslant l + m.$$

Proof. Informally speaking, we label columns and quasi-columns from left to right. Here is a formal description. Order the edge set of F arbitrarily: $E(F) = \{D_1, \ldots, D_l\}$. A label $(j-1)(l+m)+i \in [nl + (n-1)m]$ with $j \in [n]$ is assigned to $c_{D_i,j}$ if $i \in [l]$ and to $r_{i-l,j}$ if $l < i \leqslant l+m$. It is easy to see that for $n \geqslant 3$, the largest difference between adjacent labels is $m + l$; it is achieved for pairs of adjacent horizontal edges. $\square$

The *support* of a set $S \subset E(G)$ is $V(S) = \bigcup_{D \in S} D$. (Thus, for example, for $F \oplus P_n$ we have $V(R_i) = \{(i, j) : j \in [n]\}$.) Two subsets of $E(G)$ *touch* if their supports intersect.

The *complement* of a given set S of edges of G is $\overline{S} = E(G) \setminus S$. For an edge $D \in \overline{S}$, the *distance* of D from S is the order of the shortest path in G joining a vertex of D to a vertex of $V(S)$. (For example, if $D \cap V(S) \neq \emptyset$, then their distance is 1.) The tth *neighborhood* $\sigma^t(S)$ of S consists of those edges in $\overline{S}$ that are at distance at most t from S. Note that $\sigma^t(S) \cap S = \emptyset$. For $t = 1$, we simply say the *neighborhood* and write $\sigma(S)$.

The following easy observation is a very useful tool for proving lower bounds on edge-bandwidth; see Harper [14] for the vertex-bandwidth version.

Lemma 5. *For any edge labeling η of G, any $1 \leqslant j < e(G)$, and any $t \geqslant 1$, we have*

$$B'(\eta) \geqslant \frac{\max(|\sigma^t(S)|, |\sigma^t(\overline{S})|)}{t} \quad \text{where } S = \eta^{-1}([j]). \tag{5}$$

Proof. The edge D_1 in $\sigma^t(S)$ with the largest label, which is at least $j + |\sigma^t(S)|$, can be connected by a path P with at most t vertices to some edge D_2 in S, which has label at most j. Consider now the path P' obtained from P by adding D_1 at the beginning and D_2 at the end. At some vertex v of P, the labels on the two edges of P' that are adjacent to v differ by at least $|\sigma^t(S)|/t$, as required. The bound given by $\sigma^t(\overline{S})$ is proved similarly. □

3. Proof of the lower bound in Theorem 1

Our argument has to consider two very similar cases where rows and columns play different roles. To make the proof shorter, we will deal with them in one go. Namely, let $\{F, P_n\} = \{F_1, F_2\}$ where we do not specify which is which. For $i = 1, 2$, let $v_i = v(F_i)$ and $e_i = e(F_i)$. (Thus, for example, $\{v_1, v_2\} = \{m, n\}$.)

Take any edge labeling η of $G = F_1 \oplus F_2$ that achieves the edge-bandwidth. Let s be the smallest number such that $\eta^{-1}([s+1])$ contains two lines as subsets. Let $S = \eta^{-1}([s])$. Note that S contains precisely one line. We can assume without loss of generality that S contains R_p for some $p \in [v_1]$.

Let

$$K = \{i \in [v_1] : V(S) \cap V(R_i) \neq \emptyset\}$$

consist of all (indexes of) rows that touch S. Let $k = |K|$.

Suppose first that $k = v_1$. Then the neighborhood $\sigma(S)$ contains at least v_2 vertical edges: for each $j \in [v_2]$, we have $C_j \setminus S \neq \emptyset$ while C_j touches $R_p \subset S$. Also, for each $i \in [v_1] \setminus \{p\}$, $R_i \setminus S \neq \emptyset$ but R_i and S touch because $K = [v_1]$. This shows that $\sigma(S)$ has at least $v_1 - 1$ horizontal edges. By Lemma 5,

$$B'(G) \geqslant |\sigma(S)| \geqslant v_1 + v_2 - 1 = m + n - 1,$$

which is even strictly greater than the desired bound.

So assume that $k < v_1$. Let $Y = \sigma^{v_1 - k}(S)$ and $Y' = Y \setminus \sigma(S)$. To estimate $|Y|$, we break Y into three disjoint sets

$$Y = \left(\bigcup_{j \in [v_2]} (Y \cap C_j) \right) \cup \left(\bigcup_{D \in E(F_2)} (Y' \cap C'_D) \right) \cup \left(\bigcup_{i \in [v_1]} (\sigma(S) \cap R_i) \right),$$

and estimate the cardinality of each of them.

First, for any $j \in [v_2]$ at least $v_1 - k$ vertices of $V(C_j)$ do not belong to $V(S)$, which implies that $|C_j \setminus S| \geqslant v_1 - k$. As C_j is a connected graph (it is isomorphic to F_1), we have

$$|Y \cap C_j| \geqslant v_1 - k. \tag{6}$$

Consequently,

$$\left| \bigcup_{j \in [v_2]} (Y \cap C_j) \right| \geqslant (v_1 - k) v_2.$$

Our estimate of the second part is given by the following lemma.

Lemma 6. *We have*

$$|Y' \cap C'_D| \geqslant v_1 - k - 1 \tag{7}$$

for every $D \in E(F_2)$.

Proof. Let $D \in E(F_2)$. Note that if $\{i, j\} \in E(F_1)$ is such that $r_{i,D}$ and $r_{j,D}$ are not in S, then the distances from $V(S)$ of $r_{i,D}$ and $r_{j,D}$ differ by at most one. Since $k < v_1$ and F_1 is connected, there is $i \in V(F_1)$ such that the distance of $r_{i,D}$ from $V(S)$ is 2. It follows, because of the connectivity of F_1, that the set of distances of $r_{i,D}$ from $V(S)$ as i runs through $V(F_1)$ under the condition that $r_{i,D} \notin S \cup \sigma(S)$ consists of consecutive integers from 2 up to some integer. Since C'_D, has at least $v_1 - k$ elements that do not touch S, it follows that

$$|Y' \cap C'_D| \geqslant v_1 - k - 1. \qquad \square$$

Finally, since F_2 is connected, we have $\sigma(S) \cap R_i \neq \emptyset$ for each $i \in K \backslash \{p\}$ which implies that

$$\left| \bigcup_{i \in [v_1]} (\sigma(S) \cap R_i) \right| \geqslant k - 1. \tag{8}$$

Adding all these estimates together, we obtain

$$|Y| \geqslant (v_1 - k)v_2 + (v_1 - k - 1)e_2 + k - 1. \tag{9}$$

Let y denote the right-hand side of (9). If $F_1 = P_n$, then we obtain after routine calculations that

$$y = (n - k)(m + l - 1) + n - l - 1 \geqslant (n - k)(m + l - 1) + 1,$$

which implies the required bound by Lemma 5. (Recall that $n \geqslant l + 2$ by the assumption of Theorem 1.) In the case $F_1 = F$ we obtain $y = (m - k)(2n - 2) + m - n$. Using the facts that $m - k = v_1 - k \geqslant 1$ and $m \leqslant l + 1 \leqslant n - 1$, we obtain the desired bound:

$$y \geqslant (m - k)(n + m - 3) + 1 \geqslant (m - k)(l + m - 1) + 1.$$

This finishes the proof of Theorem 1 by Lemma 5.

4. Proof of the lower bound in Theorem 2

Let $G = P_n \oplus P_n$. Let us apply the proof of the lower bound of Theorem 1 to G using the same notation. (Thus, $v_1 = v_2 = n$, $e_1 = e_2 = n - 1$, etc.) Observe that in Section 3 we use the restriction $n \geqslant l + 2$ only after (9). Hence, the inequality (9) applies also to G, giving $|Y| \geqslant (n - k)(2n - 2)$. If this inequality is strict, then we immediately obtain the claimed lower bound by Lemma 5. So, let us suppose on the contrary that Theorem 2 is not true. It follows that

$$B'(P_n \oplus P_n) = 2n - 2, \tag{10}$$

and that (9) and the inequalities which led to it are all equalities. Also we have $k < n$. The overall plan is to get as much structural information about S as possible so that we can derive the final contradiction.

Lemma 7. *For every line L we have* $|\sigma(S) \cap L| \leqslant 1$.

Proof. If L is a row, then the claim follows from the fact that we have an equality in (8). So suppose that some column $L = C_j$ violates the lemma, that is, $|\sigma(S) \cap C_j| \geqslant 2$. As (6) is an equality, we conclude that $C_j \backslash S$ has at most (and hence precisely) $n - k$ edges. It follows that $k \leqslant n - 2$ and that $C_j \backslash S \subset \sigma^{n-k-1}(S)$. Consequently, $C'_D \backslash S \subset \sigma^{n-k}(S)$ for any edge D of P_n containing j. This makes (7) strict, a contradiction. $\square$

Let us call a line L *compressed* if $V(S) \cap V(L)$ is either empty or spans a connected subgraph (that is, a path) that contains at least one endpoint of L. The following claim is an obvious corollary of Lemma 7.

Lemma 8. *Every line is compressed.*

We know that $V(R_p)$ intersects every set $V(S) \cap V(C_i)$, $i \in [n]$. As $k < n$, there is a row disjoint from every such set. As each column is compressed by Lemma 8, we conclude that the intersections of $V(S)$ with the columns, if projected onto the first coordinate, form a nested family. Furthermore, since each row is compressed, we can choose one of the two canonical ways to label the vertex set of each factor P_n by $[n]$, so that for any $i_1, i_2, j_1, j_2 \in [n]$ we have

$$i_1 \leqslant i_2, \quad j_1 \leqslant j_2, \quad (i_2, j_2) \in V(S) \Rightarrow (i_1, j_1) \in V(S). \tag{11}$$

Let us assume that this *monotonicity* property (11) holds. In particular, since $k < n$, we have $n \notin K$.

We say that a line L is *full* if $V(L) \subset V(S)$. As $n \notin K$, no column is full but we may have a few full rows. A line L is *filled* (resp. *almost filled*) if no edge (resp. exactly one edge) of $L \backslash S$ has both endpoints in $V(S)$. Intuitively, a filled line has as many edges in S as possible given the set $V(S)$.

Lemma 9. *Every line that is not full is filled. All full rows are filled or almost filled.*

Proof. Suppose that a line L is not filled, that is, there is an edge $\{x, y\} \in \overline{S}$ with $x, y \in V(L) \cap V(S)$. If $V(L) \backslash V(S) \neq \emptyset$, the set $\sigma(S) \cap L$ contains the edge $\{x, y\}$ and at least one more edge. This contradicts Lemma 7. Thus the line L is full, proving the first part.

For any full row L we have $\sigma(S) \cap L = L \backslash S$, implying that the latter set has at most one element, again by Lemma 7. $\square$

Recall the notation that applies to $G = P_n \oplus P_n$:

$$r_{i,j} = r_{i,\{j,j+1\}}, \quad c_{i,j} = c_{\{i,i+1\},j}, \quad R'_i = R'_{\{i,i+1\}}, \quad C'_j = C'_{\{j,j+1\}}.$$

The following claims are proved by analyzing

$$Z = \sigma(\overline{S}),$$

the first neighborhood of the *complement* of S, so it is convenient to put them into a single lemma.

Lemma 10. *We have $p = 1$. There is at most one almost filled row; moreover, if such a row exists, then it is R_2.*

Proof. Assume that there is at least one almost filled row. (Otherwise we are done: R_p is the only full row and, by (11), $p = 1$.)

By Lemma 9 every almost filled row is full. Let $f \geqslant p$ be the largest index such that R_f is full. (It is not excluded so far that $f = p$.) We have $f \geqslant 2$ and, by (11), all rows R_i with $i \in [f]$ are full.

By Lemma 5 and the assumption (10), we have $|Z| \leqslant 2n - 2$. Observe that for every $j \in [n]$, we have $c_{n-1,j} \in \overline{S}$ (because $n \notin K$) and $c_{1,j} \in S$ (because R_1 and R_2 are full while the column C_j is filled by Lemma 9). Hence, $Z \cap C_j \neq \emptyset$ and, in total, Z contains at least n vertical edges.

Take any edge $\{x, y\} \in E(P_n)$ such that $r_{f,xy} \in S$. Choose the largest $i \geqslant f$ such that $r_{i,xy} \in S$. As $n \notin K$, we have $i < n$. Since R_{i+1} is not full, it is filled by Lemma 9. The edge $r_{i+1,xy}$ is not in S, so at least one of its endpoints is not in $V(S)$; let it be $(i + 1, x)$. This means that $c_{i,x} \in \overline{S}$ and $r_{i,xy} \in Z$. By Lemma 9 we have at least $n - 2$ choices for xy, so Z has at least $n - 2$ horizontal edges in rows $R_f, \ldots, R_{n-1}$.

This already gives us that $|Z| \geqslant 2n - 2$. Any row R_i with $i \in [f] \backslash \{p\}$ has precisely one missing edge by Lemma 9. So, in order to prevent extra horizontal edges in Z, we have to assume that $f = 2$ and $p = 1$, as required. $\square$

For $i = 1, 2, 3$, let $D_i = \eta^{-1}(s + i)$, $S_i = \eta^{-1}([s + i])$, $Y_i = \sigma(S_i)$, and $Z_i = \sigma(\overline{S_i})$. Let

$$\delta = \begin{cases} 1 & \text{if } (2, n) \in V(S), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\delta = 0$ if and only if R_1 is the only full row.

Lemma 11. *The edge D_1 is vertical.*

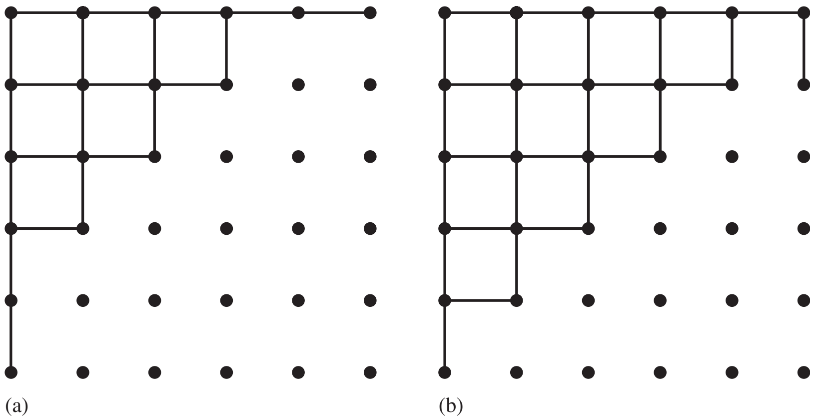

Fig. 1. The structure of S_1 given by Lemma 13.

Proof. Suppose on the contrary that D_1 is horizontal. Let it lie in the ith row. By the definition of S, we have $R_i \subset S_1$. The argument of Lemma 10 shows that Z_1 has at least $n-1+\delta$ vertical edges (at least one edge per each column C_i except C_n if $\delta = 0$).

Observe that there are no almost filled rows among $R_{i+1}, \ldots, R_n$. (Indeed, we have $i \geqslant 2$ so the existence of such a row contradicts Lemma 10.) Now, the argument of Lemma 10 shows that Z_1 contains at least one edge from each quasi-column. Furthermore, if $\delta = 0$, then the edges $r_{1,n-1}, r_{i,n-1} \in S_1$, coming from the same quasi-column C'_{n-1}, are both in Z. (Indeed, $(2,n) \notin V(S)$; so $(i,n) \notin V(S)$ by (11); as D_1 is horizontal, we have $c_{1,n}, c_{i,n} \in \overline{S_1}$.) Thus, we have exhibited at least $n-\delta$ horizontal edges in Z_1. This gives us the desired contradiction $|Z_1| \geqslant 2n-1$. $\square$

Lemma 12. *The edge D_1 belongs to C_1; thus $D_1 = c_{n-1,1}$.*

Proof. If $D_1 = c_{n-1,n}$, then (11) and Lemmas 9 and 10 imply that $n = 3$ and, furthermore, $S = \{r_{1,1}, r_{1,2}, D, c_{1,1}, c_{1,2}, c_{1,3}\}$, where D is either $r_{2,1}$ or $r_{2,2}$. If $D = r_{2,1}$, then $|Z_1| = 5$, a contradiction. If $D = r_{2,2}$, then $r_{n,n-1}$ is only choice for $D_2 = \eta^{-1}(s+2)$ that avoids the contradiction $|Z_2| = 5$. But then we obtain a contradiction in the next step: $|Z_3| = 5$ for any D_3.

So assume that $D_1 \notin C_n$. The set $Y_1 = \sigma(S_1)$ contains at least $n-1$ vertical edges and at least one edge from each of $R_2, \ldots, R_n$. If $D_1 \notin C_1$, then Y_1 has at least two edges from R_n, giving the desired contradiction $|Y_1| \geqslant 2n-1$. $\square$

Now we are able to show that S_1 must have a very restrictive structure. (The reader may refer to Fig. 1 for an illustration.) Let Σ_q consist of all edges of G spanned by $\{(i,j) \in V(G) : i+j \leqslant q\}$.

Lemma 13. *If $\delta = 0$, then $S_1 = R_1 \cup C_1 \cup \Sigma_q$ for some $3 \leqslant q \leqslant n+1$. If $\delta = 1$, then $S_1 = R_1 \cup C_1 \cup \Sigma_{n+1} \cup \{c_{1,n}\}$.*

Proof. Suppose first that $\delta = 0$. All columns and rows are filled with respect to S_1. The argument of Lemma 10 shows that Z_1 contains at least one edge from each quasi-line. Since this already gives at least $2n-2$ edges, no quasi-line can have two common edges with Z_1. It follows that for any $i \geqslant 2$ with $S_1 \cap R_i \neq \emptyset$ we have $|S_1 \cap R_{i+1}| \geqslant |S_1 \cap R_i| - 1$: otherwise $|R'_{i-1} \cap Z_1| \geqslant 2$. The analogous claim holds for the sizes of $S_1 \cap C_j$. A moment's thought reveals that S_1 has the required structure.

Let $\delta = 1$. Here, R_2 is the unique almost filled row. Let $r_{2,f}$ be the unique edge of $R_2 \backslash S_1$. Then Z_1 has a non-empty intersection with each of

$$C'_1, \ldots, C'_{n-1} \text{ (except possibly } C'_f) \quad \text{and} \quad R'_2, \ldots, R'_{n-1},$$

while $|Z_1 \cap R'_1| \geqslant 2$. This already gives us that $|Z_1| \geqslant 2n-2$. If follows that $f = n-1$ for otherwise $|Z_1 \cap R'_1| \geqslant 3$. Also, we must have $S_1 \cap C_{n-1} = c_{1,n-1}$ for otherwise we would have $|Z_1 \cap C'_{n-2}| \geqslant 2$, a contradiction. Working inductively on $j = n-2, n-3, \ldots, 1$ one argues that

$$S_1 \cap C_j = \{c_{i,j} : i = 1, \ldots, n-j\},$$

which implies the claim. □

Given so much information about S_1, we can directly analyze the next few values of η.

Suppose first that $\delta = 1$. Routine considerations show that we have $D_2 = r_{n,1}$ for otherwise $Z_2 = Z_1 \cup \{D_2\}$ and we obtain the contradiction $|Z_2| \geqslant 2n-1$. But any edge in Z_2 touches at least two edges of $\overline{S}_2$. So, as it is easy to see, we have $Z_3 = Z_2 \cup \{D_3\}$, a contradiction.

Suppose that $\delta = 0$. If $q = n+1$, then to prevent $Z_2 = Z_3 \cup \{D_2\}$ we should let D_2 equal $c_{1,n}$ or $r_{n,1}$. But either of these choices gives us a situation isomorphic to the one for $\delta = 1$, which leads to a contradiction anyway. Finally, if $q \leqslant n$, then we get a contradiction already for S_2. Indeed, if D_2 is $c_{1,q-1}$ or $r_{q-1,1}$, then $|Y_2| = 2n-1$; otherwise $|Z_2| = 2n-1$.

This completes the proof of Theorem 2.

5. Proof of Theorem 3

Since the arguments here are very similar to those in the proof of Theorem 1, we will be rather brief.

The upper bound on $B'(C_m \oplus C_n)$, for $m \geqslant n$, follows by labeling rows and quasi-rows one by one, moving in both directions along the cycle C_m. Namely, the order of rows and quasi-rows is the following:

$$R_1, R'_1, R'_m, R_2, R_m, R'_2, R'_{m-1}, R_3, R_{m-1}, \ldots,$$

while each individual (quasi-)row is labeled in the same fixed cyclic order on C_n. It is easy to see that the bandwidth of this labeling is $4n$.

On the other hand, let $m, n \geqslant 3$ be arbitrary. (We do not specify their relative order.) Take an edge-labeling η of $C_m \oplus C_n$ that achieves the edge-bandwidth. Let s be the smallest integer such that $S = \eta^{-1}([s])$ contains a whole line minus one edge. Assume without loss of generality that this is a row R_p, that is, $|R_p \backslash S| = 1$. Let K consist of those $i \in [m]$ such that R_i and S touch, and let $k = |K|$. Let $l = \lceil (m-k)/2 \rceil$ and $Y = \sigma^l(S)$.

If $k = m$, then $\sigma(S)$ contains at least two edges from each column and at least two edges from every row R_i except the row R_p, which contributes only one edge. Here $B'(\eta) \geqslant 2m + 2n - 1$, giving the required.

So suppose that $k < m$, that is, $l \geqslant 1$. For each $i \in [n]$ we have $|C_i \backslash S| \geqslant m - k + 1 \geqslant 2l$. For any proper edge-subset of a cycle, its first neighborhood has at least 2 elements or catches all remaining edges. Hence, $|Y \cap C_i| \geqslant 2l$. As each C'_j has at least $m - k > 2l - 2$ elements that do not touch S, we conclude that

$$|Y \cap C'_j| \geqslant 2l - 2 + \delta_j,$$

where $\delta_j = |\sigma(S) \cap C'_j \cap (\bigcup_{i \in K} R_i)|$. We have $\sum_{j=1}^{n} \delta_j \geqslant 2k - 1$. Indeed, the definition of S implies that for each row R_i with $i \in K \backslash \{p\}$ we have $|R_i \backslash S| \geqslant 2$ and thus $|\sigma(S) \cap R_i| \geqslant 2$; also $|\sigma(S) \cap R_p| = 1$.

We obtain

$$|Y| \geqslant 2ln + (2l-2)n + 2k - 1 \geqslant 2ln + (2l-2)n + 2(m-2l) - 1 =: y. \tag{12}$$

If $m \geqslant n$, then $y = l(4n-6) + 2m - 2n + 2l - 1 \geqslant l(4n-6) + 1$, which implies the required lower bound by Lemma 5. If $m < n$, then we obtain the desired bound on $|Y|$ as follows:

$$y = l(4n-6) + 2m - 2n + 2l - 1 \geqslant l(4m-6) + 4(n-m) + 2m - 2n + 2l - 1 \geqslant l(4m-6) + 1.$$

Theorem 3 is proved.

Remark. From (12) one can also deduce that

$$B'(C_m \oplus C_n) = 4\min(m,n) \quad \text{if } \max(m,n) \geqslant 4\min(m,n) + 4. \tag{13}$$

Indeed, if $m \geqslant n$, then we obtain (using $l = \lceil (m-k)/2 \rceil \leqslant m/2$) that

$$y = l(4n-1) + 2m - 2n - 3l - 1 \geqslant l(4n-1) + 2m - 2n - 3\frac{m}{2} - 1 \geqslant l(4n-1) + 1.$$

If $m < n$, then

$$y = l(4n-1) + 2m - 2n - 3l - 1 \geqslant l(4m-1) + 2n - 2m - 3\frac{m}{2} - 1 \geqslant l(4m-1) + 1.$$

Now, (13) follows from Lemma 5. Also, small improvements on (4) could be obtained for some other ranges of (m, n) but we do not think that this direction is worth pursuing.

6. Open problems

It would be of interest to compute the exact value of the edge-bandwidth for three-dimensional grids. Our Theorem 1, when applied to $F = P_l \oplus P_m$, gives

$$B'(P_l \oplus P_m \oplus P_n) = 3lm - m - l \quad \text{if } n \geqslant 2lm - l - m + 2.$$

However, the general case is still unsolved. Another open problem is to close the gap in Theorem 3.

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