INFINITE MATROIDAL VERSION OF HALL'S MATCHING THEOREM

JERZY WOJCIECHOWSKI

Abstract. Hall's theorem for bipartite graphs gives a necessary and sufficient condition for the existence of a matching in a given bipartite graph. Aharoni and Ziv [2] generalized the notion of matchability to a pair of possibly infinite matroids on the same set and gave a condition that is sufficient for the matchability of a given pair $(\mathcal{M}, \mathcal{W})$ of finitary matroids, where the matroid $\mathcal{M}$ is SCF — a sum of countably many matroids of finite rank. The condition of Aharoni and Ziv is not necessary for matchability. In this paper we give a condition that is necessary for the existence of a matching for any pair of matroids (not necessarily finitary) and is sufficient for any pair $(\mathcal{M}, \mathcal{W})$ of finitary matroids, where the matroid $\mathcal{M}$ is SCF.

1. Matroids

Following Higgs [6] (see also Oxley [10]), we will define *matroid* as a pair $\mathcal{S} = \big(S, \bar{\partial}\big)$ where S is a set and $\bar{\partial}$ is an IE-operator (idempotent-exchange operator) on S.

A *space* is a pair $\mathcal{S} = \big(S, \bar{\partial}\big)$ where S is a set and $\bar{\partial} : 2^S \to 2^S$ is an operator on S such that:

M1. $X \subseteq \bar{\partial}(X)$ for every $X \subseteq S$;

M2. if $X \subseteq Y \subseteq S$, then $\bar{\partial}(X) \subseteq \bar{\partial}(Y)$.

If $\mathcal{S} = \big(S, \bar{\partial}\big)$ is a space, and $\bar{\partial}^* : 2^S \to 2^S$ is defined by

$$x \in \bar{\partial}^*(X) \text{ iff } x \in X \text{ or } x \notin \bar{\partial}(S \setminus (X \cup \{x\})).$$

then $\mathcal{S}^* = \big(S, \bar{\partial}^*\big)$ is also a space (the space *dual to* $\mathcal{S}$). It is easy to see that the space $\mathcal{S}^{**}$ dual to $\mathcal{S}^*$ is equal to $\mathcal{S}$.

A space $\mathcal{S} = \big(S, \bar{\partial}\big)$ is *idempotent* if

M3. $\bar{\partial}(\bar{\partial}(X)) = \bar{\partial}(X)$ for every $X \subseteq S$;

and it is *exchange* if

M4. for every X, Y and p such that $X \subseteq Y \subseteq S$ and $p \in S \setminus Y$, if $p \in \bar{\partial}(Y) \setminus \bar{\partial}(X)$ then there is $x \in Y \setminus X$ with $x \in \bar{\partial}(Y \setminus \{x\} \cup \{p\})$.

It is a straightforward exercise to verify that a space is idempotent if and only if its dual space is exchange.

A *matroid* is a space that is both idempotent and exchange. A matroid $\mathcal{S} = \big(S, \bar{\partial}\big)$ is *finite* if S is finite, and it is *finitary* if

M5. for every $X \subseteq S$ and $x \in S$, if $x \in \bar{\partial}(X)$ then there is a finite $Y \subseteq X$ such that $x \in \bar{\partial}(Y)$.

A finitary matroid is often called an *independence space* in the literature. Obviously, every finite matroid is finitary. The space dual to a matroid is clearly also a matroid, but the matroid dual to a finitary matroid (called *cofinitary*) does not have to be finitary.

Let $\mathcal{S} = \big(S, \bar{\partial}\big)$ be a space and let $X \subseteq S$. If $x \in \bar{\partial}(X)$ or $Y \subseteq \bar{\partial}(X)$, then we say that X *spans* x or X *spans* Y, respectively. We say that X is *spanning* in $\mathcal{S}$ if X spans S, and that X is *independent* in $\mathcal{S}$ if no $x \in X$ is spanned by $X \setminus \{x\}$. Note that X is independent in $\mathcal{S}$ if and only if $S \setminus X$ is spanning in the space dual to $\mathcal{S}$. If X is not independent in $\mathcal{S}$, then we say that it is *dependent* in $\mathcal{S}$.

Given a finitary matroid $\mathcal{S} = \big(S, \bar{\partial}\big)$, let $\bar{\mathcal{S}}$ be the family of subsets of S that are independent in $\mathcal{S}$. Note that (see Oxley [10]):

I1. $\bar{\mathcal{S}} \neq \varnothing$;
I2. if $A \in \bar{\mathcal{S}}$ and $B \subseteq A$, then $B \in \bar{\mathcal{S}}$;
I3. if $I, J \in \bar{\mathcal{S}}$ are finite and $|I| = |J| + 1$, then there is an element $y \in I \setminus J$ such that $J \cup \{y\} \in \bar{\mathcal{S}}$;
I4. if $A \subseteq S$ and $I \in \bar{\mathcal{S}}$ for every finite $I \subseteq A$, then $A \in \bar{\mathcal{S}}$.

Conversely, if $\bar{\mathcal{S}}$ is a family of subsets of a set S satisfying conditions I1–I4, and $\bar{\partial} : 2^S \to 2^S$ is defined by

$$x \in \bar{\partial}(X) \quad \text{iff } x \in X \text{ or there is } A \subseteq X \text{ such that } A \in \bar{\mathcal{S}} \text{ and } A \cup \{x\} \notin \bar{\mathcal{S}},$$

then $\mathcal{S} = \big(S, \bar{\partial}\big)$ is a finitary matroid and $\bar{\mathcal{S}}$ is equal to the family of subsets of S that are independent in $\mathcal{S}$.

Let $\mathcal{S} = \big(S, \bar{\partial}\big)$ be a space and let $X \subseteq S$. If X is both spanning and independent in $\mathcal{S}$, then it is said to be a *base* of $\mathcal{S}$. It is easy to see that X is a base of $\mathcal{S}$ if and only if it is maximal in the family of independent sets of $\mathcal{S}$, and if and only if it is minimal in the family of spanning sets.

In general, a matroid may have no bases.

Example 1.1. *Let $S_0 = \mathbb{Z}$ and*

$$\bar{\partial}_0(X) = \begin{cases} X & \text{if } X \text{ is finite;} \\ S_0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathcal{S}_0 = \big(S_0, \bar{\partial}_0\big)$ is a matroid with the family of independent sets equal to the family of all finite subsets of S_0 and the family of spanning sets equal to the family of infinite subsets of S_0. Thus $\mathcal{S}_0$ has no bases.

However, if $\mathcal{S}$ is a finitary matroid, then for every independent X and spanning Y with $X \subseteq Y \subseteq S$ there is a base B of $\mathcal{S}$ with $X \subseteq B \subseteq Y$. It follows immediately that the same is true for cofinitary matroids. If $\mathcal{S} = \big(S, \bar{\partial}\big)$ is a matroid and for every $Y \subseteq X \subseteq S$ the family of subsets of X that contain Y and are independent in $\mathcal{S}$ has a maximal element, then $\mathcal{S}$ is called a *B-matroid*. Any finitary matroid is a B-matroid.

Let $\mathbb{Z}^\infty = \mathbb{Z} \cup \{-\infty, \infty\}$ be the set of *quasi-integers*. If $a_1, \dots, a_n \in \mathbb{Z}^\infty$, then let the sum $a_1 + \cdots + a_n$ be the usual sum if $a_1, \dots, a_n$ are all integers, let the sum be ∞ if at least one of them is ∞, and let it be $-\infty$ if none of $a_1, \dots, a_n$ is ∞ but at least one of them is $-\infty$. Note that it follows immediately from the above definition that the operation of addition in $\mathbb{Z}^\infty$ is commutative and associative. The difference $a - b$ of two quasi-integers a, b means $a + (-b)$; and likewise, for example, $a - b + c - d$ means $a + (-b) + c + (-d)$, etc. Let $\mathbb{Z}^\infty$ be ordered in the obvious way. Note that if $a, b, c, d \in \mathbb{Z}^\infty$ satisfy $a \leq c$ and $b \leq d$, then $a + b \leq c + d$. Given a set S, let $\|S\| \in \mathbb{Z}^\infty$ be the cardinality of S if S is finite, and $\|S\| = \infty$ if S is infinite.

Let $\mathcal{S} = \big(S, \bar{\partial}\big)$ be a matroid. The *quasirank* of $\mathcal{S}$ (denoted $r(\mathcal{S})$) is the element of $\mathbb{Z}^\infty$ that is equal to the maximal cardinality of a finite independent set of $\mathcal{S}$ if such a cardinality exists, and it is equal to ∞ otherwise. If $r(\mathcal{S})$ is finite, then $\mathcal{S}$ is said to be a *finite-rank* matroid. It is obvious that a finite-rank matroid is finitary. If $\mathcal{S}$ is finitary, then all bases of $\mathcal{S}$ have the same cardinality (denoted $\rho(\mathcal{S})$), and this cardinality is defined to be the *rank* of $\mathcal{S}$. Let $r^*(\mathcal{S})$ be the quasirank of the matroid dual to $\mathcal{S}$.

Assume that $\mathcal{S} = \left(S, \bar{\partial}\right)$ is a space and $X \subseteq S$. The *restriction* of $\mathcal{S}$ to X, denoted $\mathcal{S}|X$, is defined to be the space $\left(X, \bar{\partial}'\right)$ with $\bar{\partial}'$ being the restriction of $\bar{\partial}$ to 2^X. The *contraction* of $\mathcal{S}$ to X, denoted $\mathcal{S}.X$, is the dual space to the restriction to X of the space dual to $\mathcal{S}$. Explicitly, $\mathcal{S}.X = \left(X, \bar{\partial}''\right)$ with $x \in \bar{\partial}''(A)$ (where $A \subseteq X$) if and only if $x \in \bar{\partial}\left(A \cup (S \setminus X)\right)$. If $\mathcal{S}$ is a matroid, then both $\mathcal{S}|X$ and $\mathcal{S}.X$ are matroids. If moreover $\mathcal{S}$ is either finite, finite-rank, finitary, or is a B-matroid, then both $\mathcal{S}|X$ and $\mathcal{S}.X$ have the same property. Let $\mathcal{S} \setminus X = \mathcal{S}|\left(S \setminus X\right)$ and $\mathcal{S}/X = \mathcal{S}.\left(S \setminus X\right)$.

Let $(S_i : i \in I)$ be a family of pairwise disjoint sets and $(\mathcal{S}_i : i \in I)$ be a family of spaces with $\mathcal{S}_i = \left(S_i, \bar{\partial}_i\right)$. The *sum* of the family $(S_i : i \in I)$ is defined to be the space $\mathcal{S} = \left(S, \bar{\partial}\right)$ with $S = \bigcup_{i \in I} S_i$ and $x \in \bar{\partial}(X)$ if and only if $x \in \bar{\partial}_i(X \cap S_i)$ where $i \in I$ is such that $x \in S_i$. It is easy to see that the following lemma holds.

Lemma 1.1. *If the space $\mathcal{S} = \left(S, \bar{\partial}\right)$ is the sum of the family $(\mathcal{S}_i : i \in I)$ of spaces with $\mathcal{S}_i = \left(S_i, \bar{\partial}_i\right)$ for every $i \in I$, and $A \subseteq S$, then $\mathcal{S}|A$ is the sum of the family $(\mathcal{S}_i|A_i : i \in I)$ and $\mathcal{S}.A$ is the sum of the family $(\mathcal{S}_i.A_i : i \in I)$, where $A_i = A \cap S_i$ for every $i \in I$.*

A matroid $\mathcal{S}$ is said to be SCF if it is the sum of a countable family of finite-rank matroids.

2. A matroidal analog of Hall's Theorem

Let $\mathcal{M}$ and $\mathcal{W}$ be matroids on a set E. Aharoni and Ziv [2] defined the pair $(\mathcal{M}, \mathcal{W})$ to be *matchable* if there is a subset of E that is both spanning in $\mathcal{M}$ and independent in $\mathcal{W}$. Such a subset of E will be called a *matching* in $(\mathcal{M}, \mathcal{W})$.

The concept of matchability of a pair of matroids on the same set originated as a generalization of a matroidal interpretation of the existence of a matching in a bipartite graph. Indeed, if $G = (V, E)$ is a bipartite graph with bipartition $V = M \cup W$, then let $\mathcal{M} = \left(E, \bar{\partial}_{\mathcal{M}}\right)$ and $\mathcal{W} = \left(E, \bar{\partial}_{\mathcal{W}}\right)$ be the matroids on the set of edges E defined by:

- $y \in \bar{\partial}_{\mathcal{M}}(X)$ if and only if there exists $x \in X$ such that x and y are incident to the same vertex in M;
- $y \in \bar{\partial}_{\mathcal{W}}(X)$ if and only if there exists $x \in X$ such that x and y are incident to the same vertex in W.

It is easy to see that a subset of E is a matching in $\Gamma = (\mathcal{M}, \mathcal{W})$ if and only if it contains a matching in the graph G, implying that Γ is matchable if and only if the graph G is matchable.

Let $\mathcal{M}$ and $\mathcal{W}$ be matroids on a set E. Aharoni and Ziv define a *hindrance* in $(\mathcal{M}, \mathcal{W})$ to be a subset H of E such that H is independent in both $\mathcal{W}$ and $\mathcal{M}.\left(\bar{\partial}_{\mathcal{W}}(H)\right)$ but H is not spanning in $\mathcal{M}.\left(\bar{\partial}_{\mathcal{W}}(H)\right)$. They prove the following result.

Theorem 2.1. *Let $\mathcal{M}$ and $\mathcal{W}$ be matroids on a set E such that $\mathcal{M}$ is SCF and $\mathcal{W}$ is finitary. If there are no hindrances in $(\mathcal{M}, \mathcal{W})$, then $(\mathcal{M}, \mathcal{W})$ is matchable.*

Theorem 2.1 is used by Aharoni and Ziv to prove a special case of the following conjecture, which is the infinite version of Edmond's theorem, and is attributed to C. Nash-Williams by Aharoni in [2].

Conjecture 2.2. *If $\mathcal{M}$ and $\mathcal{W}$ are finitary matroids on the same set S, then there exists $I \subseteq S$ such that I is independent in both $\mathcal{M}$ and $\mathcal{W}$ and there is a partition of I as $I = H \cup K$ with*

$$\bar{\partial}_{\mathcal{M}}(H) \cup \bar{\partial}_{\mathcal{W}}(K) = S.$$

The condition, in Theorem 2.1, that $(\mathcal{M}, \mathcal{W})$ does not contain a hindrance is not necessary for matchability. For example, let

$$E = \{(i, j) : i \in \{0, 1\}, \ j \in \{0, 1, 2, \dots\}\},$$

with

$$(i, j) \in \bar{\partial}_{\mathcal{W}}(A) \quad \text{iff there is } i' \in \{0, 1\} \text{ such that } (i', j) \in A,$$

and $X \subseteq E$ being independent in $\mathcal{M}$ if and only if it is the set of edges of an acyclic subgraph of the graph $G = (V, E)$ with $V = \{0, 1, 2, \dots\}$, $(0, j)$ incident to j and $j+1$, and $(1, j)$ incident to j and $j+2$, $j = 0, 1, 2, \dots$. Then

$$H = \{(1, j) : j \in \{0, 1, 2, \dots\}\}$$

is a hindrance in $(\mathcal{M}, \mathcal{W})$ and

$$T = \{(0, j) : j \in \{0, 1, 2, \dots\}\}$$

is a matching in $(\mathcal{M}, \mathcal{W})$.

The condition of Aharoni and Ziv resembles the condition in the countable version of Hall's theorem proved by Podewski and Steffens [11]. Another countable version of Hall's Theorem with a condition of a somewhat different nature was given by Nash-Williams [8] [9]. A modified version of the theorem of Nash-Williams, with a condition of a similar nature, called μ-admissibility, is proved in [12]. We are going to formulate a matroidal analog of μ-admissibility after some preliminaries.

Let $\mathcal{M}$ and $\mathcal{W}$ be matroids on a set E. Let M and W be disjoint copies of E (say $M = E \times \{0\}$ and $W = E \times \{1\}$). In an obvious way, $\mathcal{M}$ and $\mathcal{W}$ can be regarded as matroids on M and W respectively. To simplify notation, we will often identify the elements of M (of W) with the elements of E when it does not lead to confusion.

A *string* is an injective function with its domain being an ordinal. In particular, the empty set $\varnothing$ is a string with domain $0 = \varnothing$. A string f is said to be *in a set* S if $\operatorname{rge} f \subseteq S$, and it is said to be an α-*string* if its domain is equal to α. A string in $\Gamma = (\mathcal{M}, \mathcal{W})$ is a string in $M \cup W$. Given a string f in Γ, let

$$\begin{aligned} \operatorname{rge}_M f &= \{a \in E : (a, 0) \in \operatorname{rge} f\}, \\ \operatorname{rge}_W f &= \{a \in E : (a, 1) \in \operatorname{rge} f\}. \end{aligned}$$

A string f in Γ is *saturated* if $\operatorname{rge}_M f_\beta \subseteq \operatorname{rge}_W f_\beta$ for every $\beta \leq \operatorname{dom} f$.

Let f be a string and β, γ be ordinals with $\beta \leq \gamma \leq \operatorname{dom} f$. The $[\beta, \gamma)$-*segment* of f is the string $f_{[\beta,\gamma)}$ defined by

$$f_{[\beta,\gamma)}(\theta) = f(\beta + \theta),$$

for all θ with $\beta+\theta<\gamma$, that is, $f_{[\beta,\gamma)}$ is obtained from f by restricting it to $[\beta,\gamma)$ and shifting the domain to start at 0. Given $\alpha \le \operatorname{dom} f$, let $f_\alpha = f_{[0,\alpha)}$.

Assume that f is a string in Γ. The *μ-margin* $\mu(f)$ of f is an element of $\mathbb{Z}^\infty$ defined by transfinite induction on $\alpha = \operatorname{dom} f$ as follows. Let $\mu(f)=0$ if $\alpha=0$, let

(1)
$$\mu(f) = \begin{cases} \mu(f_\beta)+1 & \text{if } f(\beta)\in W \text{ and } f(\beta) \text{ is not spanned by } \operatorname{rge}_W f_\beta \text{ in } \mathcal{W}, \\ \mu(f_\beta)-1 & \text{if } f(\beta)\in M \text{ and } f(\beta) \text{ is not spanned by } E\setminus \operatorname{rge}_M f \text{ in } \mathcal{M}, \\ \mu(f_\beta) & \text{otherwise} \end{cases}$$

when $\alpha=\beta+1$ is a successor ordinal, and

$$\mu(f) = \liminf_{\beta\to\alpha} \mu(f_\beta)$$

if α is a limit ordinal. We say that Γ is *μ-admissible* if $\mu(f)\ge 0$ for every saturated string f in Γ.

We will prove the following results.

Theorem 2.3. *If $\mathcal{M}$ and $\mathcal{W}$ are arbitrary matroids on the same set and $(\mathcal{M},\mathcal{W})$ is matchable, then it is μ-admissible.*

Theorem 2.4. *Let $\mathcal{M}$ and $\mathcal{W}$ be matroids on the same set. If $\mathcal{M}$ is SCF, $\mathcal{W}$ is finitary, and $(\mathcal{M},\mathcal{W})$ is μ-admissible, then it is matchable.*

3. Necessity of the condition

In this section we are going to prove Theorem 2.3. Let's start with the following preliminary lemma.

Lemma 3.1. *Let $\mathcal{S}=\big(S,\bar\partial_{\mathcal{S}}\big)$ be a matroid, $a\in S$, and $\{S_1,S_2,S_3\}$ be a partition of $S\setminus\{a\}$ (allowing the parts to be empty). Let $S_i'=S_i\cup\{a\}$, $i=1,2,3$, and*

$$\mathcal{S}_1=(\mathcal{S}/S_1')\setminus S_3, \quad \mathcal{S}_2=(\mathcal{S}/S_1)\setminus S_3, \quad \mathcal{S}_3=(\mathcal{S}/S_1)\setminus S_3'.$$

Then

(1) *$r(\mathcal{S}_1)=r(\mathcal{S}_2)=r(\mathcal{S}_3)$ and $r^*(\mathcal{S}_1)=r^*(\mathcal{S}_2)-1=r^*(\mathcal{S}_3)$ if a is spanned by S_1 in $\mathcal{S}$;*

(2) *$r(\mathcal{S}_1)+1 = r(\mathcal{S}_2) = r(\mathcal{S}_3)$ and $r^*(\mathcal{S}_1) = r^*(\mathcal{S}_2) = r^*(\mathcal{S}_3)+1$ if a is spanned by $S_1 \cup S_2$ but not by S_1 in $\mathcal{S}$;*

(3) *$r(\mathcal{S}_1) = r(\mathcal{S}_2) - 1 = r(\mathcal{S}_3)$ and $r^*(\mathcal{S}_1) = r^*(\mathcal{S}_2) = r^*(\mathcal{S}_3)$ if a is not spanned by $S_1 \cup S_2$ in $\mathcal{S}$.*

Proof. We will only prove the equations involving the quasirank r. The equations involving the dual quasirank r^* will then follow.

Assume that a is spanned by S_1 in $\mathcal{S}$. Obviously, any set independent in $\mathcal{S}_1$ is independent in $\mathcal{S}_2$. Suppose that A is independent in $\mathcal{S}_2$. Since $a \in \bar{\partial}_{\mathcal{S}_2}(\varnothing)$, it follows that $a \notin A$ and A is independent in $\mathcal{S}_1$, thus

$$r(\mathcal{S}_1) = r(\mathcal{S}_2).$$

Assume that a is not spanned by S_1 in $\mathcal{S}$. We will show that

$$r(\mathcal{S}_1) = r(\mathcal{S}_2) - 1. \tag{2}$$

If $r(\mathcal{S}_1) = \infty$, then $r(\mathcal{S}_2) = \infty$ and (2) holds. If $r(\mathcal{S}_1)$ is finite and A is a base of $\mathcal{S}_1$, then $A \cup \{a\}$ is a base of $\mathcal{S}_2$ and (2) holds as well.

Assume that a is spanned by $S_1 \cup S_2$ in $\mathcal{S}$. We will show that

$$r(\mathcal{S}_2) = r(\mathcal{S}_3). \tag{3}$$

If $r(\mathcal{S}_3) = \infty$, then $r(\mathcal{S}_2) = \infty$ and (3) holds. If $r(\mathcal{S}_3)$ is finite and A is a base of $\mathcal{S}_3$, then it is a base of $\mathcal{S}_2$ so (3) holds as well.

Assume that a is not spanned by $S_1 \cup S_2$ in $\mathcal{S}$. We will show that

$$r(\mathcal{S}_2) - 1 = r(\mathcal{S}_3). \tag{4}$$

If $r(\mathcal{S}_3) = \infty$, then $r(\mathcal{S}_2) = \infty$ and (4) holds. If $r(\mathcal{S}_3)$ is finite and A is a base of $\mathcal{S}_3$, then $A \cup \{a\}$ is a base of $\mathcal{S}_2$ so (3) holds as well. ∎

Now we are ready to prove Theorem 2.3. Let $\mathcal{M}$ and $\mathcal{W}$ be arbitrary matroids on the same set E and let $\Gamma = (\mathcal{M}, \mathcal{W})$. Assume that T is a matching in Γ and f is a saturated string in Γ. Let $\alpha = \operatorname{dom} f$, and for each $\beta \leq \alpha$ let $T_\beta = T \cap \operatorname{rge}_M f_\beta$,

$$r^*_\beta = r^*\left(\left(\mathcal{M}.\operatorname{rge}_M f_\beta\right)|T_\beta\right),$$

and

$$r_\beta = r\left((\mathcal{W}/T_\beta) \,|\, (\mathrm{rge}_W\, f_\beta \setminus T_\beta)\right).$$

Using transfinite induction on α, we will show that

$$\mu(f) \geq r^*_\alpha + r_\alpha. \tag{5}$$

Since $r^*_\alpha, r_\alpha \geq 0$, it then follows that $\mu(f) \geq 0$ and so Γ is μ-admissible.

If $\alpha = 0$, then $\mu(f) = r^*_\alpha = r_\alpha = 0$ so (5) holds. The proof of (5) will be completed in a series of lemmas.

Lemma 3.2. *If $\alpha = \beta + 1$ is a successor ordinal, $\mu(f_\beta) \geq r^*_\beta + r_\beta$, and $f(\beta) \in W$, then $\mu(f) \geq r^*_\alpha + r_\alpha$.*

Proof. Since $\mathrm{rge}_M\, f = \mathrm{rge}_M\, f_\beta$, we have $r^*_\alpha = r^*_\beta$. Let $\mathcal{S} = \mathcal{W}$, $a = f(\beta)$,

$$S_1 = T_\beta, \quad S_2 = \mathrm{rge}_W\, f_\beta \setminus T_\beta, \quad S_3 = E \setminus \mathrm{rge}_W\, f,$$

and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ be as in Lemma 3.1. Then $r_\beta = r(\mathcal{S}_3)$ and $r_\alpha = r(\mathcal{S}_2)$. If $f(\beta)$ is not spanned by $\mathrm{rge}_W\, f_\beta = S_1 \cup S_2$ in $\mathcal{W}$, then $\mu(f) = \mu(f_\beta) + 1$ and it follows from Lemma 3.1 that $r_\alpha = r_\beta + 1$. If $f(\beta)$ is spanned by $\mathrm{rge}_W\, f_\beta = S_1 \cup S_2$ in $\mathcal{W}$, then $\mu(f) = \mu(f_\beta)$ and it follows from Lemma 3.1 that $r_\alpha = r_\beta$. ∎

Lemma 3.3. *If $\alpha = \beta + 1$ is a successor ordinal, $\mu(f_\beta) \geq r^*_\beta + r_\beta$, and $f(\beta) \in M$, then $\mu(f) \geq r^*_\alpha + r_\alpha$.*

Proof. If $f(\beta) \notin T$, then $r_\alpha = r_\beta$. If $f(\beta) \in T$, than taking $\mathcal{S} = \mathcal{W}$, $a = f(\beta)$,

$$S_1 = T_\beta, \quad S_2 = \mathrm{rge}_W\, f_\beta \setminus T_\beta, \quad S_3 = E \setminus \mathrm{rge}_W\, f,$$

and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ as in Lemma 3.1, we have $r_\beta = r(\mathcal{S}_2)$ and $r_\alpha = r(\mathcal{S}_1)$. Since T is independent in $\mathcal{W}$, it follows that $f(\beta)$ is not spanned by T_β in $\mathcal{W}$, so Lemma 3.1 implies that $r_\alpha = r_\beta - 1$.

Now let $\mathcal{S} = \mathcal{M}$, $a = f(\beta)$,

$$S_1 = E \setminus \mathrm{rge}_M\, f, \quad S_2 = T_\beta, \quad S_3 = \mathrm{rge}_M\, f_\beta \setminus T_\beta,$$

and $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ be as in Lemma 3.1.

If $f(\beta) \in T$, then $r^*_\beta = r^*(\mathcal{S}_1)$ and $r^*_\alpha = r^*(\mathcal{S}_2)$ so Lemma 3.1 implies that

$$r^*_\alpha = \begin{cases} r^*_\beta + 1 & \text{if } f(\beta) \text{ is spanned by } M \setminus \operatorname{rge}_M f \text{ in } \mathcal{M} \\ r^*_\beta & \text{if } f(\beta) \text{ is not spanned by } M \setminus \operatorname{rge}_M f \text{ in } \mathcal{M} \end{cases}$$

Assume now that $f(\beta) \notin T$. Then $r^*_\beta = r^*(\mathcal{S}_1)$ and $r^*_\alpha = r^*(\mathcal{S}_3)$. If $f(\beta)$ is spanned by $S_1 = M \setminus \operatorname{rge}_M f$ in $\mathcal{M}$, then Lemma 3.1 implies that $r^*_\alpha = r^*_\beta$. Note that since $T \subseteq S_1 \cup S_2$ and T is spanning in $\mathcal{M}$, it follows that $f(\beta)$ is spanned by $S_1 \cup S_2$ in $\mathcal{M}$. Therefore, if $f(\beta)$ is not spanned by $S_1 = M \setminus \operatorname{rge}_M f$ in $\mathcal{M}$, then it follows from Lemma 3.1 that $r^*_\alpha = r^*_\beta - 1$.

Combining the cases when $f(\beta) \in T$ and $f(\beta) \notin T$, we obtain

$$r_\alpha + r^*_\alpha = \begin{cases} r_\beta + r^*_\beta & \text{if } f(\beta) \text{ is spanned by } M \setminus \operatorname{rge}_M f \text{ in } \mathcal{M} \\ r_\beta + r^*_\beta - 1 & \text{if } f(\beta) \text{ is not spanned by } M \setminus \operatorname{rge}_M f \text{ in } \mathcal{M} \end{cases}.$$

Since

$$\mu(f) = \begin{cases} \mu(f_\beta) & \text{if } f(\beta) \text{ is spanned by } M \setminus \operatorname{rge}_M f \text{ in } \mathcal{M} \\ \mu(f_\beta) - 1 & \text{if } f(\beta) \text{ is not spanned by } M \setminus \operatorname{rge}_M f \text{ in } \mathcal{M} \end{cases},$$

the proof is complete. ∎

Lemma 3.4. *If α is a limit ordinal and $\mu(f_\beta) \geq r^*_\beta + r_\beta$ for every $\beta < \alpha$, then $\mu(f) \geq r^*_\alpha + r_\alpha$.*

Proof. Suppose, by way of contradiction, that $\mu(f) < r^*_\alpha + r_\alpha$. Then $\mu(f_\beta)$ is finite for every $\beta \leq \alpha$. Therefore r^*_β and r_β are both finite for every $\beta < \alpha$. Let A_β be a base of the matroid dual to $(\mathcal{M}.\operatorname{rge}_M f)\,|T_\beta$ for every $\beta < \alpha$. We can moreover assume that $A_\beta \subseteq A_{\beta'}$ for $\beta \leq \beta' < \alpha$. Since A_β is independent in the matroid dual to $(\mathcal{M}.\operatorname{rge}_M f_\beta)\,|T_\beta$, it follows that

$$\|A_\beta\| \leq r^*_\beta$$

for $\beta < \alpha$. Note that the union $A = \bigcup_{\beta<\alpha} A_\beta$ must be finite, since otherwise we would have $\liminf_{\beta\to\alpha} r^*_\beta = \infty$, which would imply that

$$\mu(f) = \liminf_{\beta\to\alpha} \mu(f_\beta) \geq \liminf_{\beta\to\alpha} r^*_\beta = \infty.$$

Therefore $A = A_\gamma$ for some $\gamma < \alpha$. It follows that A is spanning in the matroid dual to $(\mathcal{M}.\operatorname{rge}_M f)\,|T_\alpha$ and so

$$r^*_\alpha \leq \|A\| = \liminf_{\beta\to\alpha} \|A_\beta\|\,.$$

Let D_β be a base of $(\mathcal{W}/T_\alpha)\,|\,(\operatorname{rge}_W f_\beta \setminus T_\alpha)$ for every $\beta < \alpha$, with $D_\beta \subseteq D_{\beta'}$ for $\beta \leq \beta' < \alpha$. Since D_β is independent in $(\mathcal{W}/T_\beta)\,|\,(\operatorname{rge}_W f_\beta \setminus T_\beta)$, we have

$$\|D_\beta\| \leq r_\beta$$

for every $\beta < \alpha$. Similarly as above it follows that $D = \bigcup_{\beta<\alpha} D_\beta$ must be finite and spanning in $(\mathcal{W}/T_\alpha)\,|\,(\operatorname{rge}_W f \setminus T_\alpha)$ implying that

$$r_\alpha \leq \|D\| = \liminf_{\beta\to\alpha} \|D_\beta\|\,.$$

Since

$$\mu(f_\beta) \geq r^*_\beta + r_\beta \geq \|A_\beta\| + \|D_\beta\|$$

for every $\beta < \alpha$, it follows that

$$\begin{aligned}
\mu(f) &= \liminf_{\beta\to\alpha} \mu(f_\beta)\\
&\geq \liminf_{\beta\to\alpha} (\|A_\beta\| + \|D_\beta\|)\\
&= \|A\| + \|D\|\\
&= \liminf_{\beta\to\alpha} \|A_\beta\| + \liminf_{\beta\to\alpha} \|D_\beta\|\\
&\geq r^*_\alpha + r_\alpha.
\end{aligned}$$

This contradiction completes the proof. ∎

4. Sufficiency of the condition

4.1. **Preliminary results.** Let $\mathcal{M}$ and $\mathcal{W}$ be arbitrary matroids on the same set E and let $\Gamma = (\mathcal{M}, \mathcal{W})$. If f and g are strings in Γ with domains α and β respectively, then the *concatenation* $f * g$ of f and g is defined to be the $(\alpha+\beta)$-string h such that $h_\alpha = f$ and $h_{[\alpha,\alpha+\beta)} = g$. For $u \in M \cup W$, let $[u]$ be the 1-string f with $f(0) = u$.

Let $\mathfrak{T}$ be the set of all strings in Γ and let $\preceq$ be the relation on $\mathfrak{T}$ such that $g \preceq f$ if $g = f_\beta$ for some $\beta \leq \operatorname{dom} f$. Clearly $\preceq$ is a partial order on $\mathfrak{T}$. Let $\mathfrak{R}$ be the subset of $\mathfrak{T}$ consisting of all saturated strings f in Γ with $\mu(f) = 0$.

Lemma 4.1. *Assume that $\mathcal{M}$ is the sum of matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ (on sets M_1 and M_2 respectively) with $\mathcal{M}_1$ having positive finite rank. Let $\mathfrak{R}'$ be the subset of $\mathfrak{R}$ consisting of strings g such that there is*

$$a \in M_1 \setminus \operatorname{rge}_M g$$

with $\{a\}$ being independent in $\mathcal{M}_1$.

(1) *If Γ is μ-admissible, then the set $\mathfrak{R}'$ contains a maximal element with respect to $\preceq$.*

(2) *If Γ is μ-admissible and f is maximal in $\mathfrak{R}'$ with respect to $\preceq$, then*

$$\operatorname{rge}_M f = \operatorname{rge}_W f.$$

Proof. (1) We are going to use Zorn lemma. Since the empty string belongs to $\mathfrak{R}'$, the set $\mathfrak{R}'$ is nonempty. Let $\mathfrak{B}$ be a nonempty chain in $\mathfrak{R}'$. We will show that there is an upper bound for $\mathfrak{B}$ in $\mathfrak{R}'$.

Let $\Theta = \{\operatorname{dom} g : g \in \mathfrak{B}\}$ and $\alpha = \sup \Theta$. We are going to define an α-string f in Γ that belongs to $\mathfrak{R}'$ and is an upper bound for $\mathfrak{B}$. If $\beta < \alpha$, then there is $g \in \mathfrak{B}$ with $\beta < \operatorname{dom} g$. Define $f(\beta) = g(\beta)$. Since $\mathfrak{B}$ is a chain, the value of $f(\beta)$ does not depend on the choice of g. It is clear that $g \preceq f$ for every $g \in \mathfrak{B}$ so f is an upper bound for $\mathfrak{B}$.

Now we show that $f \in \mathfrak{R}'$. Since $\mathfrak{B} \subseteq \mathfrak{R}'$, we can assume that $f \notin \mathfrak{B}$. Then α is a limit ordinal. If $\beta < \alpha$, then $f_\beta = g_\beta$ for some $g \in \mathfrak{B}$ so f is saturated. Since $\alpha = \sup \Theta$ and since $\mu(f_\beta) = 0$ for every $\beta \in \Theta$, it follows that $\mu(f) = 0$. Therefore $f \in \mathfrak{R}$.

It remains to show that there is $a \in M_1 \setminus \operatorname{rge}_M f$ with $\{a\}$ being independent in $\mathcal{M}_1$. Suppose, by way of contradiction, that such a does not exist. For each $\beta \leq \alpha$, let

$$r_\beta = r\left(\mathcal{M}_1 | \left(M_1 \setminus \operatorname{rge}_M f_\beta\right)\right).$$

Since no singleton of $M_1 \setminus \mathrm{rge}_M f$ is independent in $\mathcal{M}_1$, we have $r_\alpha = 0$. Since $\mathcal{M}_1$ is a finite-rank matroid, there is $\gamma < \alpha$ such that $r_\gamma = r_\beta$ for every β with $\gamma \leq \beta < \alpha$. Since $f_\gamma = g_\gamma$ for some $g \in \mathfrak{B}$, it follows that there is $a \in M_1 \setminus \mathrm{rge}_M f_\gamma$ with $\{a\}$ independent in $\mathcal{M}_1$. Since $a \notin M_1 \setminus \mathrm{rge}_M f$, there is δ with $\gamma \leq \delta < \alpha$ and $f(\delta) = (a, 0)$. Let f' be the string obtained from f by removing the value $(a, 0)$ and shifting down the remaining values, that is let

$$f' = f_\delta * f_{[\delta+1,\alpha)}.$$

Since $r_\delta = r_{\delta+1}$, a is spanned by $E \setminus \mathrm{rge}_M f_{\delta+1}$ in $\mathcal{M}$, so $\mu(f_{\delta+1}) = \mu(f_\delta)$. A straightforward argument by transfinite induction shows that in general

$$\mu(f_\beta) = \mu(f_\delta * f_{[\delta+1,\beta)}) \tag{6}$$

for every β such that $\delta < \beta \leq \alpha$. The only nontrivial step of this inductive argument is when $\beta = \beta' + 1$ is a successor and $f(\beta') \in M_1$. The equality $r_\beta = r_{\beta'}$ implies then that $f(\beta')$ is spanned by $E \setminus \mathrm{rge}_M f_\beta$ and so the inductive step goes through. Applying (6) for $\beta = \alpha$, we get $\mu(f') = \mu(f) = 0$. Since $r_\alpha = 0$ and $\{a\}$ is independent in $\mathcal{M}_1$, it follows that a is not spanned by $E \setminus \mathrm{rge}_M f$ in $\mathcal{M}$, so

$$\mu(f' * [(a, 0)]) = -1.$$

Since $f' * [(a, 0)]$ is saturated and Γ is μ-admissible, this is a contradiction, implying that $f \in \mathfrak{R}'$.

Since $\mathfrak{B}$ was an arbitrary nonempty chain in $\mathfrak{R}'$, it follows from Zorn lemma that $\mathfrak{R}'$ contains a maximal element with respect to $\preceq$, and hence the proof is complete.

(2) Assume that f is maximal in $\mathfrak{R}'$ with respect to $\preceq$. Suppose, by way of contradiction that there is

$$c \in \mathrm{rge}_W f \setminus \mathrm{rge}_M f.$$

Then the string

$$f' = f * [(c, 0)]$$

is saturated and $\mu(f') \leq 0$. Since Γ is μ-admissible, we have $\mu(f') = 0$ implying that $f' \in \mathfrak{R}$. Since f is maximal in $\mathfrak{R}'$ with respect to $\preceq$, it follows that $f' \notin \mathfrak{R}'$ so there is no

$$a \in M_1 \setminus \mathrm{rge}_M f'$$

with $\{a\}$ being independent in $\mathcal{M}_1$. Since $f \in \mathfrak{R}'$, $\{c\}$ is independent in $\mathcal{M}_1$. It follows that c is not spanned by $E \setminus \mathrm{rge}_M f$ in $\mathcal{M}$ and so $\mu(f') = -1$, which is a contradiction.

∎

Lemma 4.2. *Assume that $\mathcal{M}$ is the sum of matroids $\mathcal{M}'$ and $\mathcal{M}''$ (on sets E' and E'' respectively) with $\mathcal{M}'$ having positive finite rank. If Γ is μ-admissible, then there is $a \in E'$ and disjoint sets E_1, E_2 with $E_1 \cup E_2 = E \setminus \{a\}$, such that $\{a\}$ is independent in both $\mathcal{M}$ and $\mathcal{W}$, and both pairs*

$$\Gamma_1 = (\mathcal{M}.E_1, (\mathcal{W}/\{a\})\,|E_1) \quad \textit{and} \quad \Gamma_2 = ((\mathcal{M}/\{a\})\,|E_2, \mathcal{W}.E_2)$$

are μ-admissible.

Proof. Let $\mathfrak{R}'$ be the subset of $\mathfrak{R}$ consisting of strings g such that there is

$$b \in E' \setminus \mathrm{rge}_M g$$

with $\{b\}$ being independent in $\mathcal{M}'$. Let f be a maximal element in $\mathfrak{R}'$ with respect to $\preceq$ and

$$A = \{b \in E' \setminus \mathrm{rge}_M f : \{b\} \text{ is independent in } \mathcal{M}'\}.$$

Let t be a string with

$$\mathrm{rge}\, t = A \times \{0\}$$

and $\mathrm{dom}\, t = \delta + 1$ for some ordinal δ (note that A is nonempty). Since $t(\delta)$ is not spanned by $E \setminus \mathrm{rge}_M(f * t)$ in $\mathcal{M}$, we have

$$\mu(f * t) \leq \mu(f) - 1 < 0.$$

Since Γ is μ-admissible, it follows that $f * t$ is not saturated. Since f is saturated, there is

$$a \in \operatorname{rge}_M t \setminus \operatorname{rge}_W f = A \setminus \operatorname{rge}_W f.$$

Let

$$E_1 = \operatorname{rge}_W f = \operatorname{rge}_M f$$

and

$$E_2 = E \setminus (E_1 \cup \{a\}).$$

Then a is not spanned by E_1 in $\mathcal{W}$, since it follows from the maximality of f that $f * [(a,1)] \notin \mathfrak{R}'$. In particular $\{a\}$ is independent in $\mathcal{W}$. Since $a \in A$, $\{a\}$ is independent in $\mathcal{M}$.

Let g be a saturated string in

$$\Gamma_1 = (\mathcal{M}.E_1, (\mathcal{W}/\{a\})\,|E_1).$$

Then g is a saturated string in Γ so $\mu(g) \geq 0$. Since a is not spanned by E_1 in $\mathcal{W}$ and $\operatorname{rge}_W g \subseteq E_1$, it follows that for any $\beta < \operatorname{dom} g$, if $g(\beta) \in W$, then $g(\beta)$ is spanned by $\operatorname{rge}_W g_\beta$ in $(\mathcal{W}/\{a\})\,|E_1$ if and only if it is spanned by the same set in $\mathcal{W}$. Moreover, for any $\beta < \operatorname{dom} g$, if $g(\beta) \in M$, then it is spanned by $E_1 \setminus \operatorname{rge}_M g_{\beta+1}$ in $\mathcal{M}.E_1$ if and only if it is spanned by $E \setminus \operatorname{rge}_M g_{\beta+1}$ in $\mathcal{M}$. It follows that an argument by transfinite induction can be used to prove that if μ_1 is the μ-function of the pair Γ_1, then $\mu_1(g_\beta) = \mu(g_\beta)$ for every $\beta \leq \operatorname{dom} g$. In particular $\mu_1(g) = \mu(g) \geq 0$, so Γ_1 is μ-admissible.

It remains to show that

$$\Gamma_2 = ((\mathcal{M}/\{a\})\,|E_2, \mathcal{W}.E_2)$$

is μ-admissible. Suppose, by way of contradiction, that it is not. Let g be a saturated string in Γ_2 with $\mu_2(g) < 0$, where μ_2 is the μ-function of the pair Γ_2. Without loss of generality, we can assume that $\mu_2(g) = -1$ and $\mu_2(g_\beta) \geq 0$ for every $\beta < \operatorname{dom} g$. Then

$$h = f * [(a,1)] * g$$

is a saturated string in Γ. Note that if $\beta < \operatorname{dom} g$ and $g(\beta) \in W$, then $g(\beta)$ is spanned by $\operatorname{rge}_W g_\beta$ in $\mathcal{W}.E_2$ if and only if it is spanned by

$$E_1 \cup \{a\} \cup \operatorname{rge}_W g_\beta = \operatorname{rge}_W (f * [(a,1)] * g_\beta)$$

in $\mathcal{W}$. Moreover, if $\beta < \operatorname{dom} g$ and $g(\beta) \in M$, then $g(\beta)$ is spanned by $E_2 \setminus \operatorname{rge}_M g_\beta$ in $(\mathcal{M}/\{a\})|E_2$ if and only if it is spanned by

$$(E_2 \setminus \operatorname{rge}_M g_\beta) \cup \{a\} = E \setminus \operatorname{rge}_M (f * [(a,1)] * g_\beta)$$

in $\mathcal{M}$. It follows that an argument by transfinite induction can be used to prove that

$$\mu(f * [(a,1)] * g_\beta) = \mu(f) + 1 + \mu_2(g_\beta)$$

for every $\beta \le \operatorname{dom} g$. In particular

$$\mu(h) = \mu(f * [(a,1)] * g) = \mu(f) + 1 + \mu_2(g) = \mu(f) + 1 - 1 = 0.$$

Therefore $h \in \mathfrak{R}'$ contradicting the definition of f as a maximal element in $\mathfrak{R}'$ with respect to $\preceq$. Therefore Γ_2 is μ-admissible and the proof is complete. ∎

4.2. **Tree decomposition of Γ.** Now we are ready for the proof of Theorem 2.4. We need some more terminology. By a *binary tree* we will mean a finite set N of finite $0-1$ sequences (including the empty sequence) such that if $n \ge 1$ and $a_1 a_2 \ldots a_n \in N$, then $a_1 a_2 \ldots a_{n-1} \in N$ and $a_1 a_2 \ldots a_{n-1} a'_n \in N$, where $a'_n = 1 - a_n$. An element of a tree N will be called a *vertex* of N. A *leaf* of a tree N is a sequence $a_1 a_2 \ldots a_n \in N$ such that $a_1 a_2 \ldots a_n a_{n+1} \notin N$ for any $a_{n+1} \in \{0,1\}$. An *internal vertex* of a tree N is a vertex of N that is not a leaf.

If $s = a_1 a_2 \ldots a_n$ and $s' = a'_1 a'_2 \ldots a'_m$ are in N, then we say that s is *to the left* of s' (s' is *to the right* of s) if there is i with $1 \le i \le \min(n,m)$ such that $a_i = 0$, $a'_i = 1$, and $a_j = a'_j$ for every j with $1 \le j < i$. If $s', s \in N$ are as above with $n < m$ and $a_i = a'_i$ for every $i = 1, 2, \ldots, n$, then we say that s is *above* s' (s' is *below* s).

A *tree partition* of a set A is a function $\tau : N \to 2^A$ where N is a binary tree, $\tau(s_1) \cap \tau(s_2) = \varnothing$ whenever $s_1 \ne s_2$, and

$$\bigcup_{s \in N} \tau(s) = A.$$

Note that we allow the values of τ to be empty sets.

Suppose $\tau : N \to 2^A$ is a tree partition. If s is a vertex of N, then let L_s be the union of all $\tau(s')$ with s' to the left of s, let R_s be the union of all $\tau(s')$ with s' to the right of s, U_s be the union of all $\tau(s')$ with s' above s, and D_s be the union of $\tau(s)$ and all $\tau(s')$ with s' below s. Note that the sets L_s, R_s, U_s, D_s are pairwise disjoint and

$$L_s \cup R_s \cup U_s \cup D_s = A.$$

Let τ be a tree partition of the set E. With each vertex s of N we associate a pair of matroids $\Gamma_s = (\mathcal{M}_s, \mathcal{W}_s)$ with

$$\mathcal{M}_s = (\mathcal{M} \setminus L_s) .D_s \quad \text{and} \quad \mathcal{W}_s = (\mathcal{W} \setminus R_s) .D_s.$$

We say that τ is a *tree decomposition* of Γ if for every internal vertex s of N, $\tau(s)$ is a singleton that is independent in both $\mathcal{M}_s$ and $\mathcal{W}_s$. We say that a tree decomposition τ of Γ is *μ-admissible* if for every vertex s of N the pair Γ_s is μ-admissible. Note that if $\tau : N \to 2^E$ is the trivial decomposition with N containing only the empty sequence and $\tau(\varnothing) = E$, then τ is μ-admissible if and only if Γ is μ-admissible.

The *internal set* of a tree decomposition $\tau : N \to 2^E$ is the set

$$I_\tau = \{a \in E : \tau(s) = \{a\} \text{ for some internal vertex } s \text{ of } N\}.$$

Lemma 4.3. *If $\tau : N \to 2^E$ is a tree decomposition of Γ, then the internal set of τ is independent in both $\mathcal{M}$ and $\mathcal{W}$.*

Proof. Without loss of generality we can assume that the set $\bar{N}$ of internal vertices is nonempty. Let $<$ be the ordering of $\bar{N}$ defined by $s < s'$ if and only if s is above or to the left of s', and let $s \leq s'$ if and only if $s < s'$ or $s = s'$. It is easy to see that $\leq$ is a linear ordering of $\bar{N}$. Let

$$\bar{N} = \{s_1, s_2, \ldots, s_k\},$$

with

$$s_1 < s_2 < \cdots < s_k.$$

Let $i \in \{1, \ldots, k\}$. Since τ is a tree decomposition, $\tau(s_i)$ is independent in

$$\mathcal{W}_{s_i} = (\mathcal{W} \setminus R_{s_i}) . D_{s_i},$$

which implies that $\tau(s_i)$ is not spanned $U_{s_i} \cup L_{s_i}$ in $\mathcal{W}$. Since

$$\tau(s_1) \cup \tau(s_2) \cup \cdots \cup \tau(s_{i-1}) \subseteq U_{s_i} \cup L_{s_i},$$

it follows that $\tau(s_i)$ is not spanned by the set $\tau(s_1) \cup \tau(s_2) \cup \cdots \cup \tau(s_{i-1})$ in $\mathcal{W}$. This implies that

$$I_\tau = \tau(s_1) \cup \tau(s_2) \cup \cdots \cup \tau(s_k)$$

is independent in $\mathcal{W}$ since otherwise there would be a circuit C in $\mathcal{W}|I_\tau$, and taking the largest possible i with $\tau(s_i) \subseteq C$ we would get $\tau(s_i)$ that is spanned by the set $\tau(s_1) \cup \tau(s_2) \cup \cdots \cup \tau(s_{i-1})$ in $\mathcal{W}$.

The proof that I_τ is independent in $\mathcal{M}$ is similar with the ordering $<$ of $\bar{N}$ defined by $s < s'$ if and only if s is above or to the right of s'. ∎

Assume that $\mathcal{M}$ is the sum of matroids $\mathcal{M}'$ and $\mathcal{M}''$ (on sets E' and E'' respectively) with $\mathcal{M}'$ having finite rank. If $\tau : N \to 2^E$ is a tree decomposition of Γ, then for each vertex s of N, let $L'_s = L_s \cap E'$, $L''_s = L_s \cap E''$, let R'_s, R''_s, U'_s, U''_s, D'_s, D''_s be defined similarly, and let

$$\mathcal{M}'_s = (\mathcal{M}' \setminus L'_s) . D'_s, \quad \mathcal{M}''_s = (\mathcal{M}'' \setminus L''_s) . D''_s.$$

It follows then from Lemma 1.1 that for each vertex s of N, the matroid $\mathcal{M}_s$ is the sum of $\mathcal{M}'_s$ and $\mathcal{M}''_s$. Moreover, let

$$I'_\tau = I_\tau \cap E' \quad I''_\tau = I_\tau \cap E''$$

Lemma 4.4. *Let $\tau : N \to 2^E$ be a tree decomposition of Γ. Then*

$$r(\mathcal{M}') = |I'_\tau| + \sum_{s \in V} r(\mathcal{M}'_s),$$

where V is the set of leaves of N.

Proof. Let $<$ be the ordering on N defined by $s < s'$ if and only if s is above or to the right of s', and let $s \leq s'$ if and only if $s < s'$ or $s = s'$. Let $\tau' : N \to 2^E$ be defined by

$$\tau'(s) = \tau(s) \cap E'$$

if s is an internal vertex of N and let $\xi(s)$ be a base of $\mathcal{M}'_s$ if s is a leaf of N. We claim that $\bigcup_{s\in N} \tau'(s)$ is a base of $\mathcal{M}'$. Since

$$I'_\tau = \bigcup_{s\in \bar{N}} \tau'(s),$$

where $\bar{N}$ is the set of internal vertices of N, the claim implies that

$$r\left(\mathcal{M}'\right) = |I'_\tau| + \sum_{s\in V} r\left(\mathcal{M}'_s\right).$$

It remains to prove the claim. Let

$$N = \left\{s_1, s_2, \dots, s_k\right\},$$

with

$$s_1 < s_2 < \cdots < s_k.$$

We will show, by induction on j, that for every j with $1 \leq j \leq k$ the set $\bigcup_{i=1}^{j} \tau'(s_j)$ is a base of

$$\mathcal{M}| \left(R'_{s_j} \cup U'_{s_j} \cup \tau'\left(s_j\right)\right).$$

Since $s_1 = \varnothing$ and $\tau(\varnothing)$ is independent in $\mathcal{M}'_\varnothing = \mathcal{M}'$, it follows that $\tau'(s_1)$ is a base of $\mathcal{M}'|\left(R'_{s_1} \cup U'_{s_1} \cup \tau'\left(s_1\right)\right)$. Assume that $1 \leq j < k$ and that $\bigcup_{i=1}^{j} \tau'(s_i)$ is a base of

$$\mathcal{M}'| \left(R'_{s_j} \cup U'_{s_j} \cup \tau'\left(s_j\right)\right).$$

Since

$$R'_{s_j} \cup U'_{s_j} \cup \tau'\left(s_j\right) = R'_{s_{j+1}} \cup U'_{s_{j+1}}$$

and $\tau'(s_{j+1})$ is a base of

$$\mathcal{M}'_{s_{j+1}}|\tau'\left(s_{j+1}\right) = \left(\mathcal{M}'| \left(R'_{s_{j+1}} \cup U'_{s_{j+1}} \cup \tau'\left(s_{j+1}\right)\right)\right).\tau'\left(s_{j+1}\right)$$

it follows that $\bigcup_{i=1}^{j+1} \tau'(s_i)$ is a base of

$$\mathcal{M}'| \left(R'_{s_{j+1}} \cup U'_{s_{j+1}} \cup \tau' \left(s_{j+1} \right) \right).$$

Thus the proof is complete. ∎

Let $\tau : N \to 2^E$ and $\tau' : N' \to 2^E$ be tree decompositions of Γ with

$$N' = N \cup \{a_1 a_2 \ldots a_n 0, a_1 a_2 \ldots a_n 1\}$$

where $a_1 a_2 \ldots a_n$ is a fixed leaf of N, and $\tau'(s) = \tau(s)$ for every $s \in N \setminus \{a_1 a_2 \ldots a_n\}$. Then τ' will be called a *one step refinement* of τ. If $\tau' = \tau$ or τ' can be obtained from τ by a finite sequence of one step refinements, then τ' will be called a *refinement* of τ. Note that if $\tau : N \to 2^E$ is a tree decomposition of Γ and $\tau' : N' \to 2^E$ is a refinement of τ, then for every $s \in N$, the matroids $\mathcal{M}_s$ and $\mathcal{W}_s$ do not change when we replace τ with τ'.

Lemma 4.5. *Assume that $\mathcal{M}$ is the sum of matroids $\mathcal{M}'$ and $\mathcal{M}''$ (on sets E' and E'' respectively) with $\mathcal{M}'$ having finite rank. If $\tau : N \to 2^E$ is a μ-admissible tree decomposition of Γ, then there is a μ-admissible refinement τ' of τ such that the internal set $I_{\tau'}$ of τ' contains a base of $\mathcal{M}'$.*

Proof. We will prove the lemma by induction on the number

$$\ell_\tau = r\left(\mathcal{M}'\right) - \left|I'_\tau\right|,$$

where

$$I'_\tau = I_\tau \cap E'.$$

If $\ell_\tau = 0$, then I'_τ is a base of $\mathcal{M}'$ and $\tau' = \tau$ satisfies the requirements. Assume that $\ell_\tau > 0$. Then it follows from Lemma 4.4 that there is a leaf $s = a_1 a_2 \ldots a_n$ of N with $r\left(\mathcal{M}'_s\right) > 0$. Since the pair $(\mathcal{M}_s, \mathcal{W}_s)$ is μ-admissible, it follows from Lemma 4.2 that there is

$$a \in D'_s = \tau(s) \cap E'$$

and disjoint sets E_1, E_2 with $E_1 \cup E_2 = D'_s \setminus \{a\}$, such that $\{a\}$ is independent in both $\mathcal{M}_s$ and $\mathcal{W}_s$, and both pairs

$$\Gamma_1 = (\mathcal{M}_s.E_1, (\mathcal{W}_s/\{a\})|E_1) \quad \Gamma_2 = ((\mathcal{M}_s/\{a\})|E_2, \mathcal{W}_s.E_2)$$

are μ-admissible. Let $s_0 = a_1a_2\dots a_n0$, $s_1 = a_1a_2\dots a_n1$, $N' = N \cup \{s_0, s_1\}$ and $\tau' : N' \to 2^E$ be such that $\tau'(s') = \tau(s')$ for every $s' \in N \setminus \{s\}$, $\tau'(s) = \{a\}$, $\tau'(s_0) = E_1$, and $\tau'(s_1) = E_2$. Note that

$$L_{s_0} = L_s \quad L_{s_1} = L_s \cup E_1 \quad R_{s_0} = R_s \cup E_2 \quad R_{s_1} = R_s \quad D_{s_0} = E_1 \quad D_{s_1} = E_2$$

so

$$\mathcal{M}_{s_0} = (\mathcal{M} \setminus L_{s_0}).D_{s_0} = (\mathcal{M} \setminus L_s).D_{s_0} = ((\mathcal{M} \setminus L_s).D_s).E_1$$

and

$$\mathcal{W}_{s_0} = (\mathcal{W} \setminus R_{s_0}).D_{s_0} = (((\mathcal{W} \setminus R_s).D_s)/\{a\}) \setminus E_2 = (\mathcal{W}_s/\{a\})|E_1.$$

Thus

$$\Gamma_1 = (\mathcal{M}_{s_0}, \mathcal{W}_{s_0}).$$

Similarly,

$$\Gamma_2 = (\mathcal{M}_{s_1}, \mathcal{W}_{s_1}).$$

Since both Γ_1 and Γ_2 are μ-admissible and τ is μ-admissible, it follows that τ' is μ-admissible. Since

$$a \in I'_{\tau'} = I_{\tau'} \cap E',$$

it follows that $\ell_{\tau'} < \ell_\tau$ so the inductive hypothesis implies that τ' has a μ-admissible refinement τ'' with $I_{\tau''}$ containing a base of $\mathcal{M}'$. Thus the proof is complete. ∎

4.3. **Proof of Theorem 2.4.** Let $\mathcal{M}$ be the sum of finite rank matroids $\mathcal{M}_1$, $\mathcal{M}_2$, ... Let τ_0 be the trivial tree decomposition of Γ. Suppose that we have a μ-admissible tree decomposition τ_i of Γ. It follows from Lemma 4.5 that there is a μ-admissible tree decomposition τ_{i+1} of Γ that is a refinement of τ_i such that $I_{\tau_{i+1}}$ contains a base of $\mathcal{M}_{i+1}$. Let

$$I = \bigcup_{i=1}^{\infty} I_{\tau_i}.$$

Then I is spanning in $\mathcal{M}$, and it follows from Lemma 4.3 that any finite subset of I is independent in $\mathcal{W}$. Since $\mathcal{W}$ is finitary, I is independent in $\mathcal{W}$.

5. Remarks

If we remove the restrictions on $\mathcal{M}$ and $\mathcal{W}$ in Theorem 2.4, it becomes obviously false. For example, if both $\mathcal{M}$ and $\mathcal{W}$ are equal to the matroid from Example 1.1, then $\Gamma = (\mathcal{M}, \mathcal{W})$ is μ-admissible, since $\mu(f) = \|\mathrm{rge}_W f\|$ for any saturated string f in Γ. However, Γ is not matchable as any matching in Γ would be a base of $\mathcal{M}$, and $\mathcal{M}$ has no bases. It is natural to ask, how much the restrictions on $\mathcal{M}$ and $\mathcal{W}$ can be relaxed for Theorem 2.4 to remain valid.

Another natural question to ask is whether the nonexistence of a hindrance in $\Gamma = (\mathcal{M}, \mathcal{W})$ implies its μ-admissibility. This implication clearly holds for every pair Γ with $\mathcal{M}$ being SCF and $\mathcal{W}$ being finitary, or more generally, whenever the nonexistence of a hindrance in Γ implies its matchability, since matchability always implies μ-admissibility.

On the other hand, without any restrictions on $\mathcal{M}$ and $\mathcal{W}$, the nonexistence of a hindrance does not imply μ-admissibility. For example, let $\mathcal{W}$ be as in Example 1.1 and $\mathcal{M}$ be discrete, that is, let all subsets of $\mathbb{Z}$ be independent in $\mathcal{M}$. Then there are no hindrances in $\Gamma = (\mathcal{M}, \mathcal{W})$, since if H is independent in $\mathcal{W}$, then it is finite, so $\bar{\partial}_{\mathcal{W}}(H) = H$ and H is spanning in $\mathcal{M}.\big(\bar{\partial}_{\mathcal{W}}(H)\big)$. However, Γ is not μ-admissible. Indeed, let f be the $(\omega+2)$-string in Γ defined as follows. Let $M = \mathbb{Z} \times \{0\}$, $W = \mathbb{Z} \times \{1\}$ and consider $\mathcal{M}$ and $\mathcal{W}$ as matroids on M and W respectively. Let $f(i) = (i, 1)$ for $i = 0, 2, 4, \dots$, $f(i) = (i-1, 0)$ for $i = 1, 3, 5, \dots$, $f(\omega) = (1, 1)$, and $f(\omega+1) = (1, 0)$. Then f is saturated, $\mu(f_\omega) = \mu(f_{\omega+1}) = 0$, and $\mu(f) = -1$.

It would be interesting to know whether the nonexistence of a hindrance implies μ-admissibility when $\mathcal{M}$ and $\mathcal{W}$ are finitary, and if so, then how much this restriction can be relaxed for the implication to remain valid.

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Department of Mathematics, West Wirginia University, Morgantown, WV 26506-6310, USA

E-mail address: `jerzy@math.wvu.edu`