ON THE LENGTH OF SNAKES IN POWERS OF COMPLETE GRAPHS

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ABSTRACT. We prove the conjecture stated in [10] that there is a constant λ (independent from both n and k) such that $S(K_n^d) \geq \lambda n^{d-1}$ holds for every $n \geq 2$ and $d \geq 2$, where $S(K_n^d)$ is the length of the longest snake (cycle without chords) in the Cartesian product K_n^d of d copies of the complete graph K_n.

1. INTRODUCTION

By a *path* in a graph G we mean a sequence of (at least two) distinct vertices of G with every pair of consecutive vertices being adjacent. A path will be called *closed* if its first vertex is adjacent to the last one. By a *chord* of a path P in a graph G we mean an edge of G joining two nonconsecutive vertices of P. If e is a chord in a closed path P, then e is called *proper* if it is not the edge joining the first vertex of P to its last vertex. Note that a proper chord of a closed path corresponds to the standard notion of a chord in a cycle. A *snake* in a graph G is a closed path in G with at least three vertices and without proper chords, and an *open snake* in G is a path in G without chords.

If G and H are graphs, then the *Cartesian product* of G and H is the graph $G \times H$ with $V(G) \times V(H)$ as the vertex set and (g_1, h_1) adjacent to (g_2, h_2) if either $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or else $g_1 = g_2$ and $h_1 h_2 \in E(H)$. Let K_n^d be the product of d copies of the complete graph K_n, $n \geq 2$, $d \geq 2$. It is convenient to think of the vertices of K_n^d as d-tuples of n-ary digits, that is, as the d-tuples of the elements

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of the set $\{0, 1, \ldots, n-1\}$, with edges between two d-tuples differing at exactly one coordinate. Let $S(K_n^d)$ be the length of the longest snake in K_n^d.

The problem of finding a good lower bound for the value of $S(K_n^d)$ has a long history. It was first met by Kautz [7] in the case of $n = 2$ (known in the literature as the *snake-in-the-box problem*) in constructing a type of error-checking code for a certain analog-to-digital conversion systems. He showed that

$$S(K_2^d) \geq \lambda\sqrt{2^d},$$

for some positive constant λ. Several authors improved the lower bound for $S(K_2^d)$ (see the list of references in the papers [2], [8]) until Evdokimov [6], as the first one, obtained a lower bound that is linear in 2^d, that is, he showed that

$$S(K_2^d) \geq \lambda 2^d,$$

where λ is a positive constant. Other shorter proofs of such a bound were given by Abbott and Katchalski [2] and in [8]. The largest value of the constant

$$\lambda = \frac{77}{256} = 0.300781\ldots$$

was obtained by Abbott and Katchalski [4].

The general case of the problem, with an arbitrary value of n, was introduced by Abbott and Dierker [1]. Abbott and Katchalski [5] extended the linear lower bound for $S(K_n^d)$ to all even n, that is, they proved that for every even positive integer n there is a positive constant λ_n such that

$$S(K_n^d) \geq \lambda_n n^d.$$

In [9] it is proved that a similar linear lower bound holds for every odd $n \geq 3$ as well. Therefore, for every integer $n \geq 2$ there is a positive constant λ_n such that

$$S(K_n^d) \geq \lambda_n n^d.$$

In the results above, the constant λ_n is dependent on n and approaches 0 as $n \longrightarrow \infty$. For example, the result proved in [9] says that for any odd integer $n \geq 3$, and any

$d \geq 5$

$$S(K_n^d) \geq 2\,(n-1)\,n^{d-4},$$

so

$$\lambda_n = \frac{2\,(n-1)}{n^4}.$$

Actually, obtaining a linear lower bound with the coefficient independent from both n and d is not possible since Abbott and Katchalski [3] proved the following upper bound

$$S(K_n^d) \leq \left(1 + \frac{1}{d-1}\right) n^{d-1}.$$

A natural question is whether there is a positive constant λ, that is independent from both n and d, such that

$$S(K_n^d) \geq \lambda n^{d-1}.$$

In [10] it is conjectured that the answer to the above question is positive and the following partial result is proved.

Theorem 1. *Let P be a finite set of primes. Then there is a positive constant λ_P such that*

$$S(K_n^d) \geq \lambda_P n^{d-1}$$

for any integer n that is divisible by an element of P.

However, the constant λ_P approaches 0 as $\max P \longrightarrow \infty$, so the conjecture remains open. In this paper we prove the following result that affirms the conjecture.

Theorem 2. *There is a positive constant λ such that*

$$S(K_n^d) \geq \lambda n^{d-1}$$

holds for every $n \geq 2$ and $d \geq 2$.

Since $S(K_n^2) = 2n$ (see Abbott and Dierker [1]) and

$$S(K_n^3) = \frac{3}{2}n^2 + O(n)$$

as proved by Abbott and Katchalski in [3], it is enough to prove Theorem 2 for $d \geq 4$. Because of Theorem 1, it is enough to consider the case of n being odd. Theorem 2 will follow after we prove the following result.

Theorem 3. *Assume that $n \geq 9$ is odd and $d \geq 4$. Then*

$$S(K_n^d) \geq 4 \lfloor n/4 \rfloor \lfloor n/2 \rfloor n^{d-3}.$$

The proof of Theorem 3 will be given in section 5.

2. Basic Definitions

An m-path in a graph is a path containing m vertices, that is, it is a path of length $m-1$. If P is an m-path, then we will write $m = |P|$. A *chain* $\mathcal{C}$ in a graph G is a sequence $\mathcal{C} = (P_1, \ldots, P_m)$ of (at least two) paths in G such that the last vertex of P_i is equal to the first vertex of P_{i+1}, $i = 1, 2, \ldots, m-1$. When the number m of paths in a chain needs to be specified, we shall refer to it as an *m-chain.* An m-chain $(P_i)_{i=1}^m$ will be called *closed* if the first vertex of P_1 is equal to the last vertex of P_m.

Now we are going to define an important operation that will be used throughout the paper. Given an m-path $P = (g_i)_{i=1}^m$ in a graph G and an m-chain $\mathcal{C} = (P_i)_{i=1}^m$ in a graph H, let $P \otimes \mathcal{C}$ be the $(\sum_{i=1}^m |P_i|)$-path in $G \times H$ obtained as follows. For each $i = 1, \ldots, m$, if $P_i = (h_1, h_2, \ldots, h_k)$, then let P_i' be the $|P_i|$-path in $G \times H$ given by

$$P_i' = ((g_i, h_1), (g_i, h_2), \ldots, (g_i, h_k)).$$

Note that for each $i = 1, \ldots, m-1$ the last vertex of the path P_i' is adjacent in $G \times H$ to the first vertex of the path P_{i+1}'. Let $P \otimes \mathcal{C}$ be the path obtained by joining together (juxtaposing) the paths P_1', P_2', $\ldots$, P_m'. We will say that $P \otimes \mathcal{C}$ is the path *generated* by P and $\mathcal{C}$. Note that the path generated by a closed path and a closed chain is a closed path.

If $\mathcal{D}$ is a km-chain in a graph H, with $k, m \geq 2$, then the *m-splitting* of $\mathcal{D}$ is the sequence $(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m)$ of k-chains in H which joined together (juxtaposed) give $\mathcal{D}$. The above definition of the operation $\otimes$ can be generalized to the following situation. Let $\mathcal{C} = (P_i)_{i=1}^m$ be an m-chain of k-paths in a graph G, let $\mathcal{D}$ be a km-chain in H, and let $(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m)$ be the m-splitting of $\mathcal{D}$. Note that for each $i \in \{1, 2, \ldots, m-1\}$ the last vertex of the path $P_i \otimes \mathcal{D}_i$ in the graph $G \times H$ is equal to the first vertex of the path $P_{i+1} \otimes \mathcal{D}_{i+1}$. Let $\mathcal{C} \otimes \mathcal{D}$ be the chain in $G \times H$ given by

$$\mathcal{C} \otimes \mathcal{D} = (P_1 \otimes \mathcal{D}_1, P_2 \otimes \mathcal{D}_2, \ldots, P_m \otimes \mathcal{D}_m)\,.$$

We will say that $\mathcal{C} \otimes \mathcal{D}$ is the chain generated by $\mathcal{C}$ and $\mathcal{D}$. Note that the chain generated by two closed chains is also a closed chain. It is straightforward to verify that the operation $\otimes$ is associative in the following sense.

Proposition 4. *If P is an m-path in a graph G, $\mathcal{C}$ is an m-chain of k-paths in a graph H, and $\mathcal{D}$ is a km-chain in a graph J, then*

$$(P \otimes \mathcal{C}) \otimes \mathcal{D} = P \otimes (\mathcal{C} \otimes \mathcal{D})\,.$$

When we refer to a pair s_i and s_j of elements of a sequence $(s_1, s_2, \ldots, s_t)$, we say that they are *consecutive* if $j = i \pm 1$, and that they are *cyclically consecutive* if either $j = i \pm 1$ or $\{i, j\} = \{1, t\}$.

Let $\mathcal{C} = (P_i)_{i=1}^m$ be a chain in a graph G. We say that $\mathcal{C}$ is *openly separated* if any two paths of $\mathcal{C}$ have exactly one vertex in common when they are consecutive, and they are vertex disjoint otherwise. We say that $\mathcal{C}$ is *closely separated* if $\mathcal{C}$ is closed, any two paths of $\mathcal{C}$ have exactly one vertex in common when they are cyclically consecutive, and they are vertex disjoint otherwise.

The following statement is an easy consequence of the definitions.

Proposition 5. *Let P be a path in a graph G and $\mathcal{C}$ be a $|P|$-chain of open snakes in a graph H.*

(1) *If $\mathcal{C}$ is openly separated, then $P \otimes \mathcal{C}$ is an open snake in $G \times H$.*
(2) *If P is closed and $\mathcal{C}$ is closely separated, then $P \otimes \mathcal{C}$ is a snake in $G \times H$.*

If P is a path, then let $-P$ be the path obtained from P by reversing the order of vertices, and if $\mathcal{C} = (P_i)_{i=1}^m$ is a chain, then let

$$-\mathcal{C} = (-P_m, -P_{m-1}, \ldots, -P_1)$$

be the chain obtained from $\mathcal{C}$ by reversing the order of paths and reversing every path. The expression $(-1)^i X$, where X is a path or a chain, will mean X for i even and $-X$ for i odd. Obviously, the following statement is true.

Proposition 6. *If P is an m-path in a graph G and $\mathcal{C}$ is an m-chain in a graph H, then*

$$(-P) \otimes \mathcal{C} = -\left(P \otimes (-\mathcal{C})\right).$$

Let $\mathcal{C}$ be a km-chain of paths, and let $\mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ be the m-splitting of $\mathcal{C}$. Consider the following $(m \times k)$-matrix of paths:

$$\mathcal{A} = \left(\begin{array}{c} \mathcal{C}_1 \\ -\mathcal{C}_2 \\ \vdots \\ (-1)^{m-1}\,\mathcal{C}_m \end{array} \right) = \left(\begin{array}{cccc} Q_1^1 & Q_1^2 & \ldots & Q_1^k \\ Q_2^1 & Q_2^2 & \ldots & Q_2^k \\ \vdots & \vdots & & \vdots \\ Q_m^1 & Q_m^2 & \ldots & Q_m^k \end{array} \right)$$

where $(Q_i^1, Q_i^2, \ldots, Q_i^k)$ is the sequence of paths forming the k-chain $(-1)^{i-1}\mathcal{C}_i$. We will call $\mathcal{A}$ the *alternating matrix* of $\mathcal{S}$. The splitting $\mathcal{S}$ will be called *openly alternating* if for any $\ell \in \{1, 2, \ldots, k\}$ and for any two distinct paths Q_i^ℓ, Q_j^ℓ appearing in the ℓ-th column of $\mathcal{A}$, the paths Q_i^ℓ, Q_j^ℓ have exactly one vertex in common when they are consecutive in $\mathcal{C}$ and they are vertex disjoint otherwise.

Assume now that the km-chain $\mathcal{C}$ is closed and m is even. Then, we say that the splitting $\mathcal{S}$ is *closely alternating* if for any $\ell \in \{1, 2, \ldots, k\}$ and for any two distinct paths Q_i^ℓ, Q_j^ℓ appearing in the ℓ-th column of $\mathcal{A}$, the paths Q_i^ℓ, Q_j^ℓ have exactly one vertex in common when they are cyclically consecutive in $\mathcal{C}$ and they are vertex disjoint otherwise. The following statement is an easy consequence of the definitions.

Proposition 7. *Let P be a k-path in a graph G, and let $\mathcal{D}$ be a km-chain in a graph H.*

1. *If $\mathcal{C}$ is the m-chain $(P, -P, \ldots, (-1)^{m-1}P)$, and the m-splitting of $\mathcal{D}$ is openly alternating, then the m-chain $\mathcal{C} \otimes \mathcal{D}$ in $G \times H$ is openly separated.*

(2) *If m is even, $\mathcal{C}$ is the closed m-chain $(P, -P, P, -P, \dots, -P)$, and the m-splitting of $\mathcal{D}$ is closely alternating, then the closed m-chain $\mathcal{C} \otimes \mathcal{D}$ in $G \times H$ is closely separated.*

Let $n \geq 3$ be an odd integer. Let H be a graph, $r \geq 1$ be an integer, $\mathcal{C}$ be an n^r-chain of paths in H, and $\mathcal{D}$ be an $(n-1)\,n^r$-chain of paths in H. We say that $\mathcal{C}$ is *openly well distributed* if either $r = 1$ and $\mathcal{C}$ is an openly separated chain of open snakes, or $r \geq 2$, every chain $\mathcal{C}_i$ in the n-splitting $\mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$ of $\mathcal{C}$ is openly well distributed and $\mathcal{S}$ is openly alternating. We also say that $\mathcal{D}$ is *closely well distributed* if every chain $\mathcal{D}_i$ in the $(n-1)$-splitting $\mathcal{S}' = (\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{\bar{n}})$ of $\mathcal{D}$ is openly well distributed and $\mathcal{S}'$ is closely alternating. The following property can be proved by a straightforward induction with respect to r.

Proposition 8. *If $\mathcal{C}$ is an openly well distributed n^r-chain of paths in a graph H, then the chain $-\mathcal{C}$ is also openly well distributed.*

We are going now to define inductively, for each $d = 1, 2, \dots$, an n^r-path π_n^d in K_n^r and a closed $(n-1)\,n^r$-path γ_n^{r+1} in K_n^{r+1}. Let

$$\begin{aligned} \pi_n^1 &= (0, 1, \dots, n-1), \\ \pi_n^{r+1} &= \pi_n^1 \otimes (\pi_n^r, -\pi_n^r, \pi_n^r, -\pi_n^r, \dots, \pi_n^r) \end{aligned}$$

and

$$\gamma_n^{r+1} = \gamma_n \otimes (\pi_n^r, -\pi_n^r, \pi_n^r, -\pi_n^r, \dots, -\pi_n^r)\,,$$

where γ_n is the closed $(n-1)$-path $(0, 1, \dots, n-2)$ in K_n.

The following lemmas are proved in [9].

Lemma 9. *If $r \geq 1$ and $\mathcal{C}$ is an openly well distributed n^r-chain in a graph H, then the path $\pi_n^r \otimes \mathcal{C}$ is an open snake in the graph $K_n^r \times H$.*

Lemma 10. *If $r \geq 1$ and $\mathcal{C}$ is a closely well distributed $(n-1)\,n^r$-chain in a graph H, then the path $\gamma_n^{r+1} \otimes \mathcal{C}$ is a snake in the graph $K_n^{r+1} \times H$.*

3. A Family of Snakes in K_n^2

Assume that $n \geq 9$ is odd. Let $m = (n+1)/2$ and $k = \lfloor n/4 \rfloor - 1$. For each $t \in \{0, 1, \ldots, n-1\}$, each $i \in \{n-3, n-2, n-1\}$, and each

$$\alpha, \beta \in \{0, 1\}$$

let $C_i^{\alpha t \beta}$ be the open snake in K_n^2, with $2k+2 = 2\lfloor n/4 \rfloor$ vertices, defined by

$$\begin{aligned} C_i^{\alpha t \beta} = & \left(\left(\overline{t-\alpha}, i \right), \left(\overline{t-\alpha}, \overline{\overline{1-t}} \right), \left(\overline{t+1}, \overline{\overline{1-t}} \right), \left(\overline{t+1}, \overline{\overline{2-t}} \right), \left(\overline{t+2}, \overline{\overline{2-t}} \right), \right. \\ & \left(\overline{t+2}, \overline{\overline{3-t}} \right), \left(\overline{t+3}, \overline{\overline{3-t}} \right), \ldots, \left(\overline{t+k-3}, \overline{\overline{k-2-t}} \right), \\ & \left(\overline{t+k-2}, \overline{\overline{k-2-t}} \right), \left(\overline{t+k-2}, \overline{\overline{k-1-t}} \right), \left(\overline{t+k-1}, \overline{\overline{k-1-t}} \right), \\ & \left. \left(\overline{t+k-1}, \overline{\overline{k-t}} \right), \left(\overline{t+k+\beta}, \overline{\overline{k-t}} \right), \left(\overline{t+k+\beta}, n-4 \right) \right), \end{aligned}$$

where $\overline{x} = x \bmod n$ and $\overline{\overline{x}} = x \bmod m$.

For example, if $n = 17$, then $m = 9$, $k = 3$,

$$\begin{aligned} C_{14}^{0\,0\,0} &= ((0,14), (0,1), (1,1), (1,2), (2,2), (2,3), (3,3), (3,13)), \\ C_{14}^{0\,1\,0} &= ((1,14), (1,0), (2,0), (2,1), (3,1), (3,2), (4,2), (4,13)), \\ C_{14}^{0\,2\,0} &= ((2,14), (2,8), (3,8), (3,0), (4,0), (4,1), (5,1), (5,13)), \\ &\vdots \\ C_{14}^{0\,16\,0} &= ((16,14), (16,2), (0,2), (0,3), (1,3), (1,4), (2,4), (2,13)), \end{aligned}$$

and

$$\begin{aligned} C_{14}^{0\,2\,1} &= ((2,14), (2,7), (3,7), (3,0), (4,0), (4,1), (6,1), (6,13)), \\ C_{14}^{1\,2\,0} &= ((1,14), (1,7), (3,7), (3,0), (4,0), (4,1), (5,1), (5,13)), \\ C_{14}^{1\,2\,1} &= ((1,14), (1,7), (3,7), (3,0), (4,0), (4,1), (6,1), (6,13)). \end{aligned}$$

Lemma 11. *Let $t, s \in \{0, 1, \ldots, n-1\}$ with $t < s$, let $i, j \in \{n-3, n-2, n-1\}$, and let*

$$\alpha, \beta, \gamma, \delta \in \{0, 1\}.$$

If u is a vertex of $C_i^{\alpha t\beta}$ and v is a vertex of $C_j^{\gamma s\delta}$, then $u \neq v$ except in the following cases:

(1) *$s = t+1$, $\alpha = 0$, $\gamma = 1$, $i = j$, and u, v are the first vertices of $C_i^{\alpha t\beta}$ and $C_j^{\gamma s\delta}$, respectively;*

(2) *$t = 0$, $s = n-1$, $\alpha = 1$, $\gamma = 0$, $i = j$, and u, v are the first vertices of $C_i^{\alpha t\beta}$ and $C_j^{\gamma s\delta}$, respectively;*

(3) *$s = t+1$, $\beta = 1$, $\delta = 0$, and u, v are the last vertices of $C_i^{\alpha t\beta}$ and $C_j^{\gamma s\delta}$, respectively;*

(4) *$t = 0$, $s = n-1$, $\beta = 0$, $\delta = 1$, and u, v are the last vertices of $C_i^{\alpha t\beta}$ and $C_j^{\gamma s\delta}$, respectively.*

Proof. Assume that $u = v = (a, b)$. Since $m \leq n-4$, exactly one of the following conditions holds

(a) $b \in \{n-3, n-2, n-1\}$;
(b) $b = n-4$;
(c) $b \in \{0, 1, \ldots, m-1\}$.

If (a) holds, then u is the first vertex of $C_i^{\alpha t\beta}$ and v is the first vertex of $C_j^{\gamma s\delta}$. Since both i and j are equal to b, we have $i = j$. It is clear that $s = t \pm 1 \bmod n$, and since $t < s$, we must have either $s = t+1$ or $(t, s) = (0, n-1)$. Since

$$a = (t - \alpha) \bmod n = (s - \gamma) \bmod n,$$

we must have $\alpha = 0$, $\gamma = 1$ when $s = t+1$ and $\alpha = 1$, $\gamma = 0$ when $(t, s) = (0, n-1)$. Thus one of the conditions 1 or 2 above must hold.

If (b) holds, then u is the last vertex of $C_i^{\alpha t\beta}$ and v is the last vertex of $C_j^{\gamma s\delta}$ and similar analysis as above shows that one of the conditions 3 or 4 above holds.

If (c) holds, then neither u nor v is the first or the last vertex of the corresponding path. We will show that this assumption leads to a contradiction. From the definition of $C_i^{\alpha t\beta}$ we have

$$\begin{aligned} a &\in \left\{\overline{t-\alpha}, \overline{t+k+\beta}\right\} \cup \left\{\overline{t+1}, \overline{t+2}, \dots, \overline{t+k-1}\right\} \\ &\subseteq \left\{\overline{t-1}, t, \overline{t+1}, \overline{t+2}, \dots, \overline{t+k-1}, \overline{t+k}, \overline{t+k+1}\right\}. \end{aligned}$$

Therefore

$$t \in \left\{\overline{a+1}, a, \overline{a-1}, \dots, \overline{a-k-1}\right\}.$$

Let us consider how the value of b depends on the value of t. Notice that

- $b = \overline{\overline{1-t}}$ when $a = \overline{t-\alpha}$, that is when $t \in \left\{\overline{a+1}, a\right\}$;
- $b = \overline{\overline{k-t}}$ when $a = \overline{t+k+\beta}$, that is when $t \in \left\{\overline{a-k}, \overline{a-k-1}\right\}$; and
- $b = \overline{\overline{\ell-t}}$ or $b = \overline{\overline{\ell+1-t}}$ when $a = \overline{t+\ell}$, that is when $t = \overline{a-\ell}$ for any $\ell = 1, 2, \dots, k-1$.

Therefore, given the value of t, the value of b is as in the following table:

t	b
$\overline{a+1}$	$\overline{\overline{1-\overline{a+1}}}$
a	$\overline{\overline{1-a}}$
$\overline{a-1}$	$\overline{\overline{1-\overline{a-1}}}$ or $\overline{\overline{2-\overline{a-1}}}$
$\overline{a-2}$	$\overline{\overline{2-\overline{a-2}}}$ or $\overline{\overline{3-\overline{a-2}}}$
$\vdots$	$\vdots$
$\overline{a-k+1}$	$\overline{\overline{k-1-\overline{a-k+1}}}$ or $\overline{\overline{k-\overline{a-k+1}}}$
$\overline{a-k}$	$\overline{\overline{k-\overline{a-k}}}$
$\overline{a-k-1}$	$\overline{\overline{k-\overline{a-k-1}}}$

For example, take $a = 1$ and consider how the above table looks when $n = 25$ ($m = 13$, $k = 5$) — table on the left below, and $n = 27$ ($m = 14$, $k = 5$) — table on the right below.

t	b
2	12
1	0
0	1 or 2
24	4 or 5
23	6 or 7
22	8 or 9
21	10
20	11

t	b
2	13
1	0
0	1 or 2
26	4 or 5
25	6 or 7
24	8 or 9
23	10
22	11

If a is such that the set $\left\{\overline{a+1}, a, \overline{a-1}, \dots, \overline{a-k-1}\right\}$ does not contain both 0 and $n-1$, then the possible values of b in the table above range from $\overline{\overline{1-\overline{a+1}}}$ to $\overline{\overline{k-\overline{a-k-1}}}$ through consecutive numbers modulo m. There are $2k+2$ numbers in such a sequence. Since $2k+2 \le m$, they are all distinct.

If the set $\left\{\overline{a+1}, a, \overline{a-1}, \dots, \overline{a-k-1}\right\}$ contains both 0 and $n-1$, then the possible values of b in the table above range from $\overline{\overline{1-\overline{a+1}}}$ to $\overline{\overline{k-\overline{a-k-1}}}$ through consecutive numbers modulo m except for one. Since $2k+2 \le m-1$, again all the possible values for b in the table above are distinct.

It follows that the values of a and b determine the value t in a unique way. It follows that $s = t$ contradicting the assumption. Thus the proof is complete. ∎

4. An Openly Well Distributed n^r-chain in K_n^2

Let $n \ge 9$ be an odd integer and $\mathcal{M}$ be the set of all open snakes $C_i^{\alpha t \beta}$ as defined in the previous section. For any $s \in \{0, 1, \dots, n-1\}$, let $\sigma^s : \mathcal{M} \to \mathcal{M}$ be defined by

$$\sigma^s(C_i^{\alpha t \beta}) = C_i^{\alpha\, \overline{t+s}\, \beta},$$

where $\overline{x} = x \bmod n$. Let

$$-\mathcal{M} = \{-P : P \in \mathcal{M}\}$$

and

$$\overline{\mathcal{M}} = \mathcal{M} \cup (-\mathcal{M})\,.$$

We can extend σ^s to be a map $\sigma^s : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ by setting $\sigma^s(-P) = -\sigma^s(P)$.

If $\mathcal{C}$ is a chain in K_n^2 consisting of paths from $\overline{\mathcal{M}}$, then we will say that $\mathcal{C}$ is *$\overline{\mathcal{M}}$-built*. If $\mathcal{C}$ is $\overline{\mathcal{M}}$-built, then let $\sigma^s(\mathcal{C})$ be the chain obtained by applying σ^s to each path of $\mathcal{C}$. The following proposition can be proved by a straightforward induction on the length of $\mathcal{C}$ using Lemma 11.

Proposition 12. *If $\mathcal{C}$ is an $\overline{\mathcal{M}}$-built openly well distributed chain and $s \in \{0, 1, \ldots, n-1\}$, then the chains $\sigma^s(\mathcal{C})$ and $-\sigma^s(\mathcal{C})$ are also openly well distributed.*

We say that α is the *upper begin* (*upper end*) and i is the *lower begin* (*lower end*) of the path $C_i^{\alpha t \beta}$ (the path $-C_i^{\alpha t \beta}$), and that β is the *upper end* (*upper begin*) and $n-4$ is the *lower end* (*lower begin*) of $C_i^{\alpha t \beta}$ (of $-C_i^{\alpha t \beta}$). Given an $\overline{\mathcal{M}}$-built chain $\mathcal{C}$, the *upper begin* (*lower begin*) of $\mathcal{C}$ is the upper begin (lower begin) of the first path of $\mathcal{C}$ and the *upper end* (*lower end*) of $\mathcal{C}$ is the upper end (lower end) of the last path of $\mathcal{C}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be $\overline{\mathcal{M}}$-built ℓ-chains. We say that $\mathcal{C}$ and $\mathcal{D}$ are *internally compatible* if they are the same except possibly for the upper begin and the upper end, that is, if the following conditions hold:

(1) for every $p \in \{2, \ldots, \ell - 1\}$ the p-th path of $\mathcal{C}$ is the same as the p-th path of $\mathcal{D}$;
(2) if the first path of $\mathcal{C}$ is $C_i^{\alpha t \beta}$, then the first path of $\mathcal{D}$ is $C_i^{\alpha' t \beta}$ for some $\alpha' \in \{0, 1\}$;
(3) if the first path of $\mathcal{C}$ is $-C_i^{\alpha t \beta}$, then the first path of $\mathcal{D}$ is $-C_i^{\alpha t \beta'}$ for some $\beta' \in \{0, 1\}$;
(4) if the last path of $\mathcal{C}$ is $C_i^{\alpha t \beta}$, then the last path of $\mathcal{D}$ is $C_i^{\alpha t \beta'}$ for some $\beta' \in \{0, 1\}$;
(5) if the last path of $\mathcal{C}$ is $-C_i^{\alpha t \beta}$, then the last path of $\mathcal{D}$ is $-C_i^{\alpha' t \beta}$ for some $\alpha' \in \{0, 1\}$.

The following lemma, together with Lemma 9 will allow for a construction of long open snakes in powers of K_n.

Lemma 13. *For every integer $r \geq 1$, there is an $\overline{\mathcal{M}}$-built n^r-chain $\mathcal{N}_r$ such that:*

(1) *any chain that is internally compatible with* $\mathcal{N}_r$ *is openly well distributed;*
(2) *the first path of* $\mathcal{N}_r$ *is* $C^{0\,0\,1}_{n-1}$*;*
(3) *if* P *is the* p*-th path of* $\mathcal{N}_r$ *then* $P \in (-1)^{p-1}\,\mathcal{M}$*.*

Proof. Let

$$\mathcal{N}_1 = \left(C^{0\,0\,1}_{n-1}, -C^{0\,1\,0}_{n-1}, C^{1\,2\,1}_{n-1}, -C^{0\,3\,0}_{n-1}, C^{1\,4\,1}_{n-1}, -C^{0\,5\,0}_{n-1}, \ldots, C^{1\,(n-3)\,1}_{n-1}, -C^{0\,(n-2)\,0}_{n-1}, C^{1\,(n-1)\,0}_{n-1}\right).$$

It is straightforward to verify, using Lemma 11, that any chain that is internally compatible with $\mathcal{N}_1$ is openly separated and so it is openly well distributed. It is clear that the remaining conditions are also satisfied.

Assume that $r \geq 2$, and that $\mathcal{N}_{r-1}$ is an $\overline{\mathcal{M}}$-built n^{r-1}-chain satisfying the required conditions. For any $\alpha, \beta \in \{0,1\}$ let $\mathcal{N}^{\alpha\beta}_{r-1}$ be the chain that is internally compatible with $\mathcal{N}_{r-1}$ and has upper begin α and upper end β. Let $\mathcal{N}_r$ be the $\overline{\mathcal{M}}$-built n^r-chain with the following n-splitting

$$\mathcal{S}_r = \left(\sigma^0\left(\mathcal{N}^{0\,1}_{r-1}\right), -\sigma^1\left(\mathcal{N}^{0\,0}_{r-1}\right), \sigma^2\left(\mathcal{N}^{1\,1}_{r-1}\right), -\sigma^3\left(\mathcal{N}^{0\,0}_{r-1}\right), \ldots, -\sigma^{n-2}\left(\mathcal{N}^{0\,0}_{r-1}\right), \sigma^{n-1}\left(\mathcal{N}^{1\,0}_{r-1}\right)\right).$$

Let $\mathcal{N}^{\alpha\beta}_r$ be the chain that is internally compatible with $\mathcal{N}_r$ and has upper begin α and upper end β. Let $\mathcal{S}^{\alpha\beta}_r$ be the n-splitting of $\mathcal{N}^{\alpha\beta}_r$. By the inductive hypothesis and Proposition 12, every chain of $\mathcal{S}^{\alpha\beta}_r$ is openly well distributed. To prove that $\mathcal{N}^{\alpha\beta}_r$ is openly well distributed, it remains to show that $\mathcal{S}^{\alpha\beta}_r$ is openly alternating.

Let

$$\mathcal{A}^{\alpha\beta}_r = \begin{pmatrix} \sigma^0\left(\mathcal{N}^{\alpha\,1}_{r-1}\right) \\ \sigma^1\left(\mathcal{N}^{0\,0}_{r-1}\right) \\ \sigma^2\left(\mathcal{N}^{1\,1}_{r-1}\right) \\ \vdots \\ \sigma^{n-2}\left(\mathcal{N}^{0\,0}_{r-1}\right) \\ \sigma^{n-1}\left(\mathcal{N}^{1\,\beta}_{r-1}\right) \end{pmatrix} = \begin{pmatrix} Q^1_0 & Q^2_0 & \ldots & Q^{n^{r-1}}_0 \\ Q^1_1 & Q^2_1 & \ldots & Q^{n^{r-1}}_1 \\ Q^1_2 & Q^2_2 & \ldots & Q^{n^{r-1}}_2 \\ \vdots & \vdots & & \vdots \\ Q^1_{n-2} & Q^2_{n-2} & \ldots & Q^{n^{r-1}}_{n-2} \\ Q^1_{n-1} & Q^2_{n-1} & \ldots & Q^{n^{r-1}}_{n-1} \end{pmatrix}$$

be the alternating matrix of $\mathcal{S}^{\alpha\beta}_r$. Let $\ell \in \{2, 3, \ldots, n^{r-1}-1\}$ and Q^ℓ_t, Q^ℓ_s be distinct paths appearing in the ℓ-th column of $\mathcal{A}^{\alpha\beta}_r$. If $Q^\ell_t = \pm C^{\gamma t \delta}_i$ then $Q^\ell_s = \pm C^{\gamma s \delta}_i$, so it follows from Lemma 11 that the paths Q^ℓ_t, Q^ℓ_s are vertex disjoint. If $\ell \in \{1, n^{r-1}\}$

and t, s are not cyclically consecutive in the sequence $\{0, 1, \ldots, n-1\}$, then again it follows from Lemma 11 that the paths Q_t^ℓ, Q_s^ℓ are vertex disjoint.

Assume now that $\ell = 1$ and $s = (t+1) \bmod n$. Then $Q_t^\ell, Q_s^\ell \in \mathcal{M}$. If t is even with $t \neq n-1$, then the upper begin of Q_s^ℓ is 0 while the upper ends of Q_t^ℓ and Q_s^ℓ are the same. It follows from Lemma 11 that the paths Q_t^ℓ, Q_s^ℓ are vertex disjoint. If $t = n-1$, then the upper begin of Q_t^ℓ is 1 while the upper ends of Q_t^ℓ and Q_s^ℓ are the same. It follows again from Lemma 11 that the paths Q_t^ℓ, Q_s^ℓ are vertex disjoint. If t is odd, then the upper begin of Q_t^ℓ is 0, the upper begin of Q_s^ℓ is 1, and the upper ends of Q_t^ℓ and Q_s^ℓ are the same. It follows from Lemma 11 that the paths Q_t^ℓ, Q_s^ℓ have exactly one vertex in common which is the first vertex of both of them.

If $\ell = n^{r-1}$ and $s = (t+1) \bmod n$, then a similar argument shows that the paths Q_t^ℓ, Q_s^ℓ have exactly one vertex in common when they are consecutive in $\mathcal{N}_r^{\alpha\beta}$ and they are vertex disjoint otherwise.

It is clear that the remaining required conditions are satisfied, so the proof is complete. ∎

5. Proof of Theorem 3

We need to extend our definition of the path $C_i^{\alpha t \beta}$ to the case when $\alpha = 2$, $t = 0$, $\beta = 1$, and $i = n-2$. Recall that $m = (n+1)/2$ and $k = \lfloor n/4 \rfloor - 1$. Let $C_{n-2}^{2\,0\,1}$ be the open snake in K_n^2 with $2k+2 = 2\lfloor n/4 \rfloor$ vertices defined by

$$\begin{aligned} C_{n-2}^{2\,0\,1} \;=\; & ((n-2, n-2), (n-2, 1), (1,1), (1,2), (2,2), (2,3), (3,3), \ldots, (k-2, k-2), \\ & (k-2, k-1), (k-1, k-1), (k-1, k), (k+1, k), (k+1, n-4)). \end{aligned}$$

Note that this definition is exactly what you get taking $\alpha = 2$, $t = 0$, $\beta = 1$, $i = n-2$ and using the general formula defining $C_i^{\alpha t \beta}$.

Since we have $m > 2k+2$, the following modification of Lemma 11 holds.

Lemma 14. *Let $s \in \{1, 2, \ldots, n-1\}$, let $j \in \{n-3, n-2, n-1\}$, and let $\gamma, \delta \in \{0, 1\}$. If u is a vertex of $C_{n-2}^{2\,0\,1}$ and v is a vertex of $C_j^{\gamma s \delta}$, then $u \neq v$ except in the following cases:*

(1) $s = n-2$, $\gamma = 0$, $j = n-2$, *and* u, v *are the first vertices of* $C^{2\,0\,0}_{n-2}$ *and* $C^{\gamma s \delta}_{j}$, *respectively;*

(2) $s = n-1$, $\gamma = 1$, $j = n-2$, *and* u, v *are the first vertices of* $C^{2\,0\,0}_{n-2}$ *and* $C^{\gamma s \delta}_{j}$, *respectively;*

(3) $s = 1$, $\delta = 0$, *and* u, v *are the last vertices of* $C^{2\,0\,0}_{n-2}$ *and* $C^{\gamma s \delta}_{j}$, *respectively.*

Let $\mathcal{N}_{d-3}$ be an $\overline{\mathcal{M}}$-built n^{d-3}-chain satisfying Lemma 13. For any $\alpha, \beta \in \{0,1\}$ let $\mathcal{N}^{\alpha\beta}_{d-3}$ be the chain that is internally compatible with $\mathcal{N}_{d-3}$ and has upper begin α and upper end β. We also need to define two extra chains. Let $\mathcal{N}'_{d-3}$ be obtained from $\mathcal{N}^{0\,1}_{d-3}$ by replacing its first path $C^{0\,0\,1}_{n-1}$ with the path $C^{2\,0\,1}_{n-2}$, and let $\mathcal{N}''_{d-3}$ be the chain obtained from $\mathcal{N}^{0\,0}_{d-3}$ by replacing its first path $C^{0\,0\,1}_{n-1}$ with the path $C^{0\,0\,1}_{n-2}$. Let $\mathcal{C}$ be the $\overline{\mathcal{M}}$-built n^{d-2}-chain with the following n-splitting

$$\mathcal{S} = \left(\mathcal{N}'_{d-3}, -\sigma^1\left(\mathcal{N}^{0\,0}_{d-3}\right), \sigma^2\left(\mathcal{N}^{1\,1}_{d-3}\right), -\sigma^3\left(\mathcal{N}^{0\,0}_{d-3}\right), \ldots, \sigma^{n-3}\left(\mathcal{N}^{1\,1}_{d-3}\right), -\sigma^{n-2}\left(\mathcal{N}''_{d-3}\right)\right).$$

Since none of the paths of $\mathcal{N}_{d-3}$ has $n-2$ as a lower begin or a lower end, and since Lemma 13 implies that $\mathcal{N}^{0\,1}_{d-3}$ and $\mathcal{N}^{0\,0}_{d-3}$ are openly well distributed, it follows that both $\mathcal{N}'_{d-3}$ and $\mathcal{N}''_{d-3}$ are openly well distributed. By Lemma 13 and Proposition 12, it follows that every chain of $\mathcal{S}$ is openly well distributed. An argument similar to the argument used in the proof of Lemma 13 (using additionally Lemma 14) shows that $\mathcal{S}$ is closely alternating. Therefore $\mathcal{C}$ is closely well distributed and it follows from Lemma 10 that the path $\gamma^{d-2}_n \otimes \mathcal{C}$ is a snake in the graph $K^{d-2}_n \times K^2_n = K^d_n$. Since the paths in $\overline{\mathcal{M}}$ have length $2\lfloor n/4 \rfloor$, and γ^{d-2}_n has length $2\lfloor n/2 \rfloor n^{d-3}$, it follows that $\gamma^{d-2}_n \otimes \mathcal{C}$ has length $4\lfloor n/4 \rfloor \lfloor n/2 \rfloor n^{d-3}$. Thus the proof is complete.

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