MINIMAL EQUITABILITY OF HAIRY CYCLES

JERZY WOJCIECHOWSKI

Abstract. Every labeling of the vertices of a graph with distinct natural numbers induces a natural labeling of its edges: the label of an edge (x, y) is the absolute value of the difference of the labels of x and y. By analogy with graceful labelings, we say that a labeling of the vertices of a graph of order n is minimally k-equitable if the vertices are labelled with $1, 2, \ldots, n$ and in the induced labeling of its edges every label either occurs exactly k times or does not occur at all. For $m \geq 3$, let C'_m (denoted also in the literature by $C_m \circ K_1$ and called a corona graph) be a graph with $2m$ vertices such that there is a partition of them into sets U and V of cardinality m, with the property that U spans a cycle, V is independent and the edges joining U to V form a matching. Let $\mathcal{P}$ be the set of all pairs (m, k) of positive integers such that k is a proper divisor of $2m$ (*i.e.*, a divisor different from $2m$ and 1) and k is odd if m is odd. We show that C'_m is minimally k-equitable if and only if $(m, k) \in \mathcal{P}$.

1. Introduction

A *labeling* of a graph G is an assignment of distinct natural numbers to the vertices of G. Every labeling induces a natural labeling of the edges: the label of an edge (x, y) is the absolute value of the difference of the labels of x and y. Bloom [4] defined a labeling of a graph to be *k-equitable* if in the induced labeling of its edges, every label occurs exactly k times, if at all. Furthermore, a k-equitable labeling of a graph of order n is said to be *minimal* if the vertices are labelled with $1, 2, \ldots, n$. A graph is *minimally k-equitable* if it has a minimal k-equitable labeling.

The notion of minimally k-equitable labelings is a generalization of special labelings of trees called graceful. A labeling of a tree is *graceful* if and only if it is minimally 1-equitable. (Note that the induced labels of the edges must be then $1, 2, \ldots, n-1$.) Graceful labelings were defined by Rosa [6] in connection with a famous and very difficult conjecture of Ringel and Kotzig (see Ringel [5]) concerning decompositions of complete graphs with odd number of vertices into subgraphs isomorphic to trees. The open conjecture that every tree has a graceful labeling implies the conjecture of Ringel and Kotzig.

The notion of minimal k-equitability was first applied to cycles. Let C_n be the cycle on n vertices. Given natural numbers n and k, $n \geq 3$, it is easy to see that if the cycle C_n is minimally k-equitable, then k is a proper divisor of n (that is, k divides n, $k \neq n$, and $k \neq 1$). Answering a question posed by Bloom [4], Wojciechowski [7] proved that this necessary condition is also sufficient.

Barrientos, Dejter and Hevia [3] proved a number of results concerning k-equitability of forests.

Another class of graphs to which the concept of minimal k-equitability was recently applied is the class of graphs whose vertices could be partitioned into two sets U and V such that U induces a cycle, V induces no edges, and the edges between U and V form a matching (in particular, U and V are of the same cardinality). We will call such graphs *hairy cycles* and denote them by C'_m (m is the cardinality of U).

There is a general construction that, given graphs G and H, produces the *corona* graph $G \circ H$. The hairy cycle C'_m is obtained by applying that construction to the cycle C_m and the graph K_1 consisting of a single vertex. Therefore C'_m is the corona graph $C_m \circ K_1$.

Again it is easy to see that given natural numbers m and k, $m \geq 3$, if the hairy cycle C'_m is minimally k-equitable, then k is a proper divisor of $2m$ which is the number of vertices of C'_m. Going a little further, we get the following result.

Proposition 1.1. *Let $m \geq 3$ and $k \geq 1$ be integers. If m is odd and the hairy cycle C'_m is minimally k-equitable, then k is also odd.*

Proof. Assume that $m \geq 3$ is arbitrary and the hairy cycle C'_m is minimally k-equitable. Let U and V be the sets that form a partition of the vertex set of C'_m such that U spans a cycle, V spans no edges, and the edges between U and V form a matching. Let E be the set of edges of the cycle spanned by U and F be the set of the edges of the matching.. Let

$$g : U \cup V \to \{1, 2, \dots, 2m\}$$

be any labeling of C'_m, and $h : E \cup F \to \mathbb{N}$ be the induced labeling of the edges of C'_m. It is clear that $\sum_{e\in E} h(e)$ is even.

We claim that

$$\sum_{e\in F} h(e) \equiv m \bmod 2. \tag{1}$$

If $h(e) = 1$ for every $e \in F$, then $\sum_{e\in F} h(e) = m$ so (1) holds. Otherwise, there are vertices $u \in U$ and $v \in V$ such that $|g(u) - g(v)| \geq 2$. Let w, y be vertices of C'_m such that $g(w)$ is between $g(u)$ and $g(v)$ in the standard ordering of integers, and y is joined to w by an edge of F. Let $g' : U \cup V \to \mathbb{N}$ be the labeling of C'_mobtained from g by exchanging $\min\{g(w), g(y)\}$ with $\max\{g(u), g(v)\}$. For example, if

$$g(u) < g(w) < g(v) < g(y),$$

then we exchange the labels of w and v.

Let $h' : E \cup F \to \mathbb{N}$ be the labeling of the edges of C'_m induced by g'. Then

$$\sum_{e\in F} h'(e) < \sum_{e\in F} h(e)$$

and

$$\sum_{e\in F} h'(e) \equiv \sum_{e\in F} h(e) \bmod 2.$$

Therefore, induction on $\sum_{e\in F} h(e)$ can be used to prove that (1) holds in general.

If m is odd and g is a minimal k-equitable labeling, then $\sum_{e\in E\cup F} h(e)$ is odd and divisible by k implying that k is odd. ∎

Let $\mathcal{P}$ be the set of all pairs (m, k) of positive integers such that $m \geq 3$, k is a proper divisor of $2m$ and k is odd if m is odd. Then for the graph C'_m to be minimally k-equitable it is necessary that $(m, k) \in \mathcal{P}$. Acharya and Bhat-Nayak [1] [2] proved that the condition $(m, k) \in \mathcal{P}$ is sufficient for C'_m to be minimally k-equitable when $k \in \{3, 4\}$. We will prove that this condition is sufficient in general.

Theorem 1.2. *Let $m \geq 3$ and $k \geq 2$ be integers. If $(m, k) \in \mathcal{P}$, then the graph C'_m is minimally k-equitable.*

We shall call a graph G an *integer graph* if its vertex set is a finite subset of $\mathbb{N}$. If $e = (u, v)$ is an edge of G, then we will say that e has *length* $|u - v|$. Observe that a finite graph H has a k-equitable labeling if and only if it is isomorphic to an integer graph G with either 0 or k edges of any length. We will call such G a *k-equitable representation* of the graph H. Note also that a finite graph H is minimally k-equitable if it has a k-equitable representation G whose vertices are consecutive integers. We will call such G a *minimal k-equitable representation* of H.

In the following proofs, to show that C'_m is k-equitable, we will construct an integer graph G^k_m that will be a minimal k-equitable representation of C'_m. In the included figures of the graphs G^k_m the vertices will be placed on the real line, the edges that are part of the cycle (*cycle edges*) will be marked by thick lines below the real line and the edges of the matching (*matching edges*) above the real line.

The proof of Theorem 1.2 will be split into several lemmas. The proofs of these lemmas will be given in the remaining sections. Assume that $(m, k) \in \mathcal{P}$. We have then two possibilities: either k divides m or it does not divide m. In section 2 we will present the starting point of our construction in the case when k divides m, namely, we will show that C'_m is minimally k-equitable when $m = k$ or $m = 2k$. The proof that $C'_{\ell k}$ is minimally k-equitable for $\ell \geq 3$ will be given in section 4 for even k and in section 5 for odd k. If k does not divide m, then the definition of $\mathcal{P}$ implies that $k \equiv 0 \bmod 4$ and $m = \ell k/2$ for some odd integer $\ell \geq 3$. Let $k \equiv 0 \bmod 4$. We will prove in section 3 that the graph $C'_{3k/2}$ is minimally k-equitable. The proof that $C'_{\ell k/2}$ is minimally k-equitable for odd $\ell \geq 5$ will be given in section 4.

2. Minimal k-equitability of C'_k and C'_{2k}

Lemma 2.1. *Let $k \geq 3$ be an integer. The graph C'_k is minimally k-equitable.*

Proof. Let G^k_k be the integer graph with the vertex set $\{1, 2, \ldots, 2k\}$ and the edge set consisting of all the edges listed below:

- $(1, 2)$, $(2, 3)$, $\ldots$, $(k, k + 1)$

— the above edges are k edges of length 1, the first of which is a matching edge, and all the others are cycle edges;

- $(2, k+1)$, $(3, k+2)$, $\ldots$, $(k+1, 2k)$

— the above edges are k edges of length $k-1$, the first of which is a cycle edge, and all the others are matching edges.

Figure 1 shows the graph G_3^3 and Figure 2 shows the graph G_k^k for arbitrary $k \geq 3$.

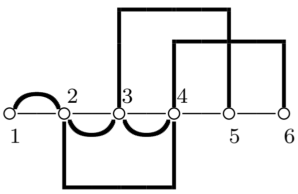

FIGURE 1. The graph G_3^3

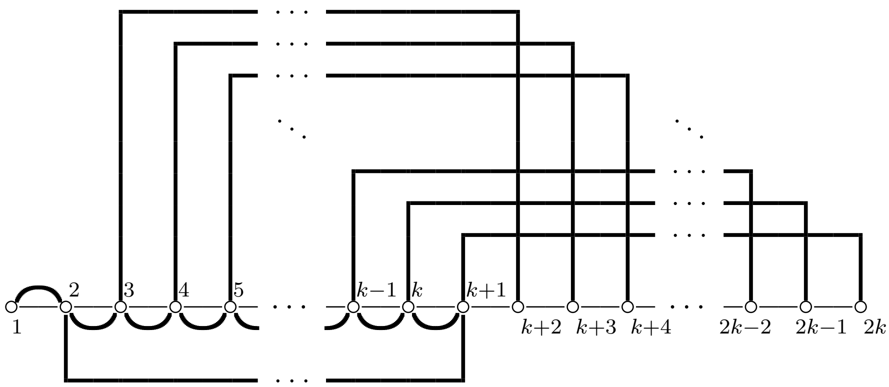

FIGURE 2. The graph G_k^k

It is clear that G_k^k is a minimal k-equitable representation of C'_k implying that C'_k is minimally k-equitable. ∎

Lemma 2.2. *Let $k \geq 3$ be an odd integer. The graph C'_{2k} is minimally k-equitable.*

Proof. Let G_{2k}^k be the integer graph with the vertex set $\{1, 2, \ldots, 4k\}$ and the edge set consisting of all the edges listed below:

- $\left(1, \frac{k+3}{2}\right)$, $\left(2, \frac{k+5}{2}\right)$, $\ldots$, $\left(\frac{k+1}{2}, k+1\right)$

— the edges listed above are $\frac{k+1}{2}$ edges of length $\frac{k+1}{2}$, all of which are matching edges;

- $\left(k+2, \frac{3k+5}{2}\right)$, $\left(k+3, \frac{3k+7}{2}\right)$, …,$\left(\frac{3k+1}{2}, 2k+1\right)$

— the edges listed above are $\frac{k-1}{2}$ edges of length $\frac{k+1}{2}$, the last of which is a cycle edge, and all the others are matching edges;

- $\left(\frac{k+3}{2}, \frac{k+5}{2}\right)$, $\left(\frac{k+7}{2}, \frac{k+9}{2}\right)$, …,$\left(\frac{3k+1}{2}, \frac{3k+3}{2}\right)$

— these are $\frac{k+1}{2}$ edges of length 1, the last of which is a matching edge, and all the others are cycle edges;

- $(2k+2, 2k+3)$, $(2k+4, 2k+5)$, …,$(3k-1, 3k)$

— these are $\frac{k-1}{2}$ edges of length 1, all of which are cycle edges;

- $\left(\frac{k+3}{2}, 2k+1\right)$, $\left(\frac{k+5}{2}, 2k+2\right)$, …,$\left(\frac{3k+1}{2}, 3k\right)$

— these are k edges of length $\frac{3k-1}{2}$, all of which are cycle edges;

- $(2k+1, 3k+1)$, $(2k+2, 3k+2)$, …,$(3k, 4k)$

— these are k edges of length k, all of which are matching edges.

Figures 3, 4, and 5 show the graphs G_6^3, G_{10}^5, and G_{14}^7 respectively; figures 6 and 7 show the general graph G_{2k}^k for arbitrary $k \equiv 1 \bmod 4$, and $k \equiv 3 \bmod 4$ respectively.

It is clear that G_{2k}^k is a minimal k-equitable representation of C'_{2k} implying that C'_{2k} is minimally k-equitable. ∎

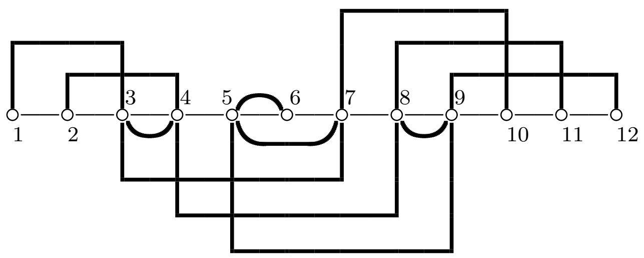

FIGURE 3. The graph G_6^3

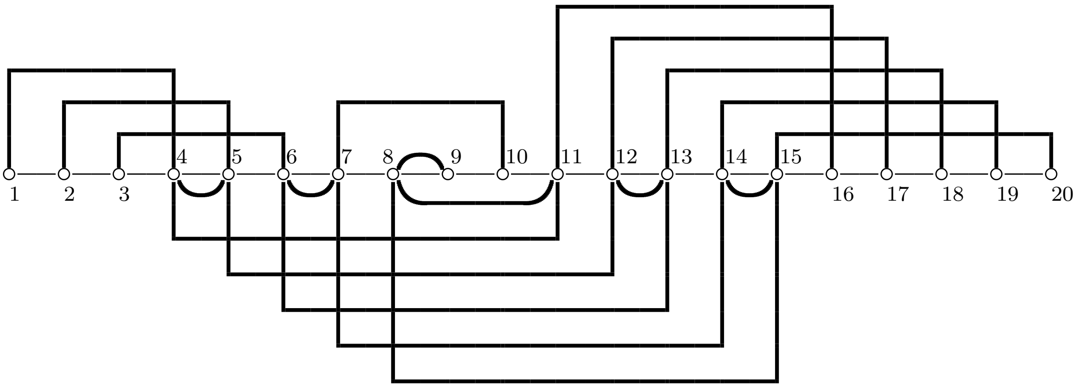

FIGURE 4. The graph G^5_{10}

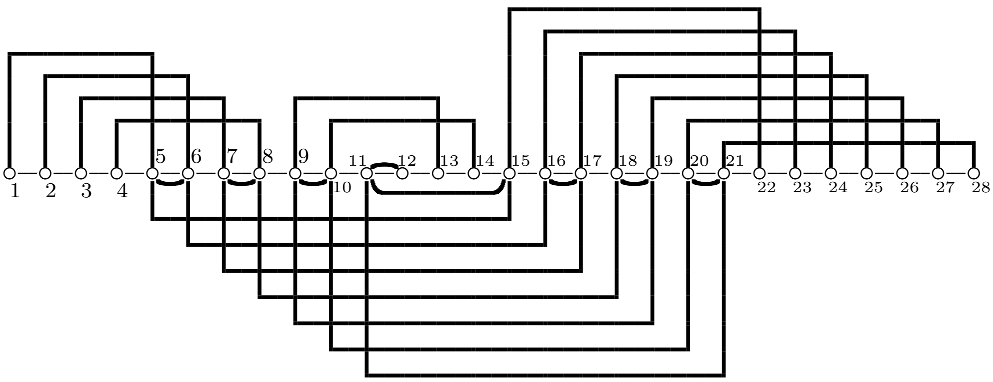

FIGURE 5. The graph G^7_{14}

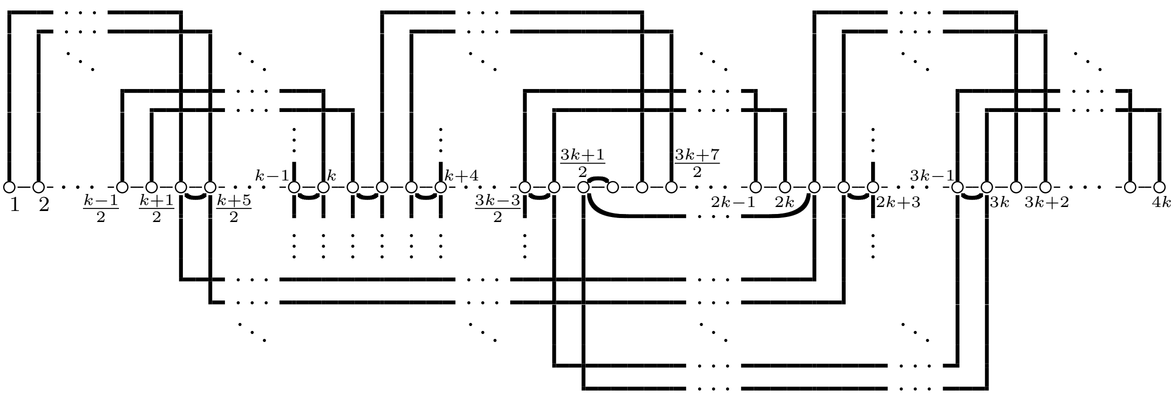

FIGURE 6. The graph G^k_{2k} when $k \equiv 1 \bmod 4$

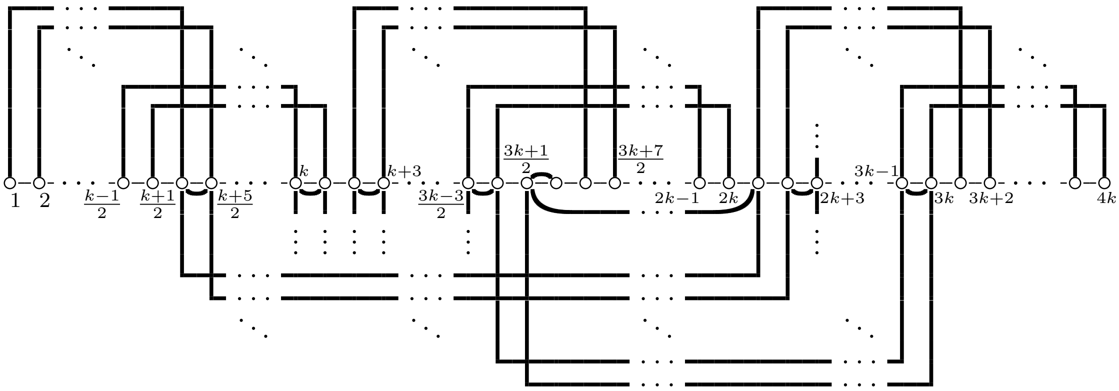

FIGURE 7. The graph G^k_{2k} when $k \equiv 3 \bmod 4$

Lemma 2.3. *Let $k \geq 2$ be an even integer. The graph C'_{2k} is minimally k-equitable.*

Proof. Let G^2_4 be the integer graph with the vertex set $\{1, 2, \dots, 8\}$ and the edge set consisting of the following edges (see Figure 8):

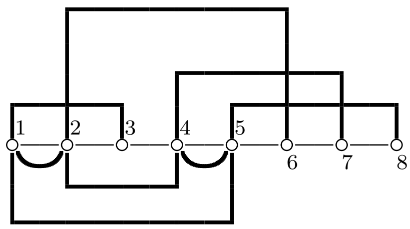

FIGURE 8. The graph G^2_4

- $(1, 2)$, $(4, 5)$ — 2 edges of length 1 — both of them cycle edges;
- $(1, 3)$, $(2, 4)$ — 2 edges of length 2 — the first a matching edge, the second a cycle edge;
- $(4, 7)$, $(5, 8)$ — 2 edges of length 3 — both of which being matching edges;
- $(1, 5)$, $(2, 6)$ — 2 edges of length 4 — the first a cycle edge, the second a matching edge.

For $k \geq 4$ let G^k_{2k} be the integer graph with the vertex set $\{1, 2, \dots, 4k\}$ and the edge set consisting of all the edges listed below:

- $\left(1, \frac{k+2}{2}\right)$, $\left(2, \frac{k+4}{2}\right)$, $\dots$, $\left(\frac{k}{2}, k\right)$

— these are $\frac{k}{2}$ edges of length $\frac{k}{2}$, all of which are matching edges;

- $\left(k+1, \frac{3k+2}{2}\right)$, $\left(k+2, \frac{3k+4}{2}\right)$, $\dots$,$\left(\frac{3k}{2}, 2k\right)$

— these are $\frac{k}{2}$ edges of length $\frac{k}{2}$, the last of which is a cycle edge, and all the others are matching edges;

- $\left(\frac{k+2}{2}, \frac{k+4}{2}\right)$, $\left(\frac{k+6}{2}, \frac{k+8}{2}\right)$, $\dots$,$\left(\frac{3k-2}{2}, \frac{3k}{2}\right)$

— these are $\frac{k}{2}$ edges of length 1, all of which are cycle edges;

- $(2k, 2k+1)$, $(2k+2, 2k+3)$, $\dots$,$(3k-2, 3k-1)$

— these are $\frac{k}{2}$ edges of length 1, all of which are cycle edges;

- $\left(\frac{k+2}{2}, 2k+1\right)$, $\left(\frac{k+4}{2}, 2k+2\right)$, $\dots$,$\left(\frac{3k}{2}, 3k\right)$

— these are k edges of length $\frac{3k}{2}$, the last of which is a matching edge, and all the others are cycle edges;

- $(2k, 3k+1)$, $(2k+1, 3k+2)$, $\dots$,$(3k-1, 4k)$

— these are k edges of length $k+1$, all of which are matching edges.

Figures 9 and 10 show the graphs G_8^4 and G_{12}^6 respectively; Figures 11 and 12 show the general graph G_{2k}^k for arbitrary $k \equiv 0 \bmod 4$, and $k \equiv 2 \bmod 4$ respectively.

It is clear that, for every even $k \geq 2$, the graph G_{2k}^k is a minimal k-equitable representation of C'_{2k}, implying that C'_{2k} is minimally k-equitable. ∎

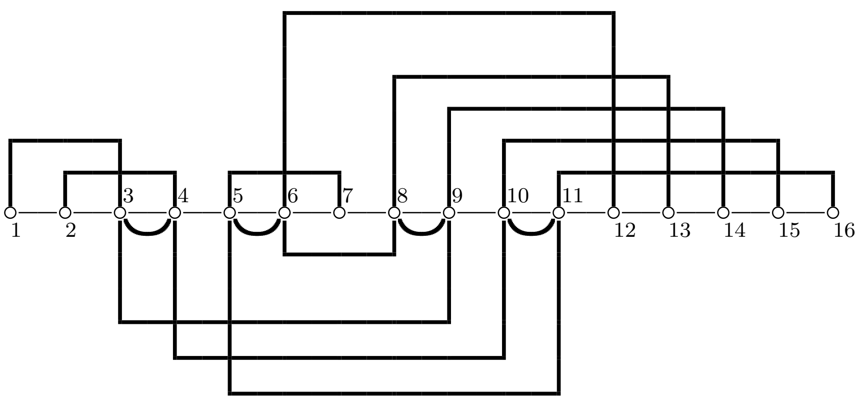

FIGURE 9. The graph G_8^4

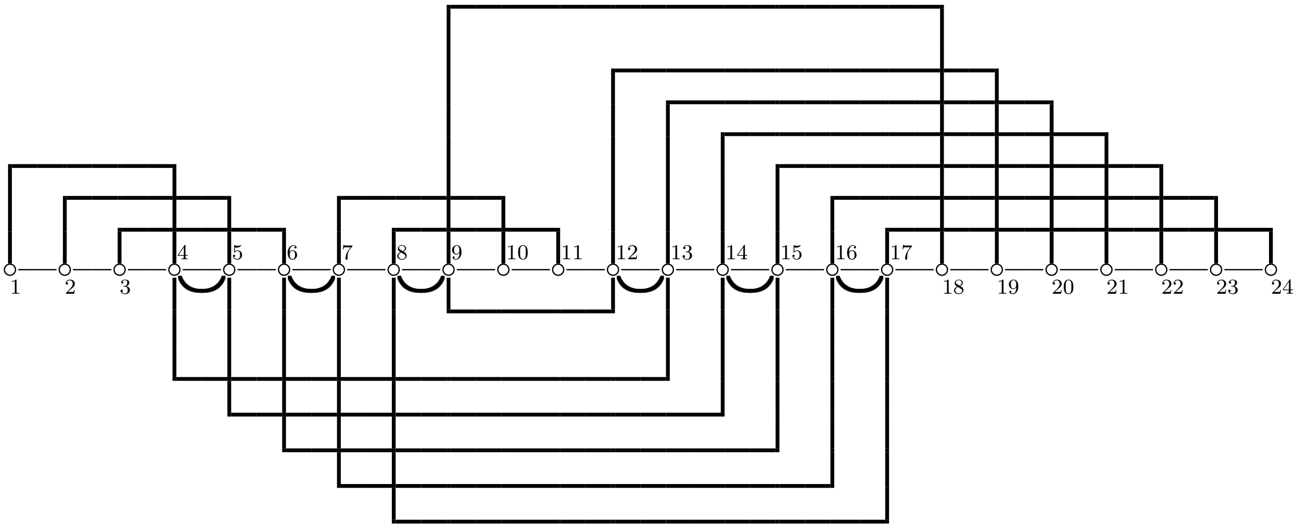

FIGURE 10. The graph G_{12}^{6}

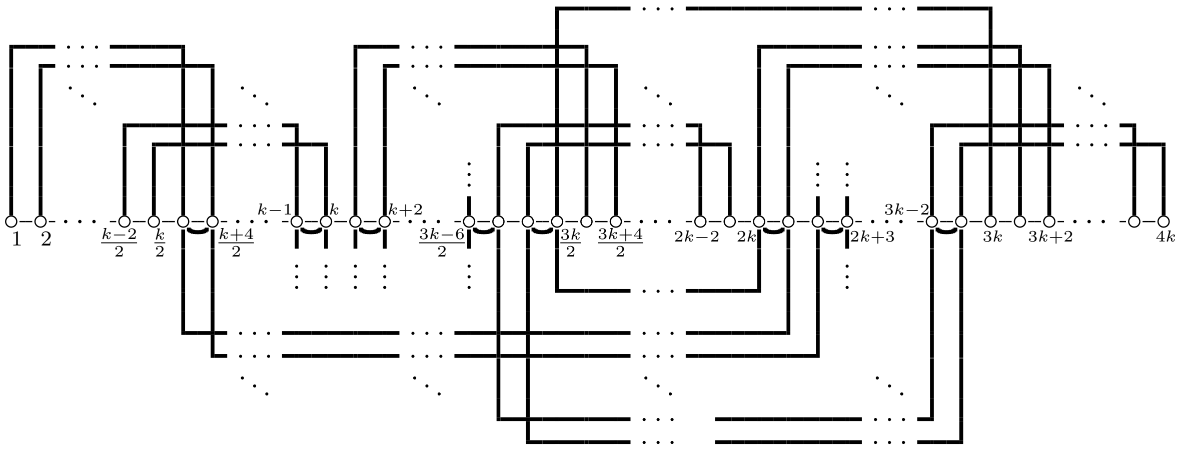

FIGURE 11. The graph G_{2k}^{k} when $k \equiv 0 \bmod 4$

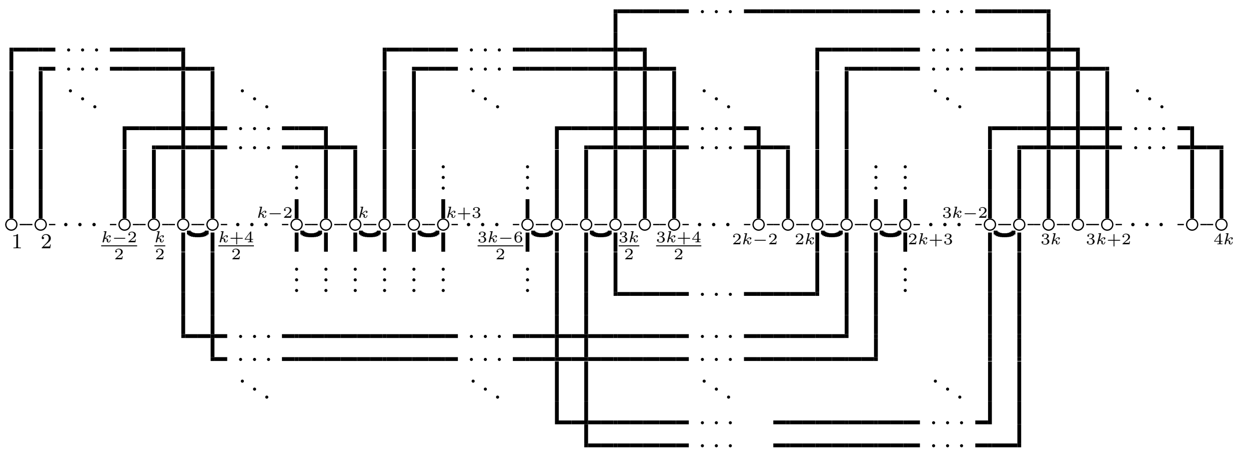

FIGURE 12. The graph G^k_{2k} when $k \equiv 2 \bmod 4$

3. MINIMAL k-EQUITABILITY OF $C'_{3k/2}$ WHEN $k \equiv 0 \bmod 4$

Lemma 3.1. *Let $k \geq 4$ be an integer such that $k \equiv 0 \bmod 4$. The graph $C'_{3k/2}$ is minimally k-equitable.*

Proof. Let $G^k_{3k/2}$ be the integer graph with the vertex set $\{1, 2, \ldots, 3k\}$ and the edge set consisting of all the edges listed below:

- $\big(1, \frac{k+2}{2}\big)$, $\big(2, \frac{k+4}{2}\big)$, $\ldots$, $\big(\frac{k-2}{2}, k-1\big)$

— these are $\frac{k-2}{2}$ edges of length $\frac{k}{2}$, all of which are matching edges;

- $\big(\frac{3k+4}{2}, 2k+2\big)$, $\big(\frac{3k+6}{2}, 2k+3\big)$, $\ldots$, $\big(2k-1, \frac{5k-2}{2}\big)$

— these are $\frac{k-4}{2}$ edges of length $\frac{k}{2}$, all of which are matching edges, (they appear only when $k \geq 8$);

- $\big(k+1, \frac{3k+2}{2}\big)$, $\big(\frac{3k}{2}, 2k\big)$, $\big(2k+1, \frac{5k+2}{2}\big)$

— these are 3 edges of length $\frac{k}{2}$, the first and the third of which are matching edges, and the second is a cycle edge;

- $\big(\frac{k+2}{2}, \frac{k+4}{2}\big)$, $\big(\frac{k+6}{2}, \frac{k+8}{2}\big)$, $\ldots$, $(k-1, k)$

— these are $\frac{k}{4}$ edges of length 1, all of which are cycle edges;

- $(k, k+1)$, $(k+1, k+2)$, $\ldots$, $\big(\frac{3k-2}{2}, \frac{3k}{2}\big)$

— these are $\frac{k}{2}$ edges of length 1, all of which are cycle edges;

- $(2k, 2k+1)$, $(2k+2, 2k+3)$, $\ldots, \left(\frac{5k-4}{2}, \frac{5k-2}{2}\right)$

— these are $\frac{k}{4}$ edges of length 1, all of which are cycle edges;

- $\left(\frac{k}{2}, 2k\right)$, $\left(\frac{k+2}{2}, 2k+1\right)$, $\ldots, \left(k, \frac{5k}{2}\right)$

— these are $\frac{k+2}{2}$ edges of length $\frac{3k}{2}$, the first and the last of which are matching edges, and all the others are cycle edges;

- $\left(k+2, \frac{5k+4}{2}\right)$, $\left(k+3, \frac{5k+6}{2}\right)$, $\ldots, \left(\frac{3k}{2}, 3k\right)$

— these are $\frac{k-2}{2}$ edges of length $\frac{3k}{2}$, all of which are matching edges.

Figures 13, 14 and 15 show the graphs G^4_6, G^8_{12}, and G^{12}_{18} respectively; figure 16 shows the general graph $G^k_{3k/2}$ for arbitrary $k \equiv 0 \bmod 4$.

It is clear that, for every $k \equiv 0 \bmod 4$, the graph $G^k_{3k/2}$ is a minimal k-equitable representation of $C'_{3k/2}$, implying that $C'_{3k/2}$ is minimally k-equitable. ∎

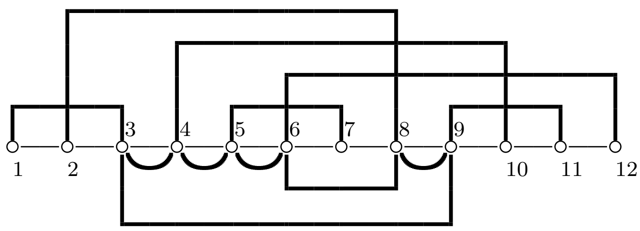

FIGURE 13. The graph G^4_6

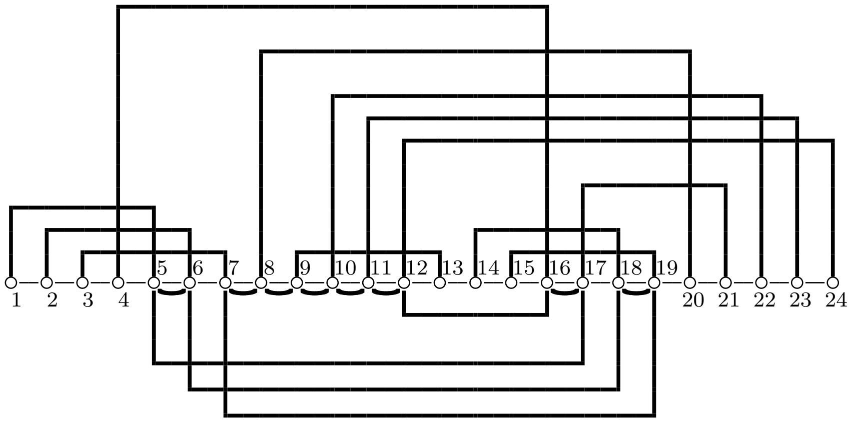

FIGURE 14. The graph G_{12}^{8}

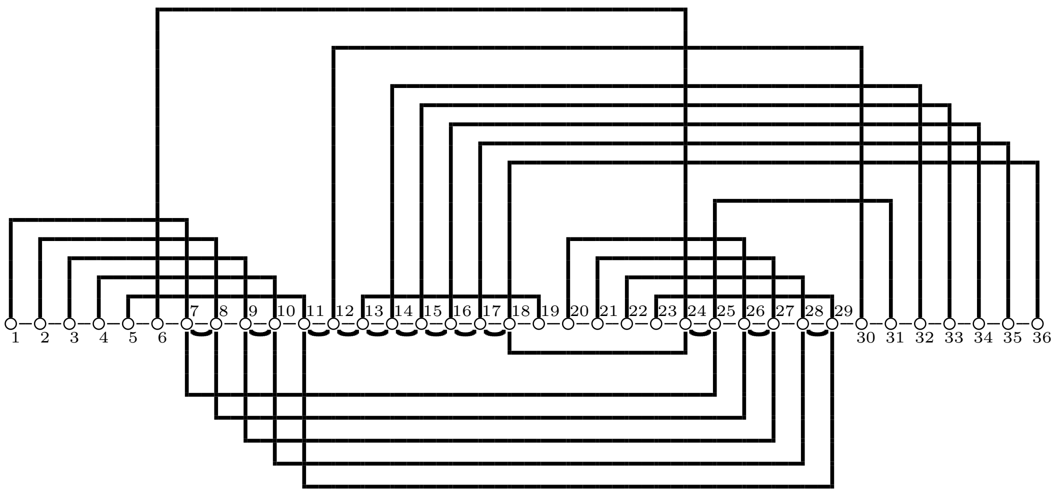

FIGURE 15. The graph G_{18}^{12}

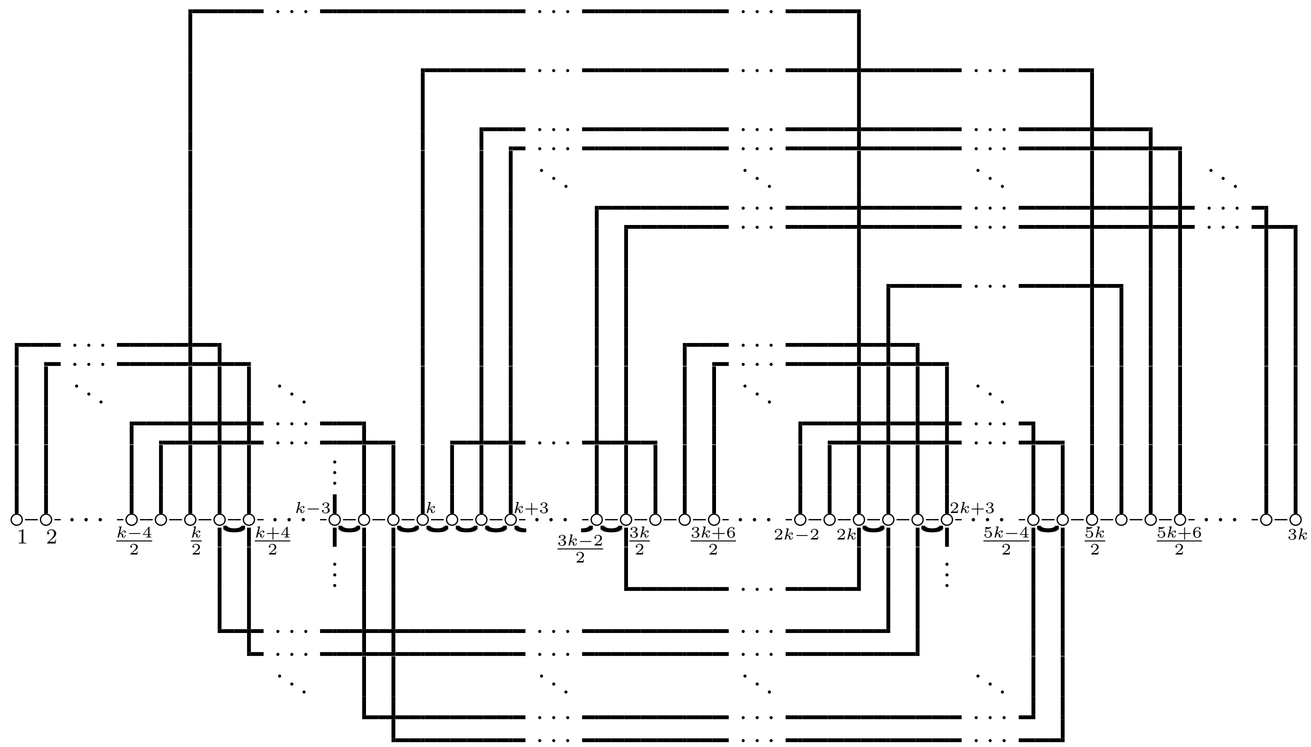

FIGURE 16. The graph $G^k_{3k/2}$ when $k \equiv 0 \bmod 4$

4. MINIMAL k-EQUITABILITY OF C'_m WHEN k IS EVEN

Assume that $k \geq 2$ is even. We are going to show that C'_m is minimally k-equitable for every m such that $(m, k) \in \mathcal{P}$.

Given a minimal k-equitable representation G of C'_m, we say that a set $\mathcal{S}$ of edges of G is a *k-socket* if the following conditions are satisfied:

(1) $\mathcal{S}$ consists of $\frac{k}{2}$ cycle edges of length 1 whose endpoints form a set of consecutive k integers s, $s+1$, $\ldots$, $s+k-1$;

(2) if a is the smallest integer in the vertex set of G and b is the largest integer in the vertex set of G, then either $b-s+1$ or $s+k-a$ is not the length of any edge of G.

Note that if the set of endpoints of the edges of $\mathcal{S}$ consists of either the largest k vertices of G or the smallest k vertices of G, then the second condition above is satisfied.

A minimal k-equitable representation of C'_m with a k-socket will be called a *k-proper representation of C'_m*.

Lemma 4.1. *The graph G^k_{2k} is a k-proper representation of C'_{2k} for every even $k \geq 2$.*

Proof. Let $k \geq 2$ be an even integer. It follows from the proof of Lemma 2.3 that G^k_{2k} is a minimally k-equitable representation of C'_{2k}. If $k = 2$, then let $\mathcal{S} = \{(4,5)\}$. (See Figure 8.) With $b = 8$ and $s = 4$ the integer $b - s + 1 = 5$ is not a length of any edge of G^2_4. Thus $\mathcal{S}$ is a 2-socket in G^2_4 implying that G^2_4 is a 2-proper representation of C'_4.

If $k \geq 4$, then let (see Figures 11 and 12)

$$\mathcal{S} = \left\{ \left(\frac{k+2}{2}, \frac{k+4}{2} \right), \left(\frac{k+6}{2}, \frac{k+8}{2} \right), \ldots, \left(\frac{3k-2}{2}, \frac{3k}{2} \right) \right\}.$$

With $b = 4k$ and $s = \frac{k+2}{2}$, the integer $b - s + 1 = \frac{7k}{2}$ is not a length of any edge of G^k_{2k}, implying that G^k_{2k} is a k-proper representation of C'_{2k}. ∎

Lemma 4.2. *The graph $G^k_{3k/2}$ is a k-proper representation of $C'_{3k/2}$ for every integer $k \geq 4$ such that $k \equiv 0 \bmod 4$.*

Proof. Let $k \geq 4$ be an integer with $k \equiv 0 \bmod 4$. It follows from the proof of Lemma 3.1 that $G^k_{3k/2}$ is a minimally k-equitable representation of $C'_{3k/2}$. Let (see Figure 16)

$$\mathcal{S} = \left\{ \left(\frac{k+2}{2}, \frac{k+4}{2} \right), \left(\frac{k+6}{2}, \frac{k+8}{2} \right), \ldots, \left(\frac{3k-6}{2}, \frac{3k-4}{2} \right), \left(\frac{3k-2}{2}, \frac{3k}{2} \right) \right\}.$$

With $b = 3k$ and $s = \frac{k+2}{2}$, the integer $b - s + 1 = \frac{5k}{2}$ is not a length of any edge of $G^k_{3k/2}$, implying that $G^k_{3k/2}$ is a k-proper representation of $C'_{3k/2}$. ∎

Lemma 4.3. *If k is even and there is a k-proper representation of C'_m, then there is a k-proper representation of C'_{m+k}.*

Proof. Let G be a k-proper representation of C'_m with a k-socket $\mathcal{S}$ and let s be the smallest integer in the set of endpoints of the edges in $\mathcal{S}$. Let a be the smallest integer which is a vertex of G and let b be the largest integer which is a vertex of G. Let H be the graph obtained from G by performing the following operations:

- remove the edges of $\mathcal{S}$;

- add 2 sets of k vertices each at both ends of the graph G, namely add the vertices $a-k$, $a-k+1$, $\ldots$, $a-1$ and $b+1$, $b+2$, $\ldots$, $b+k$;
- add k edges of length $b-a+k+1$ matching the new vertices, namely add the edges $(a-k, b+1)$, $(a-k+1, b+2)$, $\ldots$, $(a-1, b+k)$;

Case 1: if G has no edges of length $b-s+1$, then

- add k edges of length $b-s+1$ joining the endpoints of the edges in $\mathcal{S}$ to the new vertices whose value is larger than b, namely add the edges $(s, b+1)$, $(s+1, b+2)$, $\ldots$, $(s+k-1, b+k)$;
- add $\frac{k}{2}$ edges of length 1 that form a matching of the set of all the new vertices whose value is larger than b, namely add the edges $(b+1, b+2)$, $(b+3, b+4)$, $\ldots$, $(b+k-1, b+k)$.

Figure 17 shows the new edges of the graph H in case 1.

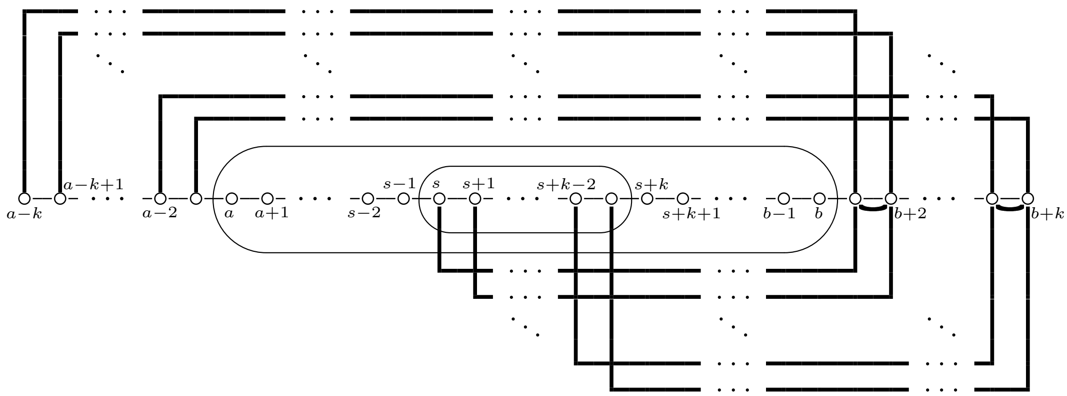

FIGURE 17. The new edges of the graph H in case 1

Case 2: if G has some edge of length $b-s+1$, (so that it has no edges of length $s+k-a$), then

- add k edges of length $s+k-a$ joining the endpoints of the edges in $\mathcal{S}$ to the new vertices whose value is smaller than a, namely add the edges $(a-k, s)$, $(a-k+1, s+1)$, $\ldots$, $(a-1, s+k-1)$;

- add $\frac{k}{2}$ edges of length 1 that form a matching of the set of the new vertices whose value is smaller than a, namely add the edges $(a-k, a-k+1)$, $(a-k+2, a-k+3)$, ..., $(a-2, a-1)$.

Figure 18 shows the new edges of the graph H in case 2. Figures 19 and 20 show the results of applying the above construction to the graphs G_4^2 and G_6^4 respectively.

Since each edge of $\mathcal{S}$ in G is replaced by a path of length 3 in H, the cycle of G gives rise to a cycle of length $m+k$ in H. Moreover, the new k edges of length $b-a+k+1$ in H are matching edges implying that H is isomorphic to C'_{m+k}. It is clear that H is a minimally k-equitable representation of C'_{m+k}. The set of the new edges of length 1 in H is a k-socket since the set of endpoints of these edges consists of either the k vertices of H having the smallest label or the k vertices of H having the largest label. Thus H is a k-proper representation of C'_{m+k}. ∎

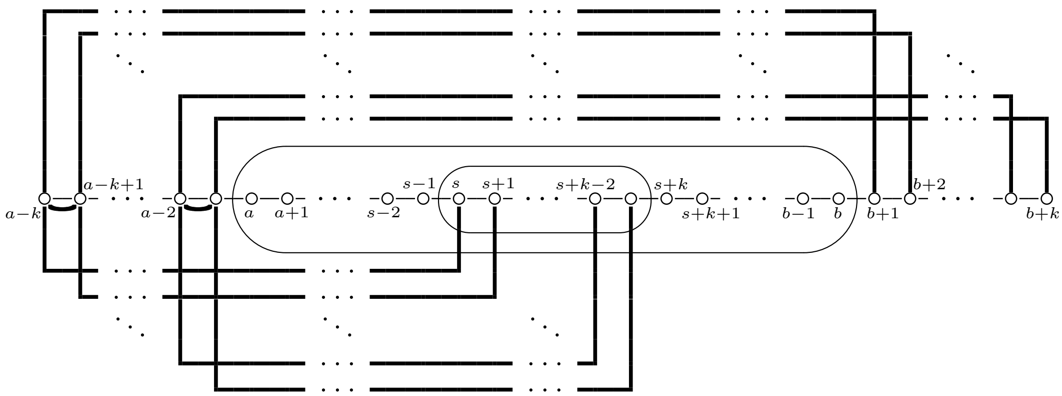

FIGURE 18. The new edges of the graph H in case 2

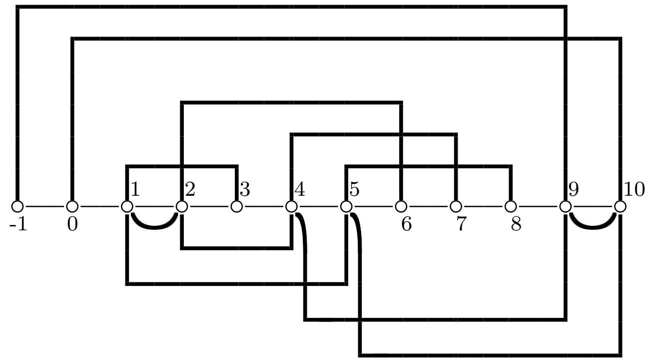

FIGURE 19. A 2-proper representation of C'_6 obtained from G^2_4

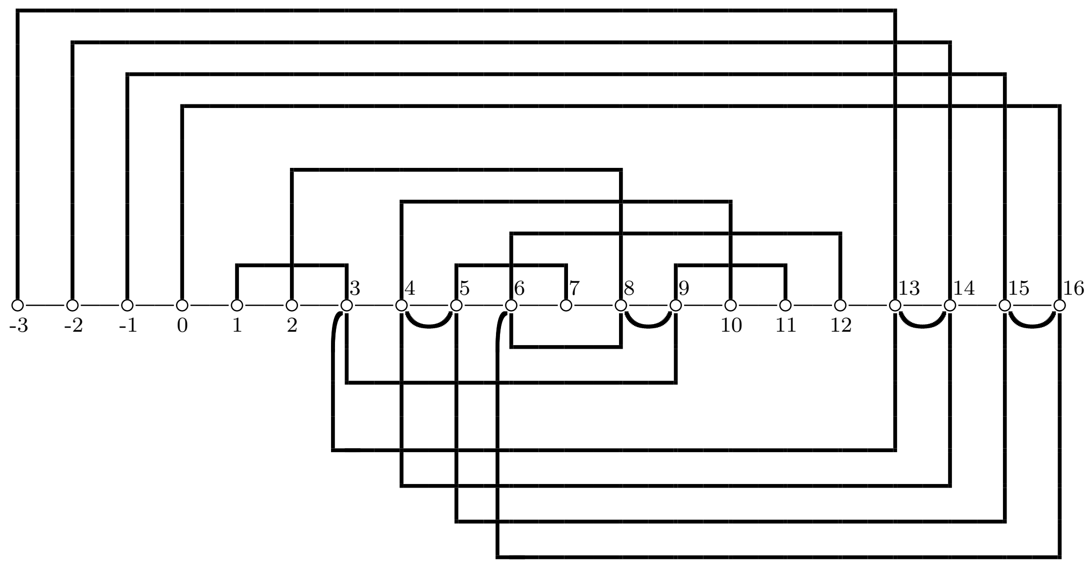

FIGURE 20. A 2-proper representation of C'_{10} obtained from G^4_6

Proposition 4.4. *Let $(m, k) \in \mathcal{P}$ with k being even. Then C'_m is minimally k-equitable.*

Proof. Assume that $m = \ell k$ for some integer ℓ. The minimal k-equitability of C'_m follows from Lemma 2.1 when $\ell = 1$, and from Lemma 2.3 when $\ell = 2$. If $\ell \geq 3$, then the minimal k-equitability of C'_m follows by induction on ℓ using Lemmas 4.1 and 4.3.

If k is not a divisor of m, then $k \equiv 0 \bmod 4$ and $m = \ell k/2$ for some odd integer $\ell \geq 3$. The minimal k-equitability of C'_m follows from Lemma 3.1 when $\ell = 3$, and follows by induction using Lemmas 4.2 and 4.3 when $\ell \geq 5$. ∎

5. Minimal k-equitability of C'_m when k is odd

Assume that $k \geq 3$ is odd. We are going to show that C'_m is minimally k-equitable for every m such that $(m, k) \in \mathcal{P}$.

If $e = (u, v)$ is an edge of an integer graph and $u < v$, then we say that u is the *left endpoint* of e and v is the *right endpoint* of e. Given a minimal k-equitable representation G of C'_m, we say that a pair $\mathbb{T} = (T, \mathcal{C})$ is a *k-thread* in G if the following conditions are satisfied:

(1) $\mathcal{C}$ consists of k edges that are cycle edges and have the same length;

(2) T is either the set of all left endpoints or the set of all right endpoints of the edges in $\mathcal{C}$ and it consists of k consecutive integers t, $t+1$, $\ldots$, $t+k-1$;

(3) if a is the vertex of G with the smallest label, b is the vertex of G with the largest label, s is the other endpoint of the edge in $\mathcal{C}$ that has t as one of its endpoints, A_G is the set of lengths of the edges of G, and $R_{G,\mathbb{T}}$, $L_{G,\mathbb{T}}$, W_G are infinite sets of integers defined as follows:

$$R_{G,\mathbb{T}} = \{b - s + 1\} \cup \{b + 2ik - t + 1 : i \geq 0\} \cup \{t - a + 2ik : i \geq 1\},$$
$$L_{G,\mathbb{T}} = \{s - a + k\} \cup \{b + (2i+1)\,k - t + 1 : i \geq 0\} \cup \{t - a + (2i+1)\,k : i \geq 0\},$$
$$W_G = \{b - a + ik + 1 : i \geq 1\},$$

then either $L_{G,\mathbb{T}}$ or $R_{G,\mathbb{T}}$ is disjoint with $A_G \cup W_G$.

The following three figures illustrate the definition of a k-thread in a graph G when $t < s$; the pictures require obvious modifications when $s < t$. Figure 21 shows the edges of the graph G that belong to the set $\mathcal{C}$. Figure 22 shows the relationship between the sets $R_{G,\mathbb{T}}$, W_G and the vertices of the graph G, where the integers in $R_{G,\mathbb{T}}$ and W_G are represented by edges of the corresponding lengths. The set $R_{G,\mathbb{T}}$ is represented above the line containing the vertices of G and the set W_G below it. Figure 23 shows analogous relationship between the sets $L_{G,\mathbb{T}}$, W_G and the graph G.

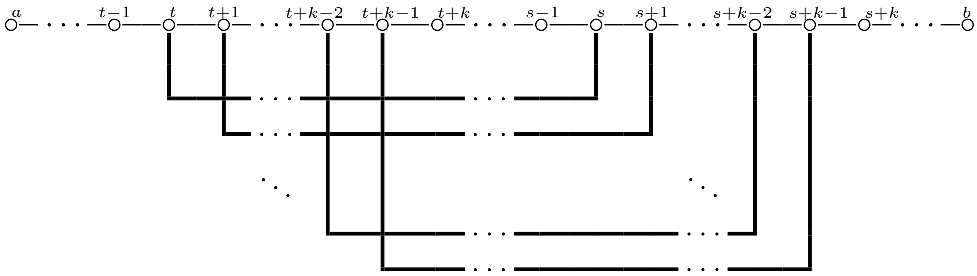

FIGURE 21. Edges of the graph G that belong to $\mathcal{C}$

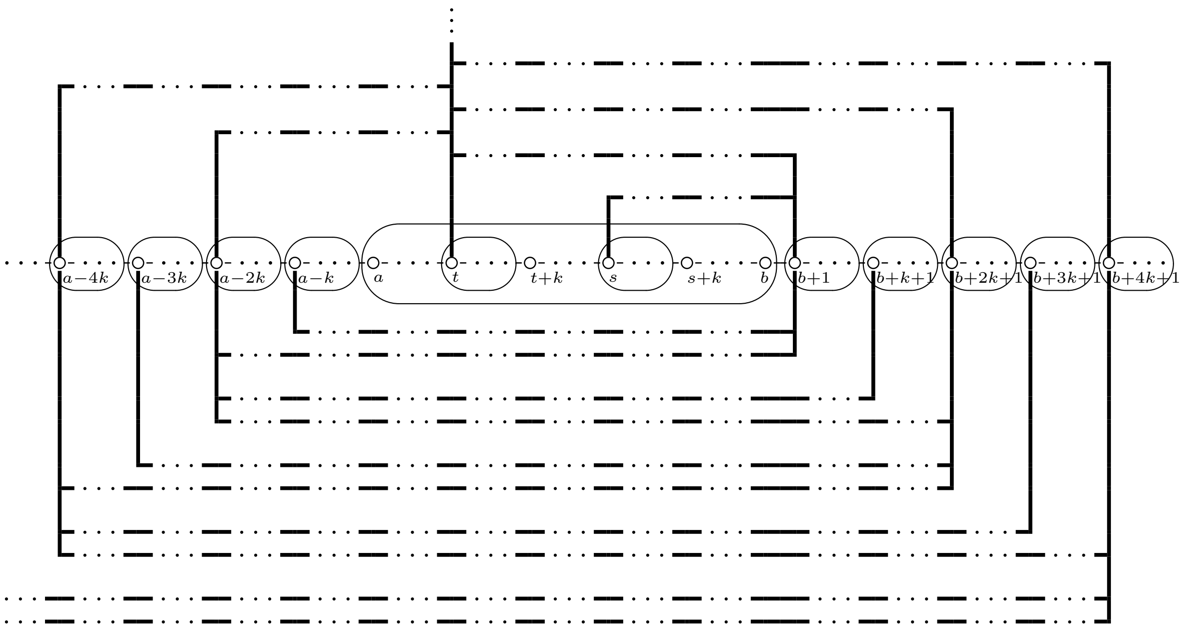

FIGURE 22. Edges whose length is in $R_{G,\mathbb{T}}$ or W_G

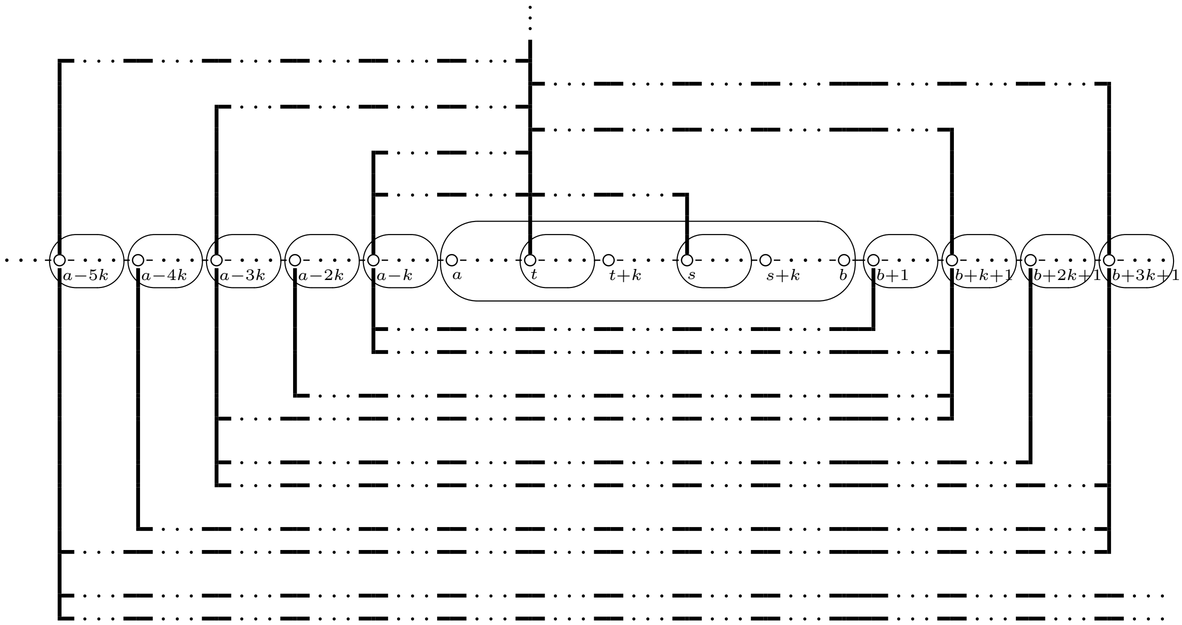

FIGURE 23. Edges whose length is in $L_{G,\mathbb{T}}$ or W_G

A minimal k-equitable representation of C'_m with a k-thread will be called a *k-proper representation of C'_m*.

Lemma 5.1. *The graph G^k_{2k} is a k-proper representation of C'_{2k} for every odd $k \geq 3$.*

Proof. Let $k \geq 3$ be an odd integer. It follows from the proof of Lemma 2.2 that G^k_{2k} is a minimally k-equitable representation of C'_{2k}.

Let (see Figures 6 and 7)

$$\mathcal{C} = \left\{ \left(\frac{k+3}{2}, 2k+1 \right), \left(\frac{k+5}{2}, 2k+2 \right), \ldots, \left(\frac{3k+1}{2}, 3k \right) \right\},$$

and

$$T = \left\{ \frac{k+3}{2}, \frac{k+5}{2}, \ldots, \frac{3k+1}{2} \right\}.$$

With $\mathbb{T} = (T, \mathcal{C})$, $a = 1$, $b = 4k$, $t = \frac{k+3}{2}$ and $s = 2k+1$, we have

$$R_{G,\mathbb{T}} = \{2k\} \cup \left\{ \frac{(4i-1)\,k-1}{2} : i \geq 2 \right\} \cup \left\{ \frac{(4i+1)\,k+1}{2} : i \geq 1 \right\}.$$

Since

$$W_G = \{ik : i \geq 5\},$$

and

$$A_G = \left\{1, \frac{k+1}{2}, k, \frac{3k-1}{2}\right\},$$

it is clear that

$$R_{G,\mathbb{T}} \cap (A_G \cup W_G) = \emptyset.$$

It follows that $\mathbb{T}$ is a k-thread in G^k_{2k}, implying that G^k_{2k} is a k-proper representation of C'_{2k}. ∎

Lemma 5.2. *If k is odd and there is a k-proper representation of C'_m, then there is a k-proper representation of C'_{m+k}.*

Proof. Let G be a k-proper representation of C'_m with a k-thread $\mathbb{T} = (T, \mathcal{C})$, let t be the smallest label of T, let s be the other endpoint of the edge in $\mathcal{C}$ that has t as one of its endpoints, and let

$$S = \{s, s+1, \ldots, s+k-1\}.$$

Let a be the smallest label of the vertices of G and let b be the largest label of the vertices of G. Let H be the graph obtained from G by performing the following operations:

- remove the edges of $\mathcal{C}$;
- add the vertices $a-k$, $a-k+1$, $\ldots$, $a-1$ and $b+1$, $b+2$, $\ldots$, $b+k$;
- add k edges of length $b-a+k+1$ matching the new vertices, namely add the edges $(a-k, b+1)$, $(a-k+1, b+2)$, $\ldots$, $(a-1, b+k)$;

Case 1: if $R_{G,\mathbb{T}} \cap (A_G \cup W_G) = \varnothing$, then

- add k edges of length $b-s+1$ joining the vertices in S to the new vertices whose label is larger than b, namely add the edges $(s, b+1)$, $(s+1, b+2)$, $\ldots$, $(s+k-1, b+k)$;
- add k edges of length $b-t+1$ joining the vertices in T to the new vertices whose label is larger than b, namely add the edges $(t, b+1)$, $(t+1, b+2)$, $\ldots$, $(t+k-1, b+k)$.

Figure 24 shows the new edges of the graph H in case 1.

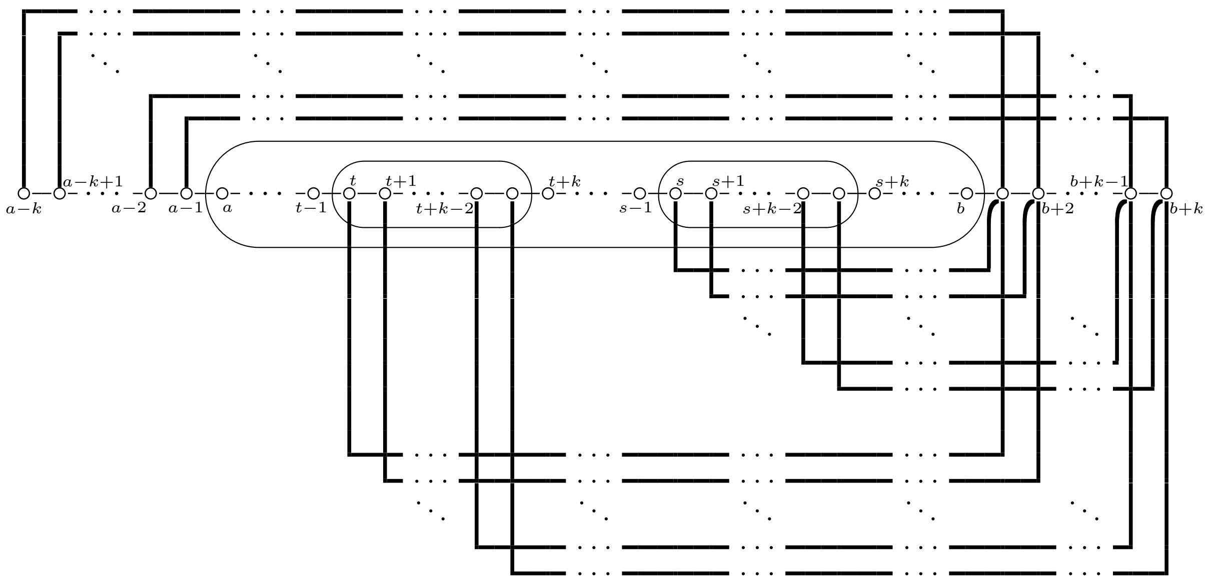

FIGURE 24. The new edges of the graph H in case 1

Case 2: if $R_{G,\mathbb{T}} \cap (A_G \cup W_G) \neq \varnothing$, (so that $L_{G,\mathbb{T}} \cap (A_G \cup W_G) = \varnothing$) then

- add k edges of length $s+k-a$ joining the vertices in S to the new vertices whose label is smaller than a, namely add the edges $(a-k, s)$, $(a-k+1, s+1)$, $\dots$, $(a-1, s+k-1)$;
- add k edges of length $t+k-a$ joining the vertices in T to the new vertices whose label is smaller than a, namely add the edges $(a-k, t)$, $(a-k+1, t+1)$, $\dots$, $(a-1, t+k-1)$.

Figure 25 shows the new edges of the graph H in case 2. Figure 26 shows the results of applying the above construction to the graph G_6^3.

Since each edge of $\mathcal{C}$ in G is replaced by a path of length 2 in H, the cycle of G gives rise to a cycle of length $m+k$ in H. Moreover, the new k edges of length $b-a+k+1$ in H are matching edges, implying that H is isomorphic to C'_{m+k}. Since

$$b-a+k+1 > b-a,$$

there are no edges in G of length $b-a+k+1$. Moreover the lengths of the new cycle edges of H are in $L_{G,\mathbb{T}}$ in case 1 and in $R_{G,\mathbb{T}}$ in case 2, implying that they are not in A_G.

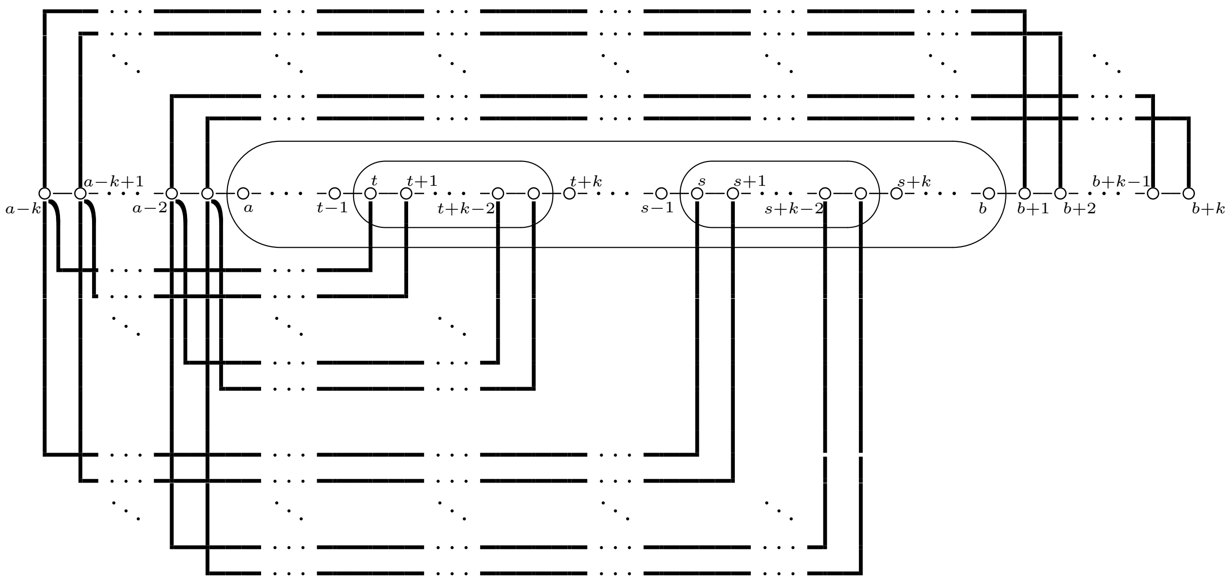

FIGURE 25. The new edges of the graph H in case 2

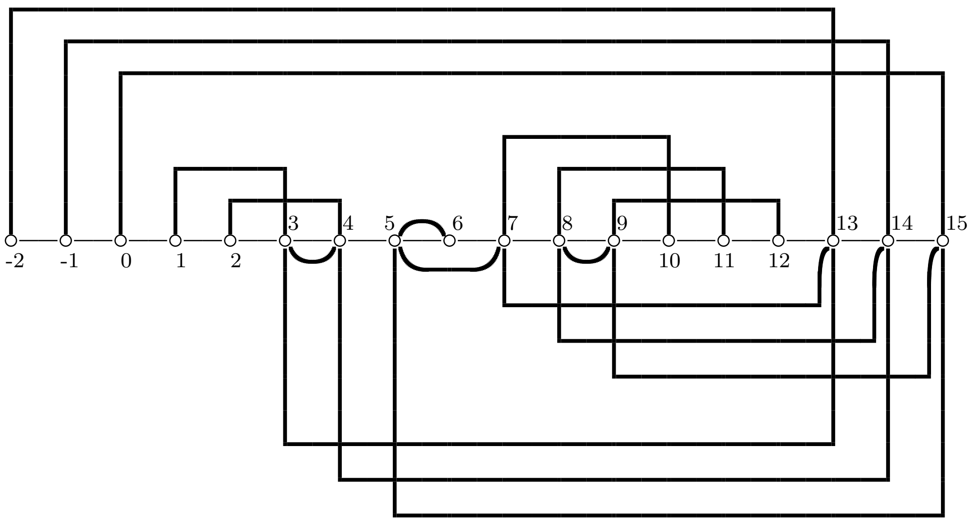

FIGURE 26. A 3-proper representation of C'_9 obtained from G_6^3

Therefore, the graph H is a minimally k-equitable representation of C'_{m+k}. To prove that H is a k-proper representation of C'_{m+k} it remains to show that there is a k-thread in H.

In case 1, let $\mathcal{C}'$ be the set of the new edges of H of length $b-t+1$ that join the vertices in T to the new vertices whose label is larger than b. We claim that the pair $\mathbb{T}' = (T, \mathcal{C}')$ is a

k-thread in the graph H. Let $a' = a - k$, $b' = b + k$, and let $s' = b + 1$ be the other endpoint of the edge of $\mathcal{C}'$ that has t as one of its endpoints. We have then

$$\begin{aligned} L_{H,\mathbb{T}'} &= \{s' - a' + k\} \cup \{b' + (2i+1)\,k - t + 1 : i \geq 0\} \cup \{t - a' + (2i+1)\,k : i \geq 0\} \\ &= \{b - a + 2k + 1\} \cup \{b + 2ik - t + 1 : i \geq 1\} \cup \{t - a + 2ik : i \geq 1\} \\ &= R_{G,\mathbb{T}} \cup \{b - a + 2k + 1\} \setminus \{b - s + 1, b - t + 1\}\,. \end{aligned}$$

Moreover

$$A_H = A_G \cup \{b - a + k + 1, b - s + 1, b - t + 1\} \setminus \{|t - s|\}\,,$$

and

$$\begin{aligned} W_H &= \{b' - a' + ik + 1 : i \geq 1\} \\ &= \{b - a + ik + 1 : i \geq 3\} \\ &= W_G \setminus \{b - a + k + 1, b - a + 2k + 1\}\,, \end{aligned}$$

so

$$A_H \cup W_H = (A_G \cup W_G) \cup \{b - s + 1, b - t + 1\} \setminus \{b - a + 2k + 1, |t - s|\}\,.$$

Since $R_{G,\mathbb{T}} \cap (A_G \cup W_G) = \emptyset$, it follows that

$$L_{H,\mathbb{T}'} \cap (A_H \cup W_H) = \emptyset,$$

so the pair $\mathbb{T}' = (T, \mathcal{C}')$ is a k-thread in the graph H.

In case 2, let $\mathcal{C}'$ be the set of the new edges of H of length $t + k - a$ that join the vertices in T to the new vertices whose label is smaller than a. We claim that the pair $\mathbb{T}' = (T, \mathcal{C}')$ is a k-thread in the graph H. Let $a' = a - k$, $b' = b + k$, and let $s' = a - k$ be the other endpoint of the edge of $\mathcal{C}'$ that has t as one of its endpoints. Then

$$\begin{aligned} R_{H,\mathbb{T}'} &= \{b' - s' + 1\} \cup \{b' + 2ik - t + 1 : i \geq 0\} \cup \{t - a' + 2ik : i \geq 1\} \\ &= \{b - a + 2k + 1\} \cup \{b + (2i+1)\,k - t + 1 : i \geq 0\} \cup \{t - a + (2i+1)\,k : i \geq 1\} \\ & \quad L_{G,\mathbb{T}} \cup \{b - a + 2k + 1\} \setminus \{s + k - a, t + k - a\}\,. \end{aligned}$$

Moreover

$$A_H = A_G \cup \{b - a + k + 1, s + k - a, t + k - a\} \setminus \{|t - s|\}\,,$$

and

$$W_H = W_G \setminus \{b - a + k + 1, b - a + 2k + 1\}$$

so

$$A_H \cup W_H = (A_G \cup W_G) \cup \{s + k - a, t + k - a\} \setminus \{b - a + 2k + 1, |t - s|\} .$$

Since $L_{G,\mathbb{T}} \cap (A_G \cup W_G) = \emptyset$, it follows that

$$R_{H,\mathbb{T}'} \cap (A_H \cup W_H) = \emptyset,$$

so the pair $\mathbb{T}' = (T, \mathcal{C}')$ is a k-thread in the graph H.

Thus H is a k-proper representation of C'_{m+k}. ∎

Proof of Theorem 1.2. Assume the $(m, k) \in \mathcal{P}$. If k is even then it follows from Proposition 4.4 that C'_m is k-equitable. If k is odd, then it is a divisor of m. The k-equitability of C'_m follows from Lemma 2.1 if $k = m$, and it follows by induction using Lemmas 5.1 and 5.2 if k is a proper divisor of m.

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Department of Mathematics, West Wirginia University, Morgantown, WV 26506-6310, USA

E-mail address: jerzy@math.wvu.edu