Semiregular Factorization of Simple Graphs

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June 16, 2005

Abstract

A graph G is a $(d, d+s)$-*graph* if the degree of each vertex of G lies in the interval $[d, d+s]$. A $(d, d+1)$-graph is said to be *semiregular*. An $(r, r+1)$-*factorization* of a graph is a decomposition of the graph into edge-disjoint $(r, r+1)$-factors.

We discuss here the state of knowledge about $(r, r+1)$-factorizations of d-regular graphs and of $(d, d+1)$-graphs.

For $r, s \geq 0$, let $\phi(r, s)$ be the least integer such that, if $d \geq \phi(r, s)$ and G is any simple $[d, d+s]$-graph, then G has an $(r, r+1)$-factorization. Akiyama and Kano (when r is even) and Cai (when r is odd) showed that $\phi(r, s)$ exists for all r, s. We show that, for $s \geq 2$, $\phi(r, s) = r(r+s+1)+1$. Earlier $\phi(r, 0)$ was determined by Egawa and Era, and $\phi(r, 1)$ was determined by Hilton.

1 Introduction

We call a graph *simple* if it has no loops or multiple edges. In this paper, *multigraphs* are graphs in which multiple edges may occur, but not loops. If multiple edges and loops may occur we use the term *pseudograph*.

An $(r, r+1)$-*pseudograph* is a pseudograph whose degrees are all either r or $r+1$; in a pseudograph, a loop counts two towards the degree of the vertex it is on. An $(r, r+1)$-*factor* of a pseudograph G is an $(r, r+1)$-subpseudograph which spans G. An $(r, r+1)$-*factorization* of a pseudograph G is a decomposition of G into edge-disjoint $(r, r+1)$-factors of G.

Let $\mathbb{N}$ be the set of non-negative integers. Given $d, s \in \mathbb{N}$ and a pseudograph G, we say that G is a $(d, d+s)$-*graph* if the degree of any vertex of G is in the interval $[d, d+s]$. Let

$\phi, \psi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be functions defined as follows. Given $r, s \in \mathbb{N}$, let $\phi(r, s)$ be the smallest integer such that $d \geq \phi(r, s)$ implies that any simple $(d, d + s)$-graph has an $(r, r + 1)$-factorization. Similarly, let $\psi(r, s)$ be the smallest integer such that $d \geq \psi(r, s)$ implies that any $(d, d + s)$-multigraph has an $(r, r + 1)$-factorization.

It is not clear at first sight that $\phi(r, s)$ and $\psi(r, s)$ exist for all values of r, s, and indeed the corresponding function for $(d, d + s)$-pseudographs does not exist for all values of r and s (see [6]). But specializations of a result of Akiyama and Kano [1] when r is even and of Cai [3] when r is odd yield the following result.

Theorem 1 *For* $r, s \in \mathbb{N}$,

$$\psi(r, s) \leq \begin{cases} (3r + 1)(r + s - 1) & \text{if } r \text{ is even,} \\ (3r + 1)(r + s) & \text{if } r \text{ is odd.} \end{cases}$$

It is clear that $\phi(r, s) \leq \psi(r, s)$ always.

Returning to $\phi(r, s)$, the value of $\phi(r, 0)$ was determined by Era [5] and Egawa [4]. A different proof of the Era-Egawa result was given in [6] where $\phi(r, 1)$ was also determined.

Theorem 2 *For* $r, s \in \mathbb{N}$, $s \in \{0, 1\}$,

$$\phi(r, s) = \begin{cases} r(r + s) & \text{if } r \text{ is even,} \\ r(r + s) + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Less precise results are known for $\psi(r, s)$ when $s = 0$ or 1. In [6] it is shown that the following result holds.

Theorem 3 *If* $r \in \mathbb{N}$ *and* $r \geq 1$, *then*

$$\frac{3}{2}r^2 - r \leq \psi(r, 0) \leq 2r^2 - 3r$$

if $r \geq 4$ *is even, and*

$$\psi(r, 0) = r^2 + 1$$

if r *is odd.*

Thus $\psi(r, 0) \neq \phi(r, 0)$ if r is even, but $\psi(r, 0) = \phi(r, 0)$ if r is odd.

In [6] bounds are also obtained for $\psi(r, 1)$.

In this paper we first describe in more detail what is known about $(r, r + 1)$-factorizations of d-regular simple graphs and simple $(d, d + 1)$-graphs, with particular emphasis on the number of factors in a factorization. We then prove the following theorem on the value of $\phi(r, s)$ when $s \geq 2$. This result stands in unexpected contrast to Theorem 2.

Theorem 4 *For $r, s \in \mathbb{N}$, $s \geq 2$*

$$\phi(r,s) = r\,(r+s+1)+1.$$

For good references on factorizations of graphs, see [2] and [8].

2 $(r, r+1)$-factorizations of simple graphs

In the cases when $s = 0$ and $s = 1$, $\phi(r,s)$ was evaluated by a novel method in [6]. A fundamental result of Hilton and de Werra [7] provided the key to this novel method. Let us give some terminology and then explain this fundamental result.

An *edge-colouring* of a pseudograph G is a map $\lambda : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours (loops being counted as edges). An edge-colouring is *equitable* if for each vertex v of G and any two colours $C_1, C_2 \in \mathcal{C}$, the number of edges incident to v and coloured C_1 differs by at most one from the corresponding number of edges coloured C_2; here a loop on v coloured C_i counts as two edges on v. For k an integer, $k \geq 2$, the *k-core* of a pseudograph G is the subpseudograph induced by the vertices of G whose degree is divisible by k. The theorem of Hilton and de Werra is:

Theorem 5 *Let k be an integer, $k \geq 2$, and let G be a simple graph. If the k-core of G contains no edges, then G has an equitable colouring with k colours.*

Using this theorem, the first author [6] proved the following result about $(r, r+1)$-factorizations of d-regular simple graphs. Theorem 5 was used to prove the 'hard' part, namely part 1.

Theorem 6 *Let G be a simple d-regular graph, and let x and r be integers with $r \geq 1$.*

1. *G has an $(r, r+1)$-factorization with exactly x $(r, r+1)$-factors if*

$$d/(r+1) < x < d/r,$$

   or if r is odd and $x = d/(r+1)$, or if r is even and $x = d/r$.

2. *If r is even and $(r+1) \mid d$, then there are d-regular simple graphs G which are, and d-regular simple graphs G which are not $(r, r+1)$-factorizable into $x = d/(r+1)$ $(r, r+1)$-factors; if r is odd and $r \mid d$, then there are d-regular simple graphs which are, and d-regular simple graphs which are not $(r, r+1)$-factorizable into $x = d/r$ $(r, r+1)$-factors.*

3. *If $x \notin [d/(r+1), d/r]$, then no d-regular simple graph is $(r, r+1)$-factorizable into x $(r, r+1)$-factors.*

For simple $(d, d+1)$-graphs the following similar theorem was also proved in [6].

Theorem 7 *Let x, d and r be integers with $d \geq r \geq 1$.*

1. *If*

$$(d+1)/(r+1) < x < d/r$$

or

$$x = \begin{cases} d/r & \text{if } r \text{ is even,} \\ (d+1)/(r+1) & \text{if } r \text{ is odd,} \end{cases}$$

then any simple $(d, d+1)$-graph G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

2. *If $x \geq 2$ and*

$$x = \begin{cases} d/r & \text{if } r \text{ is odd,} \\ (d+1)/(r+1) & \text{if } r \text{ is even,} \end{cases}$$

then some simple $(d, d+1)$-graphs do and some do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

3. *If $x \notin [(d+1)/(r+1), d/r]$, then the only simple $(d, d+1)$-graphs G having an $(r, r+1)$-factorization into x $(r, r+1)$-factors occur when*

$$\begin{cases} x = d/(r+1) & \text{and } G \text{ is } d\text{-regular,} \\ x = (d+1)/r & \text{and } G \text{ is } (d+1)\text{-regular.} \end{cases}$$

Moreover, when these conditions pertain, some but not all such graphs have an $(r, r+1)$-factorization.

Using Theorems 5, 6, and 7, it is a fairly simple matter to deduce Theorem 2 (which includes the Era-Egawa theorem).

3 $(r, r+1)$-factorization of simple $(d, d+s)$-graphs

In this section we prove Theorem 4 which says that if $r, s \in \mathbb{N}$, $s \geq 2$, then $\phi(r, s) = r(r+s+1)+1$.

Proof of Theorem 4

We first show that if

$$d \geq r(r+s+1)+1,$$

then any simple $(d, d+s)$-graph has an $(r, r+1)$-factorization. Note that

$$\frac{d}{r} - \frac{d+s}{r+1} \geq \frac{r^2+r+1}{r\,(r+1)} > 1,$$

so there is an integer x with

$$\frac{d+s}{r+1} < x < \frac{d}{r}.$$

By Theorem 5, G has an equitable coloring with x colors. Let v be a vertex of G. Since $rx < d$, there is a color class with at least $r+1$ edges incident to v. Since $(r+1)\,x > d+s$, there is a color class with at most r edges incident to v. Since the coloring is equitable the number of vertices incident to v in each color class is r or $r+1$. Thus the color classes give us an $(r, r+1)$-factorization of G.

Next we show that if

$$d = r\,(r+s+1)\,,$$

then there is a simple $(d, d+s)$-graph without an $(r, r+1)$-factorization. Note that

$$d+s = (r+1)\,(r+s)\,,$$

so any $(r, r+1)$-factorization of a $(d, d+s)$-graph contains either $r+s$ or $r+s+1$ factors. We are going to consider four cases depending on the parity of r and s. In all cases G will be a disjoint union of graphs G_1 and G_2 such that G_1 has no $(r, r+1)$-factorization into $r+s+1$ factors and G_2 has no $(r, r+1)$-factorization into $r+s$ factors.

In each case, the argument will be that if G_1 or G_2 did have such an $(r, r+1)$-factorization, then some $(r, r+1)$-factor would have to have an odd number of vertices of odd degree, which is impossible.

Assume first that r is even. Then d is even. Let G_1 be a graph with one vertex of degree $d+2$ and the remaining vertices of degree d. Some factor of an $(r, r+1)$-factorization of such a G_1 into $r+s+1$ factors would have exactly one vertex of degree $r+1$ which is impossible. For example, a suitable G_1 can be obtained by taking K_{d+2}, removing a Hamiltonian cycle, and adding a new vertex adjacent to all the other vertices. To construct G_2 we consider two cases.

If s is odd, let G_2 be a graph in which each vertex has degree $d+s$ except for one which has degree $d+s-1$. Since $d+s$ is odd, G_2 has odd order. In any $(r, r+1)$-factorization of G into $r+s$ factors, all but one of the factors of G_2 would have to be regular of degree $r+1$ which is odd. But since the order of G_2 is also odd, this is impossible. A suitable graph G_2 can be obtained by taking K_{d+s+2} and removing a spanning subgraph with $(d+s+1)\,/2$ components, where $(d+s-1)\,/2$ components are P_2's and one is a P_3.

If s is even, let G_2 be a regular graph of degree $d+s$ of odd order. In any $(r, r+1)$-factorization of G_2 into $r+s$ factors, all the $(r, r+1)$-factors would be $(r+1)$-regular, which is impossible since $r+1$ is odd and G_2 has odd order.

Now assume that r is odd (so $d+s$ is even). Let G_2 be a graph which has one vertex of degree $d+s-2$, the remainder having degree $d+s$. Some factor of an $(r, r+1)$-factorization of such a G_2 into $r+s$ factors would have exactly one vertex of degree r, the remaining vertices having degree $r+1$. This is impossible as r is odd. An example of a suitable G_2 may be obtained by taking K_{d+s+2}, marking two of its vertices as u and v, removing a 1-factor from $K_{d+s+2} - \{u, v\}$, and removing a path of length 2 with endpoints u and v. To construct G_1 we consider two cases.

If s is even (so that d is even), let G_1 be a d-regular graph of odd order. If s is odd (so that d is odd), let G_1 be a graph with one vertex of degree $d+1$ and the remaining vertices of degree d. An example of such G_1 can be obtained from K_{d+2} by removing $(d+1)/2$ independent edges.

4 $(r, r+1)$-factorizations of multigraphs

Recall that we have defined multigraphs as having no loops.

Theorem 3 shows that the upper bounds for $\psi(r, s)$ given in Theorem 1 are not best possible, at least in the case when $s = 0$. This is also true if $s = 1$, as in [6] the following is proved.

Theorem 8 *If* $r \in \mathbb{N}$, $r \geq 1$, *then*

$$\frac{3r^2}{2} - r \leq \psi(r, 1) \leq 2r^2 + r - 1$$

if r *is even, and*

$$r(r+1) + 1 \leq \psi(r, 1) \leq 2r^2 + 3r - 1$$

if r *is odd.*

Thus to determine $\psi(r, s)$ remains an open problem.

Theorem 3 also seems to suggest the surprising possibility that $\phi(r, s) = \psi(r, s)$ holds for every $r, s \in \mathbb{N}$ with r odd. However, the question if that is really true requires more evidence.

Acknowledgement. We would like to thank the referee of the first version of this note for drawing our attention to the papers by Akiyama and Kano [1] and Cai [3].

References

[1] J. Akiyama and M. Kano. Almost regular factorization of graphs. *J. Graph Theory*, 9:123–128, 1985.

[2] J. Akiyama and M. Kano. Factors and factorizations of graphs - a survey. *J. Graph Theory*, 9:1–42, 1985.

[3] M. -C. Cai. Almost regular factorization of graphs. *J. Graph Theory*, 9:123–128, 1985.

[4] Y. Egawa. Era's conjecture on $[k, k+1]$-factorizations of regular graphs. *Ars Combin.*, 21:217–220, 1986.

[5] H. Era. Semiregular factorizations of regular graphs. In *Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory (F. Harary and J. Maybee, eds.)*, pages 101–116. John Wiley and Sons, New York, 1984.

[6] A. J. W. Hilton. (r,r+1)-factorizations of $(d, d+1)$-graphs. submitted.

[7] A. J. W. Hilton and D. de Werra. A sufficient condition for equitable edge-colourings of simple graphs. *Discrete Math.*, 128:179–201, 1994.

[8] M. D. Plummer. Factors and factorizations in graphs: an update. *Discrete Math.*, to appear.