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Amalgamating infinite latin squares

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Abstract

A finite latin square is an $n \times n$ matrix whose entries are elements of the set $\{1, \ldots, n\}$ and no element is repeated in any row or column. Given equivalence relations on the set of rows, the set of columns, and the set of symbols, respectively, we can use these relations to identify equivalent rows, columns and symbols, and obtain an amalgamated latin square. There is a set of natural equations that have to be satisfied by an amalgamated latin square. Using these equations we can define the notion of an outline latin square and it follows easily that an amalgamated latin square is an outline latin square. Hilton (Math. Programming Stud. 13 (1980) 68) proved that the opposite implication holds as well, that is, every outline latin square is an amalgamated latin square. In this paper, we present a generalization of that result to infinite latin squares with the sets of rows, columns and symbols of arbitrary cardinality.

Keywords: Infinite latin squares; Amalgamations; Infinite graphs

1. Preamble

The study of amalgamations of various simple finite combinatorial structures has been pursued by the present authors and others (see for example [1–3,5–13,15,16]). Hilton on this theme concerned latin squares [5] (see [7] for a clearer account) and is the one that we extend to the infinite case in this paper. The proof in the case of finite latin squares is not particularly difficult, but the proofs for some of the other finite structures are most

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complicated, and the results themselves are quite deep. It may well be that, in the infinite as well as in the finite case, the first relatively easy result is the precursor of a number of deep and difficult results—perhaps only time will tell.

In the infinite case, the obvious analogue of the conditions in the finite latin square case are all there, but there is one further condition, "well-distributedness", that has to be included.

2. Introduction

Let X, Y, Z be disjoint sets. A *latin system* can be thought of as the complete tripartite graph G with sides X, Y, and Z with partition of its edge-set into triangles. An *amalgamated latin system* is then a tripartite multigraph together with a partition of its edge set into triangles. For fixed partitions of each of the sides X, Y, Z, we can define a multigraph G' obtained from G by identifying vertices in each part (for each of the sides) into a single "big" vertex. We use the convention that an *ordinal number* α is equal to the set of all ordinal numbers smaller than α and that a *cardinal number* is an ordinal number α such that any ordinal number smaller than α has a smaller cardinality. We denote by ω the first infinite ordinal number.

Let **Card** be the class of cardinal numbers, $\mathbf{Card}^+$ be the class of positive cardinal numbers and $\mathbf{Card}^\infty$ be the class of infinite cardinal numbers.

Let X, Y and Z be sets. Given a subset $\mathscr{S} \subseteq X \times Y \times Z$, we say that $\mathscr{S}$ is a *latin system* if

$$|\{x \in X : (x, y, z) \in \mathscr{S}\}| = 1 \quad \text{for every } y \in Y \text{ and } z \in Z,$$
$$|\{y \in Y : (x, y, z) \in \mathscr{S}\}| = 1 \quad \text{for every } x \in X \text{ and } z \in Z,$$
$$|\{z \in Z : (x, y, z) \in \mathscr{S}\}| = 1 \quad \text{for every } x \in X \text{ and } y \in Y.$$

Note that for a fixed $x \in X$ the condition that $(x, y, z) \in \mathscr{S}$ defines a bijection between Y and Z, implying that $|Y| = |Z|$, and similarly we can conclude that all the sets X, Y and Z have the same cardinality. If $X = Y = Z = \{1, 2, \dots, n\}$ for some positive integer n and we interpret the elements of X as labels of the rows of an $(n \times n)$-matrix A, the elements of Y as the labels of the columns of A and the elements of Z as the entries of A, then a latin system $\mathscr{S} \subseteq X \times Y \times Z$ corresponds to the familiar notion of a latin square (see [7]). Another way of looking at a latin system $\mathscr{S} \subseteq X \times Y \times Z$ is to interpret it as a partition of the set of edges of the complete tripartite graph with sides X, Y, and Z of the same cardinality into triangles.

In the finite case the notions discussed in this paper (of outline and amalgamated latin systems) are considered in detail in [7], where some examples are given. This paper is self-contained, but to understand it, it may help to look at [7] first. For set-theoretic concepts see [14].

Using the tripartite graph interpretation, an amalgamated latin system (defined formally later) can be thought of as being obtained from a latin system $\mathscr{S} \subseteq X \times Y \times Z$ by partitioning each of its sides in some way (independently for each side) and then identifying the vertices in each of the parts into a single "big" vertex. At the same time we preserve each of the edges and the partition of the edge-set into triangles; the new endpoints of an edge are the "big" vertices that contain the old endpoints. A system that is obtained in this way can be described formally by considering sets X', Y', and Z' of "new" vertices that are obtained from the sets X, Y, and Z, respectively together with four functions. Three of these functions, with domains X', Y', and Z' respectively and with values in $\mathbf{Card}^+$, describe how "big" each of the "new" vertices are, that is how many "old" vertices they contain. The fourth function $\phi\colon X' \times Y' \times Z' \to \mathbf{Card}$ describes the distribution of triangles so that $\phi(a, b, c)$ gives the cardinality of the set of triangles with vertices a, b, and c. There are obvious cardinality conditions that have to be satisfied by these functions. A system defined by assuming these conditions will be called an outline latin system. Now we will present the formal definitions.

Given a set A, a *weight distribution* on A is a function $f : A \to \mathbf{Card}^+$. If $\phi\colon X \times Y \times Z \to \mathbf{Card}$, then the quadruple (X, Y, Z, ϕ) will be called a 3-*weighted system* and the map ϕ will be referred to as 3-*weight*.

Let f, g, h be weight distributions on the sets X, Y, Z, respectively, and let $\Gamma = (X, Y, Z, \phi)$ be a 3-weighted system. We say that Γ is an (f, g, h)-*outline latin system* if

$$\sum_{x \in X} \phi(x, y, z) = g(y)h(z) \quad \text{for every } y \in Y \text{ and } z \in Z,$$

$$\sum_{y \in Y} \phi(x, y, z) = f(x)h(z) \quad \text{for every } x \in X \text{ and } z \in Z,$$

$$\sum_{z \in Z} \phi(x, y, z) = f(x)g(y) \quad \text{for every } x \in X \text{ and } y \in Y.$$

Note that if $\mathscr{S} \subseteq X \times Y \times Z$ is a latin system and ϕ is the characteristic function of $\mathscr{S}$, i.e. $\phi\colon X \times Y \times Z \to \{0, 1\}$ with $\phi(x, y, z) = 1$ if and only if $(x, y, z) \in \mathscr{S}$, then (X, Y, Z, ϕ) is an (f, g, h)-outline latin system where $f \equiv 1$, $g \equiv 1$ and $h \equiv 1$.

We will define the process of obtaining an outline latin system from a latin system more generally, namely we will allow the original system to be an outline system as well. This more general definition will be needed later in the proofs.

Let $\Gamma = (X, Y, Z, \phi)$ be an (f, g, h)-outline latin system, and $\alpha\colon X \to X'$, $\beta\colon Y \to Y'$, $\gamma\colon Z \to Z'$ be surjections. The (α, β, γ)-*amalgamation* of Γ is the 3-weighted system $\Xi = (X', Y', Z', \psi)$ where the 3-weight ψ is defined by

$$\psi(x, y, z) = \sum_{a \in \alpha^{-1}(x)} \sum_{b \in \beta^{-1}(y)} \sum_{c \in \gamma^{-1}(z)} \phi(a, b, c).$$

It is easy to see that the (α, β, γ)-amalgamation Ξ of Γ is an (f', g', h')-outline latin system where

$$f'(x) = \sum_{a \in \alpha^{-1}(x)} f(a), \quad g'(y) = \sum_{b \in \beta^{-1}(y)} g(b), \quad h'(z) = \sum_{c \in \gamma^{-1}(z)} h(c),$$

for every $x \in X'$, $y \in Y'$ and $z \in Z'$.

If $\mathscr{S} \subseteq X \times Y \times Z$ is a latin system and $\alpha\colon X \to X'$, $\beta\colon Y \to Y'$, $\gamma\colon Z \to Z'$ are surjections, then the (α, β, γ)-*amalgamation* of $\mathscr{S}$ is the (α, β, γ)-amalgamation of the corresponding $(1, 1, 1)$-outline latin system $\Gamma = (X, Y, Z, \phi)$, (where ϕ is the characteristic function of $\mathscr{S}$). Explicitly, the (α, β, γ)-*amalgamation* of $\mathscr{S}$ is the 3-weighted system $\Xi = (X', Y', Z', \psi)$ where the map ψ satisfies

$$\psi(x, y, z) = |\mathscr{S} \cap (\alpha^{-1}(x) \times \beta^{-1}(y) \times \gamma^{-1}(z))|,$$

for any $x \in X'$, $y \in Y'$ and $z \in Z'$. A 3-weighted system (X, Y, Z, ϕ) is an *amalgamated latin system* if it is an amalgamation of a latin system.

As we remarked before, any amalgamation of an outline latin system is an outline latin system itself, hence in particular, any amalgamated latin system is an outline latin system. To be more precise, if $\Xi = (X', Y', Z', \psi)$ is the (α, β, γ)-amalgamation of a latin system $\mathscr{S} \subseteq X \times Y \times Z$, then Ξ is an (f, g, h)-outline latin system where $f(x) = |\alpha^{-1}(x)|$, $g(y) = |\beta^{-1}(y)|$ and $h(z) = |\gamma^{-1}(z)|$ for any $x \in X'$, $y \in Y'$ and $z \in Z'$.

Hilton [5] (see [7] for a clearer account) proved that in the finite case the converse of the above statement holds as well. Namely, he proved the following theorem about latin squares.

Theorem 1. *Let $\Gamma = (X, Y, Z, \psi)$ be an (f, g, h)-outline latin system. If the sets X, Y, Z are finite and the functions f, g, h take finite values, then Γ is an amalgamated latin system, i.e. there are sets X', Y', Z', surjections $\alpha\colon X' \to X$, $\beta\colon Y' \to Y$, $\gamma\colon Z' \to Z$ and a latin system $\mathscr{S} \subseteq X' \times Y' \times Z'$ such that Γ is the (α, β, γ)-amalgamation of $\mathscr{S}$.*

In this paper, we are going to generalize Theorem 1. The following theorem is a special case of our generalization.

Theorem 2. *Let $\Gamma = (X, Y, Z, \psi)$ be an (f, g, h)-outline latin system. If two of the functions f, g, h take only finite values, then Γ is an amalgamated latin system.*

Before we state the complete generalization of Theorem 1, let us remark that Theorem 2 becomes false if we allow two of the functions f, g, h to take infinite values. Consider the following examples.

Example 3. Let $X = Y = Z = \mathbb{Z}$, and let f, g, h be the weight distributions on the sets X, Y, Z, respectively, such that $f(x) = g(y) = \omega$ for any $x \in X$ and $y \in Y$, and $h(z) = 1$ for any $z \in Z$. Define $\psi\colon X \times Y \times Z \to \mathbf{Card}$ by

$$\psi(x, y, z) = \begin{cases} \omega & \text{if } x + y = z, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Gamma = (X, Y, Z, \psi)$ is an (f, g, h)-outline latin system, but it is easy to see that Γ is not an amalgamated latin system.

The following example is a generalization of Example 3.

Example 4. Let λ be an infinite cardinal, $X = Y = Z = \lambda$, and f, g, h be weight distributions on the sets X, Y, Z, respectively, such that $f(x) = g(y) = \lambda$ for any $x \in X$ and $y \in Y$, and $h(z) < \lambda$ for any $z \in Z$. Let $\eta\colon \lambda \times \lambda \to \lambda$ be a map such that for every $\alpha, \beta < \lambda$ there are $\gamma, \delta < \lambda$ such that $\eta(\alpha, \gamma) = \beta$ and $\eta(\delta, \alpha) = \beta$. (It is easy to see that such a function η exists. At the end of this example we will give a construction of such a function.) Define a 3-weight $\psi\colon X \times Y \times Z \to \mathbf{Card}$ by

$$\psi(x, y, z) = \begin{cases} \lambda & \text{if } \eta(x, y) = z, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\Gamma = (X, Y, Z, \psi)$ is an (f, g, h)-outline latin system, but it is easy to see that Γ is not an amalgamated latin system.

Now we will show one possible way of constructing the function η. Let $\{X_1, X_2\}$ be a partition of λ (that is, the set of all ordinals less than λ) such that $|X_1| = |X_2| = \lambda$, and let τ_1, τ_2 be bijections from X_1, X_2, respectively, onto λ. Define η by

$$\eta(\alpha, \beta) = \begin{cases} \tau_1(\alpha) & \text{if } \alpha, \beta \in X_1, \\ \tau_2(\alpha) & \text{if } \alpha, \beta \in X_2, \\ \tau_1(\beta) & \text{if } \alpha \in X_2 \text{ and } \beta \in X_1, \\ \tau_2(\beta) & \text{if } \alpha \in X_1 \text{ and } \beta \in X_2. \end{cases}$$

Examples 3 and 4 show that in order to generalize Theorem 1 we need to add an extra condition to the definition of an outline latin system.

Let $\Gamma = (X, Y, Z, \phi)$ be a 3-weighted system and f, g, h be weight distributions on X, Y, Z, respectively. We say that Γ is *well X-distributed* if for every $y \in Y$ and $z \in Z$ we have

$$\sum_{x \in X} \min(f(x), \phi(x, y, z)) \geqslant \max(g(y), h(z)). \tag{1}$$

The notions of *well Y-distributed* and *well Z-distributed* 3-weighted systems are defined in a similar way. A 3-weighted system $\Gamma = (X, Y, Z, \phi)$ is *well distributed* if it is well X-distributed, well Y-distributed and well Z-distributed.

It is easy to see that any amalgamated latin system is well distributed. Indeed, let $\Xi = (X, Y, Z, \phi)$ be an (f, g, h)-outline latin system that is an amalgamation of a latin system $\mathscr{S} \subseteq A \times B \times C$. We will show that Ξ is well X-distributed. Given any $y \in Y$ and $z \in Z$ (assuming, say, that $g(y) \geqslant h(z)$) let c be any element of $\gamma^{-1}(z)$, that is an "old" vertex that was amalgamated into the "new" vertex z. Consider all the triangles of the latin system $\mathscr{S}$ where one of the vertices is c and another was amalgamated into y. There are $g(y) = \max(g(y), h(z))$ such triangles. Any two such triangles that are distinct must have different vertices inside A. The number of such triangles with vertices amalgamated into some $x \in X$ is at most $\min(f(x), \phi(x, y, z))$, implying that the required inequality is satisfied.

Assume that $\Xi = (X, Y, Z, \phi)$ is an (f, g, h)-outline latin system. We can easily observe that in such a case for Ξ to be well X-distributed it is enough to assume only that (1) is satisfied for every $y \in Y$ and $z \in Z$ such that $g(y) = h(z) \in \mathbf{Card}^\infty$ since in the remaining cases the inequality is satisfied anyway. Indeed, if $g(y) \geqslant h(z)$, and $h(z)$ is finite, then

$$f(x) \geqslant \frac{\phi(x, y, z)}{h(z)}, \quad \text{for every } x \in X,$$

where $\lambda/n = \lambda$ for any $\lambda \in \mathbf{Card}^\infty$, implying that

$$\sum_{x \in X} \min(f(x), \phi(x, y, z)) \geqslant \sum_{x \in X} \frac{\phi(x, y, z)}{h(z)} = g(y) = \max(g(y), h(z)).$$

Now assume that $g(y) > h(z) \in \mathbf{Card}^{\infty}$ and, by way of contradiction, that

$$\sum_{x\in X} \min(f(x), \phi(x,y,z)) = \sum_{x\in \bar{X}} \min(f(x), \phi(x,y,z)) < g(y),$$

where

$$\bar{X} = \{x \in X : \phi(x,y,z) \geqslant 1\}.$$

Let

$$X_1 = \{x \in \bar{X} :\ f(x) \leqslant h(z)\} \quad \text{and} \quad X_2 = \{x \in \bar{X} :\ f(x) > h(z)\}.$$

Since $\min(f(x), \phi(x,y,z)) \geqslant 1$ for $x \in \bar{X}$, it follows that

$$|X_1| \leqslant |\bar{X}| < g(y).$$

Moreover $f(x)h(z) = h(z) < g(y)$ for $x \in X_1$, so

$$\sum_{x\in X_1} f(x)h(z) = |X_1|h(z) < g(y).$$

Since $\phi(x,y,z) \leqslant f(x)h(z) = f(x)$ for $x \in X_2$, we conclude that

$$\begin{aligned} g(y) = g(y)h(z) = \sum_{x\in X} \phi(x,y,z) &= \sum_{x\in \bar{X}} \phi(x,y,z) \\ &= \sum_{x\in X_1} \phi(x,y,z) + \sum_{x\in X_2} \phi(x,y,z) \\ &\leqslant \sum_{x\in X_1} f(x)h(z) + \sum_{x\in X} \min(f(x), \phi(x,y,z)) \\ &< g(y) + g(y) = g(y) \end{aligned}$$

which is a contradiction.

The outline latin systems in Examples 3 and 4 are not amalgamated latin systems since they are not well distributed. The following theorem is the main result of our paper.

Theorem 5. *Let $\Gamma = (X, Y, Z, \phi)$ be a well-distributed (f, g, h)-outline latin system. Then Γ is an amalgamated latin system.*

Note that if Γ is an (f, g, h)-outline latin system and at least two of the functions f, g, h take only finite values, then Γ is well distributed. Hence, Theorem 2 is a special case of Theorem 5; Theorem 2 could be proved more directly, but we omit such a proof.

To prove Theorem 5 we will be splitting each of the vertices of a well-distributed (f, g, h)-outline latin square one by one into the required number of vertices making sure that at each intermediate step we have a well-distributed outline latin square. To split, say, a vertex $x \in X$, we will be considering the bipartite multigraph D with sides Y and Z obtained from the tripartite graph corresponding to our outline latin square by taking those edges that belong to a triangle with vertex x. Since we want to replace x with $f(x)$ vertices, we need to decide how to distribute the triangles that have x as a vertex between the copies of x. There is a natural one-to-one correspondence between such triangles and the edges of the bipartite multigraph D, so what we need to do is to colour the edges of D with $f(x)$ colours in a suitable way. To get such a colouring we will first temporarily split each of the vertices of D to get a $f(x)$-regular graph, apply an edge-colouring lemma, and finally identify the vertices to get back the original vertices of D. The main difficulty in this proof will be in getting the proper splitting of the vertices of D. That is where we will be using the extra assumption that our outline latin square is well distributed. This splitting process will follow from a general result, Theorem 6 in Section 3, about splitting vertices in bipartite multigraphs.

The proof of Theorem 5 will be given in Section 5.

3. Vertex-splitting in bipartite multigraphs

A *bipartite multigraph* is a quintuple $D = (Y, Z, L, \xi, \zeta)$ where Y, Z and L are disjoint sets, and ξ, ζ are maps from L into Y, Z, respectively. The elements of $Y \cup Z$ are the *vertices* of D and the elements of L are the *edges* of D. We say that the edge e of D is *incident* with $y \in Y$ (with $z \in Z$) if $\xi(e) = y$ (if $\zeta(e) = z$), and that the edges e_1, e_2 of D are *adjacent* if they are distinct and incident with the same vertex of D. If $y \in Y$ and $z \in Z$, then $|\xi^{-1}(y)|$ and $|\zeta^{-1}(z)|$ are the *degrees* of y and z, and

$$L_{yz} = \{e \in L : \xi(e) = y \text{ and } \zeta(e) = z\}.$$

If $|L_{yz}| \leqslant 1$ for every $y \in Y$ and $z \in Z$, then we say that D is a *bipartite graph*. Given $\lambda \in \mathbf{Card}$, we say that a bipartite graph is *λ-regular* if every vertex has degree λ.

Assume that $D = (Y, Z, L, \xi, \zeta)$ is a bipartite multigraph, and that g is a function $Y \cup Z \to \mathbf{Card}^+$. We say that D has a *g-splitting* $G = (M, W, L, \xi', \zeta')$ if G is a bipartite graph such that

$$M = \bigcup_{v \in Y} \Psi_v, \quad W = \bigcup_{v \in Z} \Psi_v$$

with $|\Psi_v| = g(v)$ where $\Psi_v \cap \Psi_w = \emptyset$ for every distinct $v, w \in Y \cup Z$, and $\xi'(e) \in \Psi_{\xi(e)}$, $\zeta'(e) \in \Psi_{\zeta(e)}$ for every $e \in L$.

Now we will state and prove our main auxiliary result on bipartite multigraphs.

Theorem 6. *Let $D = (Y, Z, L, \xi, \zeta)$ be a bipartite multigraph, $g : Y \cup Z \to \mathbf{Card}^+$, and $\lambda \in \mathbf{Card}^+$ be such that the degree of every $v \in Y \cup Z$ is $\lambda g(v)$. Then there exists a λ-regular g-splitting of D if and only if the following two conditions are satisfied.*

1. *For every $y \in Y$ and $z \in Z$ we have $|L_{yz}| \leqslant g(y)g(z)$.*
2. *If $\lambda \in \mathbf{Card}^\infty$, then for every $v \in Y \cup Z$ with $g(v) = \lambda$ we have*

$$\sum_{z \in Z} \min(g(z), |L_{vz}|) = \lambda$$

when $v \in Y$, *and*

$$\sum_{y \in Y} \min(g(y), |L_{yv}|) = \lambda$$

when $v \in Z$.

Proof. Let us first prove that the two conditions are necessary for D to have a λ-regular g-splitting. Assume that $G = (M, W, L, \xi', \zeta')$ is a g-splitting of D with

$$M = \bigcup_{v \in Y} \Psi_v, \quad W = \bigcup_{v \in Z} \Psi_v.$$

Since the first condition is obviously satisfied, we are only going to prove the second condition. If $y \in Y$ and $v \in \Psi_y$, then v is adjacent in G to λ vertices in W since G has no multiple edges. On the other hand, for each $z \in Z$, the number of vertices in Ψ_z adjacent to v cannot be larger than either $g(z)$ or $|L_{yz}|$. Therefore

$$\sum_{z \in Z} \min(g(z), |L_{yz}|) \geqslant \lambda.$$

If $\lambda \in \mathbf{Card}^{\infty}$ and $g(y) = \lambda$, then

$$\lambda = g(y)\lambda = \sum_{z \in Z} |L_{yz}| \geqslant \sum_{z \in Z} \min(g(z), |L_{yz}|) \geqslant \lambda,$$

and so we have equality.

Now we are going to prove that the two conditions are sufficient for D to have a λ-regular g-splitting. Let us assume that the conditions are satisfied. To prove the existence of a λ-regular g-splitting of D we can do the splitting in two stages, splitting the vertices in Y first and the vertices in Z later. Note that it is enough to show that it is possible to split each vertex in Y so that if we think of the obtained set M as a new version of Y with $g(y) = 1$ for every $y \in Y$, then each $y \in Y$ will have degree $\lambda = g(y)\lambda$ and both conditions will be still satisfied. Since, by symmetry, the splitting operation can be applied to the vertices in Z (with $g(y) = 1$ for every $y \in Y$), it will follow that we can do both stages of the splitting obtaining a λ-regular bipartite multigraph with the value of g being 1 for every vertex. Since this multigraph will satisfy the first condition, it will be a graph.

Let us now show that the splitting of Y described above is possible. Since there are no interactions between vertices, it is enough to define the splitting for one vertex in Y. We shall need only to make sure that the multigraph we obtain is λ-regular on the Y-side, satisfies the first condition, and satisfies the second condition on the Z-side. The second condition will be satisfied on the Y-side since the function g will take the value 1 only, and thus the second condition will be satisfied vacuously on the Y-side. To ensure that the second condition is satisfied on the Z-side, we note first that it is satisfied on the Z-side at the beginning. Therefore it will be satisfied at the end provided that, in case $\lambda \in \mathbf{Card}^{\infty}$, when splitting $y \in Y$ we ensure that:

$(*)$ every $z \in Z$ with $g(z) = \lambda$ is joined to $\min\{|L_{yz}|, g(y)\}$ of the new vertices replacing y.

Fix $y \in Y$ and let $\mu = g(y)$. Our general strategy to obtain a splitting of y satisfying the required conditions will be to distribute the edges in L_{yv}, for each $v \in Z$, as equally as possible between the new vertices replacing y. Let Ψ_y be a set of cardinality μ. We want to redefine the values of ξ on $\bigcup_{z\in Z} L_{yz}$ replacing the old value y with elements of Ψ_y.

First assume that $\lambda > \mu \in \mathbf{Card}^\infty$. Let

$$Z' = \{z \in Z : |L_{yz}| \geqslant \mu\} \quad \text{and} \quad Z'' = \{z \in Z : 1 \leqslant |L_{yz}| < \mu\}.$$

For each $z \in Z'$ define ξ on L_{yz} so that its inverse image on every element of Ψ_y is a set of the same cardinality $\kappa_z \leqslant g(z)$. This is possible since $|L_{yz}| \leqslant \mu g(z)$. If $|\bigcup_{z\in Z''} L_{yz}| \geqslant \mu$, then $|Z''| \geqslant \mu$ and we can partition Z'' into sets Z_i, $i \in I$, of cardinality μ. Define ξ on $\bigcup_{z\in Z''} L_{yz}$ so that its restriction to $\bigcup_{z\in Z_i} L_{yz}$ is a bijection onto Ψ_y for every $i \in I$. If $|\bigcup_{z\in Z''} L_{yz}| < \mu$, then define ξ so that its restriction to $\bigcup_{z\in Z''} L_{yz}$ is an injection, and set $I = \emptyset$. Then the degree of each $v \in \Psi_y$ is equal to $\sum_{z\in Z'} \kappa_z + |I|$. This number must equal λ since $\lambda > \mu$ and the degree of y was equal to $\lambda g(v) = \mu\lambda$. Moreover, for each $z \in Z'$ we have at most $g(z)$ edges between v and z and for each $z \in Z''$ we have at most 1 edge between v and z. It is clear that condition $(*)$ also holds so the splitting of y satisfies all the required conditions.

Now assume that both λ and μ are finite. Let

$$\bar{Z} = \{z \in Z : |L_{yz}| \geqslant 1\}.$$

Then

$$\left|\bigcup_{z\in\bar{Z}} L_{yz}\right| = \left|\bigcup_{z\in Z} L_{yz}\right| = \lambda\mu$$

is finite, implying that $\bar{Z}$ is finite. Let $\bar{Z} = \{z_0, z_1, \dots, z_r\}$, and let $e_0, e_1, \dots, e_{\lambda\mu-1}$ be an enumeration of the set $\bigcup_{z\in Z} L_{yz}$ such that all the elements of L_{yz_0} are listed first, then all the elements of L_{yz_1}, and so on. Define ξ on $\bigcup_{z\in Z} L_{yz}$ by setting $\xi(e_j) = a_{j \bmod \mu}$ for every $j \in \{0, 1, \dots, \mu\lambda - 1\}$, where $\Psi_y = \{a_0, a_1, \dots, a_{\mu-1}\}$. Then every vertex of Ψ_y has degree λ. It is clear that the required conditions are satisfied.

Next assume that $\lambda \in \mathbf{Card}^\infty$ and μ is finite. Let

$$Z' = \{z \in Z : |L_{yz}| \in \mathbf{Card}^\infty\} \quad \text{and} \quad Z'' = \{z \in Z : 1 \leqslant |L_{yz}| < \omega\}.$$

For each $z \in Z'$ define ξ on L_{yz} so that its inverse image on every element of Ψ_y is a set of cardinality $|L_{yz}|$. If $|\bigcup_{z\in Z''} L_{yz}|$ is infinite, then Z'' is infinite and we can partition Z'' into sets Z_i, $i \in I$, of cardinality ω. Given $i \in I$, let

$$Z_i = \{z_{i,0}, z_{i,1}, \dots\}, \qquad L_i = \bigcup_{j=0}^{\infty} L_{yz_{i,j}}.$$

Let $e_{i,0}, e_{i,1}, \dots$ be an enumeration of the set L_i such that all the elements of $L_{yz_{i,0}}$ are listed first, then all the elements of $L_{yz_{i,1}}$, and so on. Define ξ on $\bigcup_{z\in Z''} L_{yz}$ by setting $\xi(e_{i,j}) = a_{j \bmod \mu}$ for every $i \in I$ and $j < \omega$, where $\Psi_y = \{a_0, a_1, \dots, a_{\mu-1}\}$. If $|\bigcup_{z\in Z''} L_{yz}|$ is finite, then let $Z'' = \{z_0, z_1, \dots, z_r\}$, and let $e_0, e_1, \dots, e_p$ be an enumeration of the set

$\bigcup_{z\in Z''} L_{yz}$ such that all the elements of L_{yz_0} are listed first, then all the elements of L_{yz_1}, and so on. Define ξ on $\bigcup_{z\in Z''} L_{yz}$ by setting $\xi(e_j) = a_{j \bmod \mu}$ for every $j \in \{0, 1, \ldots, p\}$. Set $I = \emptyset$. Then the degree of each $v \in \Psi_y$ is the same and is equal to $\sum_{z\in Z'} |L_{yz}| + \omega|I|$. This number must be equal to λ since λ is infinite and μ is finite and the degree of y was equal to $\mu\lambda = \lambda$.Moreover, for each $z \in Z'$ we have $|L_{yz}| \leqslant \mu g(z) = g(z)$ edges between v and z (since $g(z) \in \mathbf{Card}^\infty$ for $z \in Z'$) and for each $z \in Z''$ we have at most $\lceil |L_{yz}|/\mu\rceil \leqslant g(z)$ edges between v and z (where $\lceil |L_{yz}|/\mu\rceil = |L_{yz}|$ if $|L_{yz}| \in \mathbf{Card}^\infty$, and is the smallest integer that is greater or equal to $|L_{yz}|/\mu$ if $|L_{yz}|$ is finite). It is clear that condition $(*)$ also holds so the splitting of y satisfies all the required conditions.

Finally consider the case when $\mu \geqslant \lambda$ and $\mu \in \mathbf{Card}^\infty$. Let $\theta_z = \min\{|L_{yz}|, g(z)\}$ for every $z \in Z$. We will show first that $\sum_{z\in Z} \theta_z = \mu$. It follows from the second condition that we need only to prove that equality when $\mu > \lambda$ (the argument will be similar to the argument preceding Theorem 5 in the introduction). Since the degree of y is $\sum_{z\in Z} |L_{yz}|$ and is also $\lambda g(y) = \lambda\mu = \mu$, the inequality $\sum_{z\in Z} \theta_z \leqslant \mu$ clearly follows. Therefore it is enough to show that $\sum_{z\in Z} \theta_z \geqslant \mu$. If λ is finite, then $g(z) \geqslant |L_{yz}|/\lambda$ for every $z \in Z$ (where $\kappa/\lambda = \kappa$ for $\kappa \in \mathbf{Card}^\infty$) since the degree of z is $g(z)\lambda$. Therefore

$$\sum_{z\in Z} \theta_z \geqslant \sum_{z\in Z} \frac{|L_{yz}|}{\lambda} = \frac{\sum_{z\in Z} |L_{yz}|}{\lambda} = \frac{\lambda\mu}{\lambda} = \mu.$$

Now assume that $\lambda \in \mathbf{Card}^\infty$, and suppose, by way of contradiction, that $\sum_{z\in Z} \theta_z < \mu$. Let

$$\bar{Z} = \{z \in Z : |L_{yz}| \geqslant 1 \text{ and } g(z) \leqslant \lambda\} \quad \text{and} \quad \hat{Z} = \{z \in Z : g(z) > \lambda\}.$$

Then $|\bar{Z}| < \mu$ and $g(z)\lambda = \lambda < \mu$ for $z \in \bar{Z}$ implying that

$$\sum_{z\in \bar{Z}} g(z)\lambda = |\bar{Z}|\lambda < \mu.$$

Since $|L_{yz}| \leqslant g(z)\lambda = g(z)$ for $z \in \hat{Z}$, we have $|L_{yz}| \leqslant \theta_z$ for $z \in \hat{Z}$ and so

$$\begin{aligned}\mu = \mu\lambda = \sum_{z\in Z} |L_{yz}| &= \sum_{z\in \bar{Z}} |L_{yz}| + \sum_{z\in \hat{Z}} |L_{yz}| \\ &\leqslant \sum_{z\in \bar{Z}} g(z)\lambda + \sum_{z\in \hat{Z}} \theta_z < \mu + \mu = \mu\end{aligned}$$

which is a contradiction proving that $\sum_{z\in Z} \theta_z = \mu$.

Now we will complete the proof of this last case. If $\lambda = 1$, then define ξ on $\bigcup_{z\in Z} L_{yz}$ so that it is a bijection onto Ψ_y. This will clearly satisfy all the required conditions. Otherwise, let

$$Z_1 = \{z \in Z : \theta_z \in \mathbf{Card}^\infty\} \quad \text{and} \quad Z_2 = \{z \in Z : 1 \leqslant \theta_z < \omega\}.$$

Since $\sum_{z\in Z} \theta_z = \mu \in \mathbf{Card}^\infty$, either $\sum_{z\in Z_1} \theta_z = \mu$ or $\sum_{z\in Z_2} \theta_z = \mu$ (and then $|Z_2| = \mu$).

Suppose $\bigcup_{z\in Z_1}\theta_z=\mu$. For every $z\in Z_1$ the set L_{yz} is infinite of cardinality at least θ_z, so there is a partition $\{L'_{yz},L''_{yz}\}$ of L_{yz} such that $|L'_{yz}|=\theta_z$ and $|L''_{yz}|\geqslant\theta_z$. Let

$$L'=\bigcup_{z\in Z_1}L'_{yz}\quad\text{and}\quad L''=\bigcup_{z\in Z_1}L''_{yz}\cup\bigcup_{z\in Z_2}L_{yz}.$$

Since $|\bigcup_{z\in Z}L_{yz}|=\mu$, the cardinalities of L' and L'' are at most μ. Since

$$\bigcup_{z\in Z_1}|L''_{yz}|\geqslant\bigcup_{z\in Z_1}|L'_{yz}|=\bigcup_{z\in Z_1}\theta_z=\mu$$

it follows that $|L'|=|L''|=\mu$. Define ξ on $L'\cup L''$ so that its inverse image on any element of Ψ_y has $\lambda-1$ elements from L' (where $\lambda-1=\lambda$ for $\lambda\in\mathbf{Card}^{\infty}$) and one element from L''.

Let $v\in\Psi_y$. Then $|\xi^{-1}(v)|=\lambda$. Moreover, for each $z\in Z_1$, there are at most

$$|L'_{yz}|+1=\theta_z+1=\theta_z\leqslant g(z)$$

edges between v and z, and for each $z\in Z_2$ there is at most 1 edge between v and z. Thus condition 1 is satisfied. To see that $(*)$ holds note that if $z\in Z_1$, then every edge of L''_{yz} joins z to a different vertex of Ψ_y, and if $z\in Z_2$, then every edge of L_{yz} joins z to a different vertex of Ψ_y. Since $|L''_{yz}|=|L_{yz}|$ for every $z\in Z_1$, each $z\in Z$ is joined to at least $|L_{yz}|$ of the new vertices replacing y. Thus the splitting of y satisfies all the required conditions.

If $|Z_2|=\mu$, then $|Z_1\cup Z_2|=\mu$. Let $\{\bar{Z}_\alpha:\alpha<\lambda\}$ be a partition of $Z_1\cup Z_2$ into λ sets of cardinality μ. Then $|\bigcup_{z\in\bar{Z}_\alpha}L_{yz}|=\mu$ for every $\alpha<\lambda$. Define ξ on $\bigcup_{z\in Z}L_{yz}$ so that its restriction to $\bigcup_{z\in\bar{Z}_\alpha}L_{yz}$ for every $\alpha<\lambda$ is a bijection. Let $v\in\Psi_y$. Then $|\xi^{-1}(v)|=\lambda$ and, for each $z\in Z$, there is at most 1 edge between v and z (this in particular implies that $(*)$ holds). Thus the splitting of y satisfies all the required conditions and the proof is complete. □

4. Perfect edge-colourings of λ-regular bipartite graphs

Let $\lambda\in\mathbf{Card}^{+}$. A (*partial*) *edge λ-colouring* of D is a (partial) function $\eta: L\to\lambda$ such that $\eta(e_1)\neq\eta(e_2)$ for any adjacent edges e_1 and e_2 of D that are in the domain of η. Given an edge λ-colouring η of D we say that η is *perfect* if for any vertex v of D and any colour $c\in\lambda$, there is an edge e of D which is incident to v and $\eta(e)=c$. The following lemma follows easily from a result of Hall [4].

Lemma 7. *Let $D=(M,W,L,\xi,\zeta)$ be a bipartite multigraph. If D is n-regular, for some positive integer n, then there is a perfect n-colouring of D.*

Proof. It follows from the condition of Hall for the existence of a perfect matching in a locally finite bipartite graph that if D is an n-regular bipartite multigraph, then it has a perfect matching. The lemma now follows by induction on n. □

We actually only use Lemma 7 in the case when D is a bipartite graph. We now give the corresponding result for λ-regular bipartite graphs, including in particular the case when

$\lambda \in \mathbf{Card}^{\infty}$. Perhaps, we might remark that Lemma 8 is not true for λ-regular bipartite multigraphs in general when $\lambda \in \mathbf{Card}^{\infty}$. But it is true if the underlying bipartite graph is also λ-regular (essentially the same proof can be used to prove this).

Lemma 8. *For any $\lambda \in \mathbf{Card}^{+}$, there is a perfect λ-colouring of any λ-regular bipartite graph.*

Proof. Let $D = (M, W, L, \xi, \zeta)$ be a λ-regular bipartite graph. Because of Lemma 7, we can assume that $\lambda \in \mathbf{Card}^{\infty}$. Without loss of generality, we can assume that D is connected (i.e. between any two vertices there is a finite path). Then the cardinalities of M and W are at most λ and $|L| = \lambda$. Let $(x_\alpha)_{\alpha<\lambda}$ be a sequence enumerating the elements of the set $(M \times \lambda) \cup (W \times \lambda) \cup L$ such that if $x_\alpha = (y, \beta)$ and $x_{\alpha'} = (y, \beta')$ are elements of $(M \times \lambda) \cup (W \times \lambda)$ with $\alpha < \alpha' < \lambda$, then $\beta < \beta'$. We define, by induction on α, an increasing sequence $(\eta_\alpha)_{\alpha<\lambda}$ of partial edge λ-colourings of D such that the following conditions hold for every $\alpha < \lambda$:

(i) if $x_\alpha \in L$, then $x_\alpha \in \operatorname{dom} \eta_\alpha$,
(ii) if $x_\alpha = (y, \gamma) \in M \times \lambda$, then the set of values of the restriction of η_α to $\xi^{-1}(y)$ contains γ,
(iii) if $x_\alpha = (y, \gamma) \in W \times \lambda$, then the set of values of the restriction of η_α to $\zeta^{-1}(y)$ contains γ.

Thus condition (i) ensures that each edge is coloured, and conditions (ii) and (iii) ensure that each colour occurs on an edge incident with each vertex.

Let $\alpha < \lambda$ and assume that η_β is defined for every $\beta < \alpha$. Let

$$\eta'_\alpha = \bigcup_{\beta<\alpha} \eta_\beta.$$

Note that if $\alpha = \alpha' + 1$ is a successor ordinal, then $\eta'_\alpha = \eta_{\alpha'}$.

To define η_α suppose first that $x_\alpha \in L$. If $x_\alpha \in \operatorname{dom} \eta'_\alpha$, then let $\eta_\alpha = \eta'_\alpha$. If $x_\alpha \notin \operatorname{dom} \eta'_\alpha$, then there is $\delta \in \lambda$ such that δ is not a value of the restriction of η'_α to $\xi^{-1}(\xi(x_\alpha)) \cup \zeta^{-1}(\zeta(x_\alpha))$. Then let

$$\eta_\alpha = \eta'_\alpha \cup \{(x_\alpha, \delta)\}.$$

It follows from the inductive hypothesis that η_α is a partial edge λ-colouring of D satisfying conditions (i)–(iii).

Now assume that $x_\alpha = (y, \gamma) \in M \times \lambda$. If the set of values of the restriction of η'_α to $\xi^{-1}(y)$ contains γ, then let $\eta_\alpha = \eta'_\alpha$. Otherwise, since D is regular, there is $e \in \xi^{-1}(y)$ such that γ is not a value of the restriction of η'_α to $\zeta^{-1}(\zeta(e))$. Thus γ is not used to colour any edge incident with either end vertex of e. Then let

$$\eta_\alpha = \eta'_\alpha \cup \{(e, \gamma)\}.$$

It follows from the inductive hypothesis that η_α is a partial edge λ-colouring of D satisfying conditions (i)–(iii). If $x_\alpha = (y, \gamma) \in W \times \lambda$, then the definition of η_α is similar.

It is clear that if $(\eta_\alpha)_{\alpha<\lambda}$ satisfies conditions (i)–(iii), then

$$\eta = \bigcup_{\alpha<\lambda} \eta_\alpha : L \to \lambda$$

is a perfect edge λ-colouring of D, so the proof is complete. $\square$

5. Proof of the main result

In this section we prove our main result, Theorem 5, that every well-distributed outline latin system is an amalgamated latin system. Assume that $\Gamma = (X, Y, Z, \phi)$ is a well-distributed (f, g, h)-outline latin system. To show that Γ is an amalgamated latin system we need to split each element of $X \cup Y \cup Z$ into a suitable number of elements given by the functions f, g, and h and distribute the triangles with vertices x, y, and z between all the copies of x, y, and z in a way to get a latin system. We will take care of the elements of X first, following with Y and Z. It is enough to show how to split a single element of X since there are no interactions between the elements of X.

Let $x_0 \in X$. The function ϕ determines a bipartite multigraph with sides Y and Z having $\phi(x_0, y, z)$ edges incident to $y \in Y$ and $z \in Z$ (the set L_{yz}). Since the vertex x_0 will be split into $f(x_0)$ vertices, we need to colour the edges of this multigraph with $f(x_0)$ colours; each colour will correspond to one of the vertices, and the edges of a particular colour class will be in triangles with the corresponding vertex. This colouring will be defined using a $(g \cup h)$-splitting of this multigraph called a bipartite presentation of Γ over x_0. Thus the splitting for $f(x_0)$ is found using a splitting of a corresponding multigraph. This latter splitting is subsequently forgotten.

A *bipartite presentation of* Γ *over* x_0 is an $f(x_0)$-regular bipartite graph $D = (M, W, L, \xi, \zeta)$ such that

$$M = \bigcup_{y\in Y} M_y, \quad W = \bigcup_{z\in Z} W_z, \quad L = \bigcup_{y\in Y} \bigcup_{z\in Z} L_{yz},$$

where $|M_y| = g(y)$, $|W_z| = h(z)$, $|L_{yz}| = \phi(x_0, y, z)$, $\xi(e) \in M_y$, and $\zeta(e) \in W_z$ for any $y \in Y$, $z \in Z$ and $e \in L_{yz}$, and all the sets M_y, W_z, L_{yz} are mutually disjoint.

The following result follows immediately from Theorem 6.

Lemma 9. *There exists a bipartite presentation of* Γ *over* x_0.

Proof. Let $\lambda = f(x_0)$. Let $D = (Y, Z, L, \xi, \zeta)$ be a bipartite multigraph such that

$$L = \bigcup_{y\in Y} \bigcup_{z\in Z} L_{yz},$$

where $|L_{yz}| = \phi(x_0, y, z)$, $\xi(e) = y$, and $\zeta(e) = z$ for every $y \in Y$, $z \in Z$, and $e \in L_{yz}$. In particular, all the sets L_{yz} are mutually disjoint. Note that the degree of every $y \in Y$ is equal to $\sum_{z\in Z} \phi(x_0, y, z) = \lambda g(y)$ and, similarly, the degree of every $z \in Z$ is equal to

$\lambda h(z)$. Since for every $y \in Y$ and $z \in Z$ we have

$$\phi(x_0, y, z) \leqslant \sum_{x \in X} \phi(x, y, z) = g(y)h(z),$$

it follows that $|L_{yz}| \leqslant g(y)h(z)$. Assume that $\lambda \in \mathbf{Card}^{\infty}$. Then for every $y \in Y$ with $g(y) = \lambda$ we have

$$\sum_{z \in Z} \min(h(z), |L_{yz}|) = \sum_{z \in Z} \min(h(z), \phi(x_0, y, z)) \geqslant \max(f(x_0), g(y)) = \lambda$$

since Γ is well Z-distributed, and

$$\sum_{z \in Z} \min(h(z), |L_{yz}|) \leqslant \sum_{z \in Z} \phi(x_0, y, z) = \lambda g(y) = \lambda,$$

so

$$\sum_{z \in Z} \min(h(z), |L_{yz}|) = \lambda.$$

Similarly

$$\sum_{y \in Y} \min(g(y), |L_{yz}|) = \lambda$$

for every $z \in Z$ with $g(z) = \lambda$. By Theorem 6 there exists a λ-regular $(g \cup h)$-splitting of D, hence a bipartite presentation of Γ over x_0, and so the proof is complete. $\square$

Proof of Theorem 5. For each $x \in X$ let X_x be a set of cardinality $f(x)$, for each $y \in Y$ let Y_y be a set of cardinality $g(y)$, and for each $z \in Z$ let Z_z be a set of cardinality $h(z)$. Assume that all the sets X_x, Y_y, Z_z are mutually disjoint. Let

$$X' = \bigcup_{x \in X} X_x, \quad Y' = \bigcup_{y \in Y} Y_y, \quad Z' = \bigcup_{z \in Z} Z_z.$$

By Lemmas 9 and 8, given $x \in X$, there is a bipartite presentation $D_x = (M^x, W^x, L^x, \xi^x, \zeta^x)$ of Γ over x and a perfect $f(x)$-colouring $\eta_x : L^x \to f(x)$ of D_x. Let $\Gamma' = (X', Y, Z, \phi')$ be the 3-weighted system such that if $x \in X$, $\vartheta_x : X_x \to f(x)$ is a bijection and $a \in X_x$, then

$$\phi'(a, y, z) = |L^x_{yz} \cap \eta_x^{-1}(\vartheta_x(a))|.$$

Then Γ' is a $(1, g, h)$-outline latin system such that Γ is the (α, i_Y, i_Z)-amalgamation of Γ', where i_Y and i_Z are the identity functions on Y and Z respectively and $\alpha : X' \to X$ is defined by $\alpha(x') = x$ if $x' \in X_x$.

Repeating the same argument we get next a $(1, 1, h)$-outline latin system $\Gamma'' = (X', Y', Z, \phi'')$ such that Γ' is the $(i_{X'}, \beta, i_Z)$-amalgamation of Γ'', where $i_{X'}$ and i_Z are the identity functions on X' and Z, respectively, and $\beta : Y' \to Y$ is defined by $\beta(y') = y$ if $y' \in Y_y$; and then we get finally a $(1, 1, 1)$-outline latin system $\Gamma''' = (X', Y', Z', \phi''')$ such that Γ'' is the $(i_{X'}, i_{Y'}, \gamma)$-amalgamation of Γ''' where $i_{X'}$ and $i_{Y'}$ are the identity functions on X' and Y', respectively, and $\gamma : Z' \to Z$ is defined by $\gamma(z') = z$ if $z' \in Z_z$. Then Γ is the (α, β, γ)-amalgamation of Γ''', hence it is the (α, β, γ)-amalgamation of the latin system corresponding to Γ'''. Therefore Γ is an amalgamated latin system and the proof is complete. $\square$

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